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Multi-soliton rational solutions for some nonlinear evolution equations

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Abstract: The Korteweg-de Vries equation (KdV) and the (2+ 1)-dimensional Nizhnik-Novikov-Veselov system (NNV) are presented. Multi-soliton rational solutions of these equations are obtained via the generalized unified method. The analysis emphasizes the power of this method and its capability of handling completely (or partially) integrable equations. Compared with Hirota's method and the inverse scattering method, the proposed method gives more general exact multi-wave solutions without much additional effort. The results show that, by virtue of symbolic computation, the generalized unified method may provide us with a straightforward and effective mathematical tool for seeking multi-soliton rational solutions for solving many nonlinear evolution equations arising in different branches of sciences.

Keywords: multi-soliton rational solution; generalized unified method; Korteweg-de Vries equation; (2+ 1)-dimensional Nizhnik-Novikov-Veselov equation

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1 Introduction

Many complex phenomena and dynamic processes in physics, mechanics, chemistry and biology can be represented by nonlinear evolution equations (NEEs) [1–7]. When we want to understand the physical mechanism of nature phenomena, described by NEEs, exact solutions for the NEEs have to be explored. Therefore, it is crucial to obtain the most general solutions of the corresponding NEEs describing the evolution of such nonlinear systems. The general solutions of the NEEs provide a lot of information about the intrinsic structure of such equations.

There are various types of wave solutions that are revealed for NEEs. Among these types: the cnoidal waves, snoidal waves, shock waves, periodic waves, solitary waves and soliton waves.

Nowadays, solitons are studied in various areas of non-linear science and many researchers focuss themselves to find a single soliton solution as well as the shock wave solution for NEEs by the aid of the solitary wave ansatz method [8–10].

In this paper, we search for multi-soliton rational solutions of NEEs which they are playing an important role in treating nonlinear problems.

A variety of methods for studying the integrability of nonlinear partial differential equations and for constructing multiple-solitary wave solutions have been developed. Among these methods, the inverse scattering method [11–13], Hirota's bilinear method and its simplified form [14–17].

The inverse scattering method represents a nonlinear partial differential equation as a condition of compatibility between two linear operators, the so-called Lax pairs [18]. In fact, this method requires heavy calculation work.

In Hirota's method, we use the bilinear transformation equation where solitary wave solutions can be constructed by using exponentials. Equivalently, this method may be thought as a rational function solution of nonlinear combination of exponential functions. That is in some sorts, it is a generalization to the well known exponential function method. While the simplified Hirota method does not depend on the construction of bilinear forms; instead it assumes that the multi-solitary wave solutions can be expressed as polynomials in exponential functions. Hirota's bilinear method and the simplified Hirota approach are rather heuristic and are significant for handling nonlinear equations. Although Hirota's method and the simplified Hirota method need simplest calculations but they assert only multi-solitary wave solutions as polynomials in exponential functions.

The main aim in this work is to present the generalized unified method which is accomplished by presenting a new algorithm to construct multi-wave solutions for NEEs. This method generalizes the unified method in [19–21].

Here, we use the idea of the generalized unified method to find multi-soliton wave rational solutions for KdV [22, 23] and NNV [24–26].

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The remainder of this paper is organized as follows. In Section 2, a description of the generalized unified method is given in detail. In Section 3, the applications of the generalized unified method to KdV equation and NNV equations are illustrated. Conclusions are presented in Section 4.

2 A methodology to the generalized unified method

In this Section, we present the outline of the generalized unified method.

Consider the NEEs of the type (q+1)-dimension

$$\begin{aligned} &F_i(u_j, (u_j)_t, (u_j)_{x_1}, \ldots, (u_j)_{x_q}, (u_j)_{x_1 x_2}, (u_j)_{x_1 x_3}, \ldots) \\ &= 0, \\ &i,\ j = 1,\ 2, \ldots\ m, \end{aligned} \tag{1}$$

where $u_j = u_j(t, x_1, \ldots, x_q)$.

Each physical observable u_j possess $(q+1)$ basic traveling wave solutions that satisfy the equation

$$\begin{aligned} &H_i(U_j,\ (U_j)_{z_1}, \ldots,\ (U_j)_{z_q},\ (U_j)_{z_1 z_2},\ (U_j)_{z_1 z_3}, \ldots) = 0, \\ &z_j = \alpha_j\, t + \sum_{s=1}^{q} \alpha_{j,s}\, x_s, \end{aligned} \tag{2}$$

where $U_j = U_j(z_1, \ldots, z_{q+1})$, α_j and $\alpha_{j,s}$ are arbitrary constants.

