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Logical entropy of quantum dynamical systems

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Abstract: This paper introduces the concepts of logical entropy and conditional logical entropy of finite partitions on a quantum logic. Some of their ergodic properties are presented. Also logical entropy of a quantum dynamical system is defined and ergodic properties of dynamical systems on a quantum logic are investigated. Finally, the version of Kolmogorov-Sinai theorem is proved.

Keywords: logical entropy; quantum logic; dynamical system; partition

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1 Introduction

Birkhoff and Von Neumann in [1] have introduced the quantum logic approach. Entropy is a tool to measure the amount of uncertainty in random event. Entropy has been applied in a variety of problem areas including physics, computer science, general systems theory, information theory, statistics, biology, chemistry, sociology and many other fields. Hejun Yuan, Mona Khare and Shraddha Roy, using the notion of state of quantum logic, introduced Shanon entropy of finite partitions on a quantum logic [2–4]. The definition of entropy of a dynamical system might be in three stages [2, 5, 6]. Logical entropy is a measure on set of ordered pairs [7]. In 1982, Rao, Good, Patil and Taillie defined and studied the notion logical entropy [8–10]. Rao introduced precisely this concept as quadratic entropy [10] and in the years 2009 and 2013, this concept was discussed by Ellerman in [7, 11, 12].

The notion of Shanon entropy of quantum dynamical systems with Bayessian state was studied by Mona Khare and Shraddha Roy in [2]. In this paper, the notion of logical entropy with finite partitions is defined and then, logical entropy of quantum dynamical systems with Bayessian state is presented and studied. In Section 2, some basic definitions are presented. In Section 3, logical entropy and coditional logical entropy of partitions of a quantum logic with respect to state s are defined and a few results about them will be presented. In the subsequent section the relations s-refinement and $\overset{\circ}{=}_s$ are defined and logical entropy and coditional logical entropy under the relations are studied. In Section 4, logical entropy of a quantum dynamical system (L, s, φ) is defined where L is a quantum logic and s is a Bayessian state. At the end, the version of Kolmogorov-Sinai theorem is proved.

2 Finite Partitions

At first, some basic definitions are presented that will be useful in further considerations.

Definition 1. [4] A quantum logic QL is a σ-orthomodular lattice, i.e., a lattice L $(L, \leq, \vee, \wedge, 0, 1)$ with the smallest element 0 and the greatest element 1, an operation $' : L \to L$ such that the following properties hold for all $a, b, c \in L$:
i) $a'' = a$, $a \leq b \Rightarrow b' \leq a'$, $a \vee a' = 1$, $a \wedge a' = 0$;
ii) Given any finite sequence $(a_i)_{i\in I}$, $a_i \leq a_{j'}$, $i \neq j$, the join $\vee_{i\in\mathbb{N}} a_i$ exists in L;
iii) L is orthomodular: $a \leq b \Rightarrow b = a \vee (b \wedge a')$.
Two elements $a, b \in QL$ are called orthogonal if $a \leq b'$ and denoted by $a \perp b$. A sequence $(a_i)_{i\in I}$ is said orthogonal if $a_i \perp a_j$, $\forall i \neq j$.

Definition 2. [4] Let L be a quantum logic. A map $s : L \to [0, 1]$ is a state iff $s(1) = 1$ and for any orthogonal sequence $(a_i)_{i\in I}$, $s(\vee_{i\in I} a_i) = \sum_{i\in I} s(a_i)$.

Definition 3. [4] Let $P = \{a_1, \ldots, a_n\}$ be a finite system of elements of a quantum logic. P is called to be a $\vee$-orthogonal system iff $\vee_{i=1}^{k} a_i \perp a_{k+1}$, $\forall k$.

Definition 4. [4] A system $P = \{a_1, \ldots, a_n\} \subset L$ is said to be a partition of L corresponding to a state s if:
i) P is a $\vee$-orthogonal system;
ii) $s(\vee_{i=1}^{n} a_i) = 1$.

Note that from definition 2, we obtain $\sum_{i=1}^{n} s(a_i) = 1$.

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Definition 5. [4] Let the system $(b_1, \ldots, b_m)$ be any partition corresponding to a state s and $a \in L$. The state s is said has Bayes' Property if $s(\vee_{j=1}^{m}(a \wedge b_j)) = s(a)$.