The fundamental rules and objectives of the unified method are used here (for details see [19]). The only distinction is that the main aim in [19] is to search for a single traveling wave solution, namely $U_j = U_j(z)$, $z = \alpha_0\, t + \sum_{j=1}^{q} \alpha_j\, x_j$.

For N-soliton wave solutions of (1), we have to construct the solutions in the form

$$u(x_1, \ldots, x_q, t) = U(z_1, \ldots, z_{N+q}). \tag{3}$$

By using the unified method [19], we obtain solutions in the form;

(i) Polynomial function solutions

(ii) Rational function solutions

In this paper, we confine ourselves to find rational function solutions.

2.1 The rational function solutions

Here, we search for a rational function solution of Equation (2) which is a bilinear transform in a linear or a nonlinear combinations of the auxiliary functions $\phi_l(z_l)$, $l = 1, 2, \ldots, N+q-1$. To this end, we introduce the steps of computations of N-wave rational solutions as follows:

Step 1. The generalized unified method asserts that, the N-wave solutions of (2)

$$\begin{aligned} &U(z_1, z_2, \ldots, z_{N+q-1}) \\ &= \frac{P_n(\phi_1(z_1),\ \phi_2(z_2), \ldots, \phi_{N+q-1}(z_{N+q-1}))}{Q_r(\phi_1(z_1),\ \phi_2(z_2), \ldots, \phi_{N+q-1}(z_{N+q-1}))}, \\ &n \geq r, \end{aligned} \tag{4}$$

where P_n and Q_r are polynomials in the auxiliary functions $\phi_j(z_j)$, $j = 1, 2, \ldots, N+q-1$ which satisfy the auxiliary equations

$$\begin{aligned} \left(\phi_j'(z_j)\right)^p &= \sum_{r=0}^{p\,k} b_{j,r}\, \phi_j^r(z_j),\ z_j = \alpha_{j,0}\, t + \sum_{s=1}^{q} \alpha_{j,s}\, x_s, \\ p &= 1, 2,\ k \geq 1, \end{aligned} \tag{5}$$

where $b_{j,r}$, $\alpha_{j,s}$ and $\alpha_{j,0}$ are constants. It is worth to be noticing that, n and k are determined from the balance equation by the criteria given in [19–21].

Also, a second condition (the consistency condition), which asserts that the constants in Equations (4) and (5) could be consistently determined, is used.

When $p = 1$, (5) solves to elementary solutions (explicit or implicit) while when $p = 2$, it solves to elliptic solutions.

When $p = 1$ and $n = r$, then $k = 1$ and the solutions of the auxiliary Equations (5) are called "jet streams".

The polynomial in the numerator of the rational function solutions when $n = r,\ k = 1$ takes the form

$$\begin{aligned} &P_n\left(\phi_1(z_1),\ \phi_2(z_2), \ldots, \phi_{N+q-1}(z_{N+q-1})\right) = a_0 \\ &+ \sum_{i_1=1}^{n} a_{i_1}\, \phi_{i_1}(z_{i_1}) + \sum_{i_1,\, i_2=1}^{n} a_{i_1, i_2}\, \phi_{i_1}(z_{i_1})\, \phi_{i_2}(z_{i_2}) + \ldots \\ &+ \sum_{i_1, i_2, \ldots, i_{N+q-1}=1}^{n} \left(a_{i_1, i_2, \ldots, i_{N+q-1}}\, \phi_{i_1}(z_{i_1}) \phi_{i_2}(z_{i_2})\right. \\ &\left.\ldots \phi_{i_{N+q}}(z_{i_{N+q-1}})\right) + b_N \prod_{k=1}^{N} \phi_k(z_k),\ n = N+q-1, \end{aligned} \tag{6}$$

where $i_1 < i_2 < \ldots < i_{N+q-1}$, $N \geq 2$ and $a_0,\ a_{i_1},\ a_{i_1, i_2}, \ldots, a_{i_1, i_2, \ldots, i_{N+q-1}},\ b_N$ are arbitrary constants to be determined latter. The polynomial $Q_r(\phi_1(z_1),\ \phi_2(z_2), \ldots, \phi_{N+q-1}(z_{N+q-1}))$ takes a similar form as in (6).

Now, we introduce the following theorem:

Theorem 1. *The N-soliton solutions via rational function solutions (when $k = 1$) are given by*

$$\begin{aligned} &U(z_1, z_2, \ldots, z_{N+q-1}) \\ &= \frac{P_n(\phi_1(z_1),\ \phi_2(z_2), \ldots, \phi_{N+q-1}(z_{N+q-1}))}{Q_n(\phi_1(z_1),\ \phi_2(z_2), \ldots, \phi_{N+q-1}(z_{N+q-1}))}, \end{aligned} \tag{7}$$

where $P_n(\phi_1(z_1),\ \phi_2(z_2), \dots, \phi_{N+q-1}(z_{N+q-1}))$ *is given by* (6) *and the auxiliary equations are* $\phi_l'(z_l) = c_l\,\phi_l(z_l)$, *where* c_l *are arbitrary constants and* $l = 1, 2, \dots, N+q-1$.

Step 2. By inserting (4) together the auxiliary equations $\phi_l'(z_l) = c_l\,\phi_l(z_l)$ into (2), we get an equation which is splitting to a set of nonlinear algebraic equations namely "the principle equations". They are solved by any computer algebra system.
Step 3. Solving the auxiliary equations.
Step 4. Finding the formal exact solutions which is given in (4).

3 Models and applications

In this section, we will apply the method described in Section 2 to find the exact multi-soliton rational solutions of KdV equation and NNV equations which are very important in the mathematical physics and have been paid attention by many researchers.

Model 1. *The Korteweg-de Vries equation (KdV)*
Consider the KdV equation [22, 23]

$$u_t + \lambda\,u_{xxx} + \nu\,u\,u_x = 0, \tag{8}$$

where λ *and* ν *are arbitrary constants. We mention that* (8) *is a fundamental mathematical model for the description of weakly nonlinear wave propagation in dispersive media. Here* $u = u(x,t)$ *is an appropriate field variable and* x, t *are space coordinate and time respectively. The coefficients* λ *and* ν *are determined by the medium properties and can be either constants or functions of* x, t. *An incomplete list of physical applications of the KdV equation includes shallow-water gravity waves, ion-acoustic waves in collisionless plasma, internal waves in the atmosphere and ocean, and waves in bubbly fluids [27].*

By using the new dependent variable transformation $u(x,t) = w_x(x,t)$ in Equation (8) and integrating both sides with respect to x, Equation (8) can be written as

$$w_t + \lambda\,w_{xxx} + \frac{\nu}{2}\,w_x^2 = 0, \tag{9}$$

where the constant of integration is considered to be zero.

Multi-soliton rational solutions (when $N = 2$)

From equations (4) and (6) when $N = 2$, we have

$$\begin{aligned} w(x,t) &= W(z_1, z_2) \\ &= \frac{p_0 + p_1\,\phi_1(z_1) + p_2\,\phi_2(z_2) + p_3\,\phi_1(z_1)\,\phi_2(z_2)}{r_0 + r_1\,\phi_1(z_1) + r_2\,\phi_2(z_2) + r_3\,\phi_1(z_1)\,\phi_2(z_2)}, \end{aligned} \tag{10}$$

where $z_1 = \alpha_1\,x + \alpha_2\,t$, $z_2 = \beta_1\,x + \beta_2\,t$ and α_k, β_k, p_i, r_i, $i = 0, 1, 2, 3$, $k = 1, 2$ are arbitrary constants. The auxiliary functions $\phi_j(z_j)$ satisfy the auxiliary equations $\phi_j'(z_j) = c_j\,\phi_j(z_j)$, where c_j are arbitrary constants, $j = 1, 2$.

By substituting from (10) into (9) and by using any package in symbolic computations, we get

$$\begin{gathered} r_3 = \frac{\nu^2\,p_1\,p_2\,R_-^2\,r_0}{R_1\,R_2\,R_+^2},\quad r_2 = \frac{\nu\,p_2\,r_0}{R_2},\quad r_1 = \frac{\nu\,p_1\,r_0}{R_1}, \\ p_3 = \frac{\nu\,p_1\,p_2\,R_-^2\,(R_1 + R_2 - \nu\,p_0)}{R_1\,R_2\,R_+^2}, \\ \alpha_2 = -\lambda\,c_1^2\,\alpha_1^3,\quad \beta_2 = -\lambda\,c_2^2\,\beta_1^3, \end{gathered} \tag{11}$$

where $R_\pm = (c_1\,\alpha_1 \pm c_2\,\beta_1)$, $R_1 = p_0\,\nu + 12\,\lambda\,c_1\,\alpha_1\,r_0$ and $R_2 = p_0\,\nu + 12\,\lambda\,c_2\,\beta_1\,r_0$.
p_0, p_1, p_2, r_0, c_1, c_2, λ, ν, α_1 and β_1 are arbitrary constants.