Lemma 6. *[4] Let $Q = (b_1, \ldots, b_m)$ be a partition on L, and $a \in L$, and the state s has Bayes' Property. Then $\sum_{j=1}^{m} s(a \wedge b_j) = s(a)$.*

3 Logical entropy of finite partitions

Let $P = \{a_1, \ldots, a_n\}$ and $Q = \{b_1, \ldots, b_m\}$ be two finite partitions of a quantum logic corresponding to a state s. The common refinement of these partitions is:

$$P \vee Q = \{a_i \wedge b_j : a_i \in P, b_j \in R\}.$$

Definition 7. Let $P = \{a_1, \ldots, a_n\}$ be a partition of a quantum logic corresponding to a state s. The logical entropy of P with respect to state s is defined by:

$$h_s^l(P) = \sum_{i=1}^{n} s(a_i)(1 - s(a_i)).$$

Since $\sum_{i=1}^{n} s(a_i) = 1$, we have $h_s^l(P) = 1 - \sum_{i=1}^{n}(s(a_i))^2$.

Definition 8. Let $P = \{a_1, \ldots, a_n\}$ and $Q = \{b_1, \ldots, b_m\}$ be two partitions of a quantum logic corresponding to a state s. The conditional logical entropy of P given Q with respect to state s is defined as:

$$h_s^l(P|Q) = \sum_{i=1}^{n}\sum_{j=1}^{m} s(a_i \wedge b_j)(s(b_j) - s(a_i \wedge b_j)).$$

In the next theorem an upper bound for logical entropy on a quantum logic is presented.

Theorem 9. *Let P be a finite partition of a quantum logic corresponding to a state s. Then $0 \le h_s^l(P) \le 1 - \frac{1}{n}$.*

Proof. Let $P = (p_1, \ldots, p_n) \in \mathbf{R}^n$ be a probability distribution, then from [7], maximum value of the logical entropy is $1 - \dfrac{1}{n}$. Since $\sum_{i=1}^{n} s(a_i) = 1$, $P = (s(a_1), \ldots, s(a_n))$ is a probability distribution and hence the proof is complete. □

In the following theorem the conditional logical entropy under the common refinement of partitions is studied.

Theorem 10. *Let P, Q and R be finite partitions of a quantum logic corresponding to a state s having Bayes' Property. Then $h_s^l(P \vee Q|R) = h_s^l(P|R) + h_s^l(Q|P \vee R)$.*

Proof. Let $P = \{a_1, \ldots, a_n\}$, $Q = \{b_1, \ldots, b_m\}$ and $R = \{c_1, \ldots, c_r\}$. Since s has Bayes' Property, by Lemma 6 we have
$h_s^l(P \vee Q|R) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r} s(a_i \wedge b_j \wedge c_k)(s(c_k) - s(a_i \wedge b_j \wedge c_k)) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r} s(a_i \wedge b_j \wedge c_k)s(c_k)$
$- \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r}(s(a_i \wedge b_j \wedge c_k))^2$
$= \sum_{i=1}^{n}\sum_{k=1}^{r} s(a_i \wedge c_k)s(c_k) - \sum_{i=1}^{n}\sum_{k=1}^{r}(s(a_i \wedge c_k))^2$
$+ \sum_{i=1}^{n}\sum_{k=1}^{r}(s(a_i \wedge c_k))^2$
$- \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r}(s(a_i \wedge b_j \wedge c_k))^2.$
On the other hand we can write

$$h_s^l(P|R) = \sum_{i=1}^{n}\sum_{k=1}^{r} s(a_i \wedge c_k)s(c_k) - \sum_{i=1}^{n}\sum_{k=1}^{r}(s(a_i \wedge c_k))^2,$$

and
$h_s^l(Q|P \vee R) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r}(s(a_i \wedge b_j \wedge c_k)$
$s(a_i \wedge c_k)) - \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r}(s(a_i \wedge b_j \wedge c_k))^2$
$= \sum_{i=1}^{n}\sum_{k=1}^{r}(s(a_i \wedge c_k))^2$
$- \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r}(s(a_i \wedge b_j \wedge c_k))^2.$
Thus the proof is complete. □

Now the assertion of the following theorem will be proved that will be useful in further theorems.