By solving the auxiliary equations $\phi_j'(z_j) = c_j\,\phi_j(z_j)$, $j = 1, 2$ and substituting together with (11) into (10), we get the solution of Equation (8) namely

$$\begin{aligned} u(x,t) &= w_x(x,t), \\ w(x,t) &= W(z_1, z_2) \\ &= \frac{R_1\,R_2\,R_+^2\,(p_0 + p_1\,e^{c_1\,z_1} + p_2\,e^{c_2\,z_2})}{r_0\,\big(R_1\,R_2\,R_+^2 + \nu\,R_+^2\,(p_1\,R_2\,e^{c_1\,z_1} + p_2\,R_1\,e^{c_2\,z_2})} \\ &\quad \frac{+\nu\,p_1\,p_2\,R_-^2 e^{c_1\,z_1 + c_2\,z_2}\,(R_1 + R_2 - \nu\,p_0)}{+\nu^2\,p_1\,p_2\,R_-^2 e^{c_1\,z_1 + c_2\,z_2}\big)}, \end{aligned} \tag{12}$$

where $z_1 = \alpha_1\,(x - \lambda\,c_1^2\,\alpha_1\,t)$ and $z_2 = \beta_1\,(x - \lambda\,c_2^2\,\beta_1\,t)$.

Shape and motion of the solution given by (12) is depicted in Figure 1

Multi-soliton rational solutions (when $N = 3$)

Now, we find a multi-soliton rational solution of Equation (8) when $N = 3$.