Theorem 11. *Let P and Q be finite partitions of a quantum logic corresponding to a state s having Bayes' Property. Then $h_s^l(P \vee Q) = h_s^l(Q) + h_s^l(P|Q)$.*

Proof. Let $P = \{a_1, \ldots, a_n\}$ and $Q = \{b_1, \ldots, b_m\}$. From Lemma 6 we can write

$$\begin{aligned} h_s^l(P \vee Q) &= \sum_{i=1}^{n}\sum_{j=1}^{m} s(a_i \wedge b_j)(1 - s(a_i \wedge b_j)) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} s(a_i \wedge b_j) - \sum_{i=1}^{n}\sum_{j=1}^{m}(s(a_i \wedge b_j))^2 \\ &= \sum_{j=1}^{m} s(b_j) - \sum_{i=1}^{n}\sum_{j=1}^{m}(s(a_i \wedge b_j))^2. \end{aligned}$$

On the other hand $h_s^l(Q) = \sum_{j=1}^{m} s(b_j)(1 - s(b_j)) = \sum_{j=1}^{m} s(b_j) - \sum_{j=1}^{m}(s(b_j))^2$. Also,

$$\begin{aligned} h_s^l(P|Q) &= \sum_{i=1}^{n}\sum_{j=1}^{m} s(a_i \wedge b_j)(s(b_j) - s(a_i \wedge b_j)) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} s(a_i \wedge b_j)(s(b_j)) - \sum_{i=1}^{n}\sum_{j=1}^{m}(s(a_i \wedge b_j))^2 \\ &= \sum_{j=1}^{m}(s(b_j))^2 - \sum_{i=1}^{n}\sum_{j=1}^{m}(s(a_i \wedge b_j))^2. \end{aligned}$$

Hence the proof is complete. □

In the next theorem it is proved subadditivity of logical entropy of partitions on a quantum logic.

Theorem 12. *Let P and Q be finite partitions of a quantum logic corresponding to a state s having Bayes' Property. Then*
i) $h_s^l(P|Q) \le h_s^l(P)$;
ii) $\max\{h_s^l(P), h_s^l(Q)\} \le h_s^l(P \vee Q) \le h_s^l(P) + h_s^l(Q)$.

Proof. i) Let $P = \{a_1, \dots, a_n\}$ and $Q = \{b_1, \dots, b_m\}$. For each $a_i \in P$, we can write
$\sum_{j=1}^m s(a_i \wedge b_j)(s(b_j) - s(a_i \wedge b_j))$
$\le \sum_{j=1}^m s(a_i \wedge b_j)(\sum_{j=1}^m s(b_j) - s(a_i \wedge b_j))$
$= s(a_i)(\sum_{j=1}^m s(b_j) - s(a_i \wedge b_j))$
$= s(a_i)(1 - \sum_{j=1}^m s(a_i \wedge b_j)) = s(a_i)(1 - s(a_i))$.
So
$h_s^l(P|Q) = \sum_{i=1}^n \sum_{j=1}^m s(a_i \wedge b_j)(s(b_j) - s(a_i \wedge b_j)) \le \sum_{i=1}^n s(a_i)(1 - s(a_i)) = h_s^l(P)$.
ii) From Theorem 11 and part i),
$h_s^l(P \vee Q) = h_s^l(Q) + h_s^l(P|Q) \le h_s^l(Q) + h_s^l(P)$.
By Theorem 11, $\max\{h_s^l(P), h_s^l(Q)\} \le h_s^l(P \vee Q)$. □

Let s be a state. Two finite partitions P and Q of a quantum logic are called s-independent if $s(a \wedge b) = s(a)s(b)$ for all $a \in P$, and $b \in Q$.
In the next theorem you observe that, for two s-independent finite partitions P and Q of a QL,
$h_s^l(P \vee Q) \neq h_s^l(P) + h_s^l(Q)$ necessarily. Also, in this case $h_s^l(P|Q) \neq h_s^l(P)$ necessarily.

Theorem 13. *Let s be a state and let P and Q be s-independent finite partitions of a quantum logic. Then*
i) $h_s^l(P \vee Q) = h_s^l(P) + h_s^l(Q) - h_s^l(P)h_s^l(Q)$;
ii) $h_s^l(P|Q) = h_s^l(P)(1 - h_s^l(Q))$.

Proof. i) Since $h_s^l(P) = 1 - \sum_{i=1}^n (s(a_i))^2$ and $h_s^l(Q) = 1 - \sum_{j=1}^m (s(b_j))^2$ and P, Q are s-independent, we can write
$h_s^l(P \vee Q) = 1 - \sum_{i=1}^n \sum_{j=1}^m (s(a_i \wedge b_j))^2$
$= 1 - \sum_{i=1}^n \sum_{j=1}^m (s(a_i))^2 (s(b_j))^2$
$= 1 - (\sum_{i=1}^n (s(a_i))^2)(\sum_{j=1}^m (s(b_j))^2)$
$= 1 - (\sum_{i=1}^n (s(a_i))^2)(\sum_{j=1}^m (s(b_j))^2)$
$- \sum_{i=1}^n (s(a_i))^2 + \sum_{i=1}^n (s(a_i))^2$
$= h_s^l(P) + \sum_{i=1}^n (s(a_i))^2 (1 - \sum_{j=1}^m (s(b_j))^2)$
$= h_s^l(P) + (1 - h_s^l(P))h_s^l(Q)$
$= h_s^l(P) + h_s^l(Q) - h_s^l(P)h_s^l(Q)$.
ii) Follows from i) and Theorem 11. □

Definition 14. Let $P = \{a_1, \dots, a_n\}$ and $Q = \{b_1, \dots, b_m\}$ be two partitions of a quantum logic corresponding to a state s. We say Q is a s-refinement of P, denoted by $P \preceq_s Q$, if there exists a partition $I(1), \dots, I(n)$ of the set $\{1, \dots, m\}$ such that $a_i = \vee_{j \in I(i)} b_j$ for every $i = 1, \dots, n$.

Now the relation between the s-refinement and the logical entropy of finite partitions will be studied.

Theorem 15. *Let $P = \{a_1, \dots, a_n\}$, $Q = \{b_1, \dots, b_m\}$ and $R = \{c_1, \dots, c_r\}$ be partitions of a quantum logic corresponding to a state s. Then*
i) $P \preceq_s Q$ implies that $h_s^l(P) \le h_s^l(Q)$;
ii) If $P \preceq_s Q$ and the quantum logic be distributive then $h_s^l(P|R) \le h_s^l(Q|R)$.

Proof. i) Since $P \preceq_s Q$, there exists a partition $I(1), \dots, I(n)$ of the set $\{1, \dots, m\}$ such that $a_i = \vee_{j \in I(i)} b_j$ for every $i = 1, \dots, n$. So from definition 2, $s(a_i) = \sum_{j \in I(i)} s(b_j)$, therefore $\sum_{j=1}^m (s(b_j))^2 \le \sum_{i=1}^n (s(a_i))^2$, so

$$h_s^l(P) = 1 - \sum_{i=1}^n (s(a_i))^2 \le 1 - \sum_{j=1}^m (s(b_j))^2 = h_s^l(Q).$$

ii) $P \preceq_s Q$ implies that $P \vee R \preceq_s Q \vee R$, because let $a_i \wedge c$ be an arbitrary element of $P \vee R$, then there exists a partition $I(1), \dots, I(n)$ of the set $\{1, \dots, m\}$ such that $a_i = \vee_{j \in I(i)} b_j$ for every $i = 1, \dots, n$. Therefore $a_i \wedge c = (\vee_{j \in I(i)} b_j) \wedge c = \vee_{j \in I(i)} (b_j \wedge c)$, hence $P \vee R \preceq_s Q \vee R$. Now by Theorems 11 and 15, it will be obtained $h_s^l(P|R) = h_s^l(P \vee R) - h_s^l(R) \le h_s^l(Q \vee R) - h_s^l(R) = h_s^l(Q|R)$. □

Definition 16. Let $P = \{a_1, \dots, a_n\}$ and $Q = \{b_1, \dots, b_m\}$ be two partitions of a quantum logic corresponding to a state s. $P \subset_s^o Q$ if for each $b_j \in Q$ there exists $a_i \in P$ with $s(a_i \wedge b_j) = s(b_j)$. $P \doteq_s Q$ if $P \subset_s^o Q$ and $Q \subset_s^o P$.