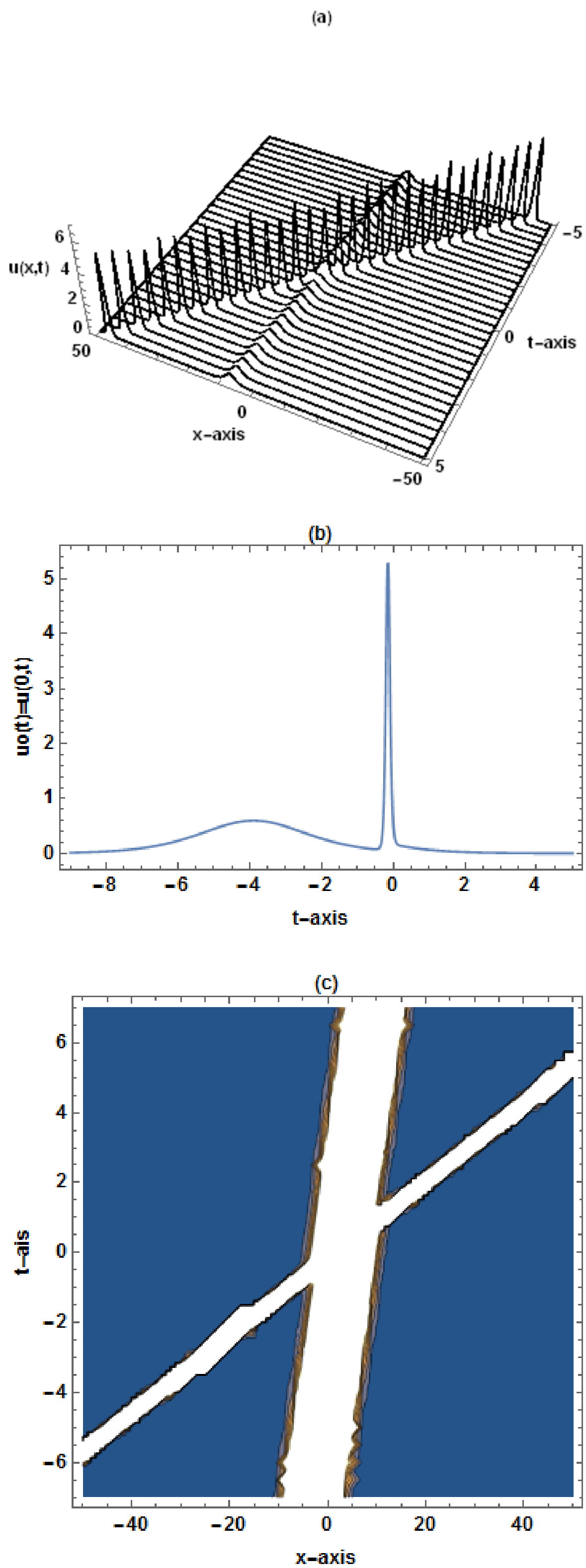

Figure 1: (a) 3D-plot for $u(x,t)$. (b) 2D-plot for $u(x,t)$ when $x = 0$. (c) the contour plot for $u(x,t)$. $\alpha_1 = 1, \beta_1 = 2, \lambda = 1, \nu = 5$ and $p_0 = 1/20,\ r_0 = 1,\ p_1 = 3/20,\ p_2 = 1/5,\ c_1 = 3,\ c_2 = 1/2$.

From Equations (4) and (6) when $N = 3$, we have

$$\begin{aligned} w(x,t) &= W(z_1, z_2, z_3) \\ &= \frac{p_0 + \sum_{s=1}^{3} p_s\,\phi_s(z_s) + \sum_{i,j=1}^{3} p_{i,j}\,\phi_i(z_i)\,\phi_j(z_j) + p_4\,\phi_1(z_1)\,\phi_2(z_2)\,\phi_3(z_3)}{r_0 + \sum_{s=1}^{3} r_s\,\phi_s(z_s) + \sum_{i,j=1}^{3} r_{i,j}\,\phi_i(z_i)\,\phi_j(z_j) + r_4\,\phi_1(z_1)\,\phi_2(z_2)\,\phi_3(z_3)}, \end{aligned} \tag{13}$$

where $i < j$, $z_1 = \alpha_1 x + \alpha_2 t$, $z_2 = \beta_1 x + \beta_2 t$, $z_3 = \gamma_1 x + \gamma_2 t$ and α_k, β_k, γ_k p_0, r_0, p_s, r_s, $p_{i,j}$, $r_{i,j}$, $i, j = 1,2,3$, $s = 1,2,3,4$, $k = 1,2$ are arbitrary constants. The auxiliary functions $\phi_l(z_l)$ satisfy the auxiliary equations $\phi_l'(z_l) = c_l\,\phi_l(z_l)$, where c_l are arbitrary constants, $l = 1,2,3$.

By substituting from (13) into (9) and by a similar way as we did in the last case (when $N = 2$), we get the solution of (8) in the form

$$\begin{aligned} u(x,t) &= w_x(x,t), \\ w(x,t) &= W(z_1,z_2,z_3) \\ &= \frac{\psi_1(z_2,z_3) + \psi_2(z_1,z_2,z_3)}{\nu\, r_0(\psi_3(z_2,z_3) + \psi_4(z_1,z_2,z_3))}, \\ \psi_1 &= r_0\,H_+^2\,R_+^2\,\Big(r_0\,\big(\nu\,p_0\,r_2\,M_-^2 + r_{2,3}\,M_+^2\,R_3\,e^{c_3 z_3}\big) \\ &\quad + r_2\,M_-^2\,e^{c_2 z_2}\,\big(r_2\,R_2 + r_{2,3}\,(R_2 + R_3 - p_0\,\nu)\,e^{c_3 z_3}\big)\Big), \\ \psi_2 &= r_1\,e^{c_1 z_1}\,\Big(R_+^2\,r_0\,\big(r_2\,R_1\,M_-^2\,H_+^2 \\ &\quad + r_{2,3}\,H_-^2\,M_+^2\,(R_1 + R_3 - p_0\,\nu)\,e^{c_3 z_3}\big) \\ &\quad + R_-^2\,r_2\,M_-^2\,e^{c_2 z_2}\,(r_2\,(R_1 + R_2 - p_0\,\nu) \\ &\quad H_+^2 + r_{2,3}\,H_-^2\,(R_1 + R_2 + R_3 - 2p_0\,\nu)\,e^{c_3 z_3})\Big), \\ \psi_3 &= R_+^2\,r_0\,H_+^2\,\Big(r_2\,M_-^2\,e^{c_2 z_2}\,\big(r_2 + r_{2,3}\,e^{c_3 z_3}\big) \\ &\quad + r_0\,\big(r_2\,M_-^2 + r_{2,3}\,M_+^2\,e^{c_3 z_3}\big)\Big), \\ \psi_4 &= r_1\,e^{c_1 z_1}\,\Big(R_-^2\,r_2\,M_-^2\,e^{c_2 z_2}\,\big(r_2\,H_+^2 + r_{2,3}\,H_-^2\,e^{c_3 z_3}\big) \\ &\quad + r_0\,R_+^2\,\big(r_2\,M_-^2\,H_+^2 + r_{2,3}\,H_-^2\,M_+^2\,e^{c_3 z_3}\big)\Big), \end{aligned} \tag{14}$$