The next theorem shows that, the logical entropy and logical conditional entropy of finite partitions of a quantum logic corresponding to a state s having Bayes' Property, are invariant under the relation $\doteq_s$.

Theorem 17. *Let P and Q be finite partitions of a quantum logic corresponding to a state s having Bayes' Property. Then*
i) $P \subset_s^o Q$ if and only if $h_s^l(P|Q) = 0$;
ii) if $P \doteq_s Q$ then $h_s^l(P) = h_s^l(Q)$;
iii) if $P \doteq_s Q$ then $h_s^l(P|R) = h_s^l(Q|R)$;
iv) if $Q \doteq_s R$ then $h_s^l(P|Q) = h_s^l(P|R)$.

Proof. i) Let $P = \{a_1, \dots, a_n\}$ and $Q = \{b_1, \dots, b_m\}$ and $P \subset_s^o Q$, then for each $b_j \in Q$ there exists $a_{i_0} \in P$ such that $s(b_j) = s(a_{i_0} \wedge b_j)$. Since $s(b_j) = \sum_{i=1}^n s(a_i \wedge b_j)$, we obtain $s(b_j) = s(a_{i_0} \wedge b_j)$ and for each $i \neq i_0$, $s(a_i \wedge b_j) = 0$ and so $h_s^l(P|Q) = 0$. Conversely, if $h_s^l(P|Q) = 0$ then for each i, j, $s(b_j) = s(a_i \wedge b_j)$ or $s(a_i \wedge b_j) = 0$. For an arbitrary element $b_j \in Q$, since $0 \neq s(b_j) = \sum_{i=1}^n s(a_i \wedge b_j)$ we deduce that there exists an i_1, $1 \le i_1 \le n$, such that $s(b_j) = s(a_{i_1} \wedge b_j)$.
ii) Since $P \subset_s^o Q$, by i) we have $h_s^l(P|Q) = 0$. So by theorem 11, $h_s^l(P|Q) = h_s^l(P \vee Q) - h_s^l(Q) = 0$ and therefore $h_s^l(P \vee Q) = h_s^l(Q)$. Similarly if $Q \subset_s^o P$, then $h_s^l(Q|P) = h_s^l(P \vee Q) -$

$h_s^l(P) = 0$ and so $h_s^l(P \vee Q) = h_s^l(P)$. Hence we imply that $h_s^l(P) = h_s^l(Q)$.
iii) We first show that $P \subset_s^o Q$ implies that $P \vee R \subset_s^o Q \vee R$. Let $b_{j_0} \wedge c_{k_0}$ be an arbitrary element of $Q \vee R$. Since $P \subset_s^o Q$, there exists $a_{i_0} \in P$ such that $s(a_{i_0} \wedge b_{j_0}) = s(b_{j_0})$. Now $s((a_{i_0} \wedge c_{k_0}) \wedge (b_{j_0} \wedge c_{k_0})) = s(a_{i_0} \wedge b_{j_0} \wedge (c_{k_0} \wedge c_{k_0})) = s(a_{i_0} \wedge b_{j_0} \wedge c_{k_0})$, it is sufficient to show that $s(a_{i_0} \wedge b_{j_0} \wedge c_{k_0}) = s(b_{j_0} \wedge c_{k_0})$. Since s has Bayes' Property, $s(a_{i_0} \wedge b_{j_0}) = s(b_{j_0}) = \sum_{i=1}^n s(a_i \wedge b_{j_0})$. So for each $i \neq i_0$, $s(a_i \wedge b_{j_0}) = 0$, therefore for each $i \neq i_0$, $s(a_i \wedge b_{j_0} \wedge c_{k_0}) = 0$ and this implies that

$$s(a_{i_0} \wedge b_{j_0} \wedge c_{k_0}) = \sum_{i=1}^n s(a_i \wedge b_{j_0} \wedge c_{k_0}) = s(b_{j_0} \wedge c_{k_0}).$$

Thus $P \vee R \subset_s^o Q \vee R$. By changing the role of P and Q, $P \doteq_s Q$ implies that $P \vee R \doteq_s Q \vee R$. Hence from ii), $h_s^l(P|R) = h_s^l(P \vee R) - h_s^l(R) = h_s^l(Q \vee R) - h_s^l(R) = h_s^l(Q|R)$. iv) We need to show that $Q \subset_s^o R$ implies that $P \vee Q \subset_s^o P \vee R$. Let $a_{i_0} \wedge c_{k_0}$ be an arbitrary element of $P \vee R$. Since $Q \subset_s^o R$, there exists $b_{j_0} \in Q$ such that $s(b_{j_0} \wedge c_{k_0}) = s(c_{k_0})$. Now $s((a_{i_0} \wedge b_{j_0}) \wedge (a_{i_0} \wedge c_{k_0})) = s((a_{i_0} \wedge a_{i_0}) \wedge b_{j_0} \wedge c_{k_0}) = s(a_{i_0} \wedge b_{j_0} \wedge c_{k_0})$, it is sufficient to show that $s(a_{i_0} \wedge b_{j_0} \wedge c_{k_0}) = s(a_{i_0} \wedge c_{k_0})$. Since s has Bayes' Property, $s(b_{j_0} \wedge c_{k_0}) = s(c_{k_0}) = \sum_{j=1}^m s(b_j \wedge c_{k_0})$. So for each $j \neq j_0$, $s(b_j \wedge c_{k_0}) = 0$, therefore for each $j \neq j_0$, $s(a_{i_0} \wedge b_j \wedge c_{k_0}) = 0$ and this implies that

$$s(a_{i_0} \wedge b_{j_0} \wedge c_{k_0}) = \sum_{j=1}^m s(a_{i_0} \wedge b_j \wedge c_{k_0}) = s(a_{i_0} \wedge c_{k_0}).$$

Thus $P \vee Q \subset_s^o P \vee R$. By changing the role of Q and R, $Q \doteq_s R$ implies that $P \vee Q \doteq_s P \vee R$. Now from ii), $h_s^l(P|Q) = h_s^l(P \vee Q) - h_s^l(Q) = h_s^l(P \vee R) - h_s^l(R) = h_s^l(P|R)$. □

4 Logical entropy of quantum dynamical systems

Definition 18. [2] Let L be a quantum logic and $\varphi : L \to L$ be a map with the following properties:
i) $\varphi(a \vee b) = \varphi(a) \vee \varphi(b), \forall a, b \in L$;
ii) $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b), \forall a, b \in L$;
iii) $\varphi(a') = (\varphi(a))', \forall a \in L$.

$\varphi : L \to L$ with respect to a state s is called state preserving if $s(\varphi(a)) = s(a)$ for every $a \in L$. Then the triple (L, s, φ) is said a quantum dynamical system where the state s having Bayes' Property. In the following theorem the existence of the limit in Definition 20 is shown.

Theorem 19. *Let (L, s, φ) be a quantum dynamical system and P be a partition of (L, s), then $\lim_{n\to\infty} \frac{1}{n} h_s^l(\vee_{i=1}^n \varphi^i P)$ exists.*

Proof. Let $a_n = h_s^l(\vee_{i=1}^n \varphi^i P)$. It will be shown that for $p \in \mathbf{N}$, $a_{n+p} \leq a_n + a_p$ and then by Theorem 4.9 in [13], $\lim_{n\to\infty} \frac{1}{n} a_n$ exists and equals $\inf_n \frac{a_n}{n}$. By Theorem 12, ii) we have

$$\begin{aligned} a_{n+p} &= h_s^l(\vee_{i=0}^{n+p-1} \varphi^i P) \\ &\leq h_s^l(\vee_{i=0}^{n-1} \varphi^i P) + h_s^l(\vee_{i=n}^{n+p-1} \varphi^i P) \\ &= a_n + h_s^l(\vee_{i=0}^{p-1} \varphi^{n+i} P) \\ &= a_n + h_s^l(\varphi^n(\vee_{i=0}^{p-1} \varphi^i P) \\ &= a_n + h_s^l(\vee_{i=0}^{p-1} \varphi^i P) \\ &= a_n + a_p. \end{aligned}$$

□

The second stage and the final stage of the definition of the logical entropy of a quantum dynamical system (L, s, φ) is given in the next definition.

Definition 20. Let (L, s, φ) be a quantum dynamical system and P be a partition of (L, s). The logical entropy of T respect to P is defined by:

$$h_s^l(\varphi, P) = \lim_{n\to\infty} \frac{1}{n} h_s^l(\vee_{i=1}^n \varphi^i P).$$

The logical entropy of φ is defined as:

$$h_s^l(\varphi) = \sup_P h_s^l(\varphi, P)$$

where the supremum is taken over all finite partitions of (L, s).

In the following proposition some ergodic properties of $h_s^l(\varphi, P)$ and $h_s^l(\varphi)$ will be studied.

Proposition 21. *If (L, s, φ) is a quantum dynamical system and P is a partition of (L, s), then*
i) $h_s^l(\varphi, P) = h_s^l(\varphi, \vee_{i=1}^k \varphi^i P)$;
ii) For $k \in \mathbb{N}$, $h_s^l(\varphi^k) = k h_s^l(\varphi)$.

Proof. i) $h_s^l(\varphi, \vee_{i=1}^k \varphi^i P) = \lim_{n\to\infty} \frac{1}{n} h_s^l(\vee_{j=1}^n \varphi^j(\vee_{i=1}^k \varphi^i P))$
$= \lim_{n\to\infty} \frac{1}{n} h_s^l(\vee_{i=1}^{k+n-1} \varphi^i P)$
$= \lim_{n\to\infty} (\frac{k+n-1}{n})(\frac{1}{k+n-1}) h_s^l(\vee_{i=1}^{k+n-1} \varphi^i P) = h_s^l(\varphi, P)$.
ii) Let P be an arbitrarary finite partition of (L, s). we can

write

$$\begin{aligned} h_s^l(\varphi^k, \vee_{i=1}^n \varphi^i P) &= \lim_{n\to\infty} \frac{1}{n} h_s^l(\vee_{j=1}^n (\varphi^k)^j (\vee_{i=1}^n \varphi^i P)) \\ &= \lim_{n\to\infty} \frac{1}{n} h_s^l(\vee_{j=1}^n \vee_{i=1}^k \varphi^{kj+i} P) \\ &= \lim_{n\to\infty} \frac{1}{n} h_s^l(\vee_{i=1}^{nk-1} \varphi^i P) \\ &= \lim_{n\to\infty} \frac{nk}{n} \frac{1}{nk} h_s^l(\vee_{i=1}^{nk-1} \varphi^i P) \\ &= k h_s^l(\varphi, P) \end{aligned}$$

So $kh_s^l(\varphi) = k \sup_P h_s^l(\varphi, P) = \sup_P h_s^l(\varphi^k, \vee_{i=1}^n \varphi^i P) \leq \sup_P h_s^l(\varphi^k, P) = h_s^l(\varphi^k)$. Since $P \preceq_s \vee_{i=1}^k \varphi^i P$, it will be obtained $h_s^l(\varphi^k, P) \leq h_s^l(\varphi^k, \vee_{i=1}^n \varphi^i P) = kh_s^l(\varphi, P)$. □

Definition 22. Let (L, s, φ) be a quantum dynamical system. A finite partition R of (L, s), is said to be an s-generator of φ, if there exists $r \in \mathbb{N}$ such that $P \preceq_s \vee_{i=1}^r \varphi^i R$ for each finite partition P of (L, s).

The main aim of this theorem is to prove an analogue of the Kolmogorov-Sinaj theorem on logical entropy and generators.

Theorem 23. *Let (L, s, φ) be a quantum dynamical system and let R be an s-generator of φ, then $h_s^l(\varphi) = h_s^l(\varphi, R)$.*

Proof. Let P be an arbitrary finite partition of (L, s). Since R is an s-generator, $P \preceq_s \vee_{i=1}^r \varphi^i R$. By Theorem 15, you get $h_s^l(\varphi, P) \leq h_s^l(\varphi, \vee_{i=1}^r \varphi^i R) = h_s^l(\varphi, R)$. Hence $h_s^l(\varphi) = \sup_P h_s^l(\varphi, P) \leq h_s^l(\varphi, R)$. On the other hand $h_s^l(\varphi, R) \leq h_s^l(\varphi)$. □

5 Conclusion

This paper has introduced logical entropy and conditional logical entropy of finite partitions on a quantum logic and has presented some of their ergodic properties. Also, logical entropy of a quantum dynamical system with finite partitions studied and some of its properties proved.

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