where $R_\pm = (c_1\,\alpha_1 \pm c_2\,\beta_1)$, $H_\pm = (c_1\,\alpha_1 \pm c_3\,\gamma_1)$, $M_\pm = (c_2\,\beta_1 \pm c_3\,\gamma_1)$, $R_1 = p_0\,\nu + 12\,\lambda\,c_1\,\alpha_1\,r_0$, $R_2 = p_0\,\nu + 12\,\lambda\,c_2\,\beta_1\,r_0$ and $R_3 = p_0\,\nu + 12\,\lambda\,c_3\,\gamma_1\,r_0$.
$z_1 = \alpha_1\,(x - \lambda\,c_1^2\,\alpha_1\,t)$, $z_2 = \beta_1\,(x - \lambda\,c_2^2\,\beta_1\,t)$ and $z_3 = \gamma_1\,(x - \lambda\,c_3^2\,\gamma_1\,t)$, where p_0, r_0, r_1, r_2, $r_{2,3}$, c_1, c_2, c_3, λ, ν, α_1, β_1 and γ_1 are arbitrary constants.

Shape and motion of the solution given by (14) is depicted in Figure 2

Model 2. *The (2+ 1)-dimensional Nizhnik-Novikov-Veselov equations (NNV)*
Here, we apply the generalized unified method described in Section 2 to find multi-soliton rational wave solutions of NNV which read [24–26]

$$\begin{aligned} u_t - u_{xxx} + \alpha\,(u\,v)_x &= 0, \\ u_x + \beta\,v_y &= 0, \end{aligned} \tag{15}$$

where α and β are arbitrary constants. The NVV system may be considered as a model for an incompressible fluid where u and v are components of the (dimensionless) velocity [28]. Boiti and et al. solved this system of equations via the inverse scattering transformation [29]. It is well known that, the system in (15) is an isotropic Lax integrable extension of the well known (1+ 1)-dimensional KdV equations and has physical significance [30]. Also, NNV system can also be obtained from the inner parameter-dependent symmetry constraint of the KP equation [31].

By using the new dependent variable transformations $u(x, y, t) = u_{1x}(x, y, t)$ and $v(x, y, t) = v_{1x}(x, y, t)$ in Equation (15) and integrating both sides with respect to x, Equation (15) can be written as

$$\begin{aligned} u_{1t} - u_{1xxx} + \alpha\,u_{1x}\,v_{1x} &= 0, \\ u_{1x} + \beta\,v_{1y} &= 0, \end{aligned} \tag{16}$$

where the constants of integration are considered to be zero.

Multi-soliton rational solutions (when $N = 2$)

From Equations (4) and (6) when $N = 2$, we have

$$\begin{aligned} u_1(x, y, t) &= U(z_1, z_2) \\ &= \frac{p_0 + p_1\,\phi_1(z_1) + p_2\,\phi_2(z_2) + p_3\,\phi_1(z_1)\,\phi_2(z_2)}{q_0 + q_1\,\phi_1(z_1) + q_2\,\phi_2(z_2) + q_3\,\phi_1(z_1)\,\phi_2(z_2)}, \\ v_1(x, y, t) &= V(z_1, z_2) \\ &= \frac{r_0 + r_1\,\phi_1(z_1) + r_2\,\phi_2(z_2) + r_3\,\phi_1(z_1)\,\phi_2(z_2)}{q_0 + q_1\,\phi_1(z_1) + q_2\,\phi_2(z_2) + q_3\,\phi_1(z_1)\,\phi_2(z_2)}, \end{aligned} \tag{17}$$

where $z_1 = \alpha_1\,x+\alpha_2\,y+\alpha_3\,t$, $z_2 = \beta_1\,x+\beta_2\,y+\beta_3\,t$ and α_k, β_k, p_i, q_i, r_i, $i = 0, 1, 2, 3$, $k = 1, 2, 3$ are arbitrary constants. The auxiliary functions $\phi_j(z_j)$ satisfy the auxiliary equations $\phi_j'(z_j) = c_j\,\phi_j(z_j)$, where c_j are arbitrary constants, $j = 1, 2$.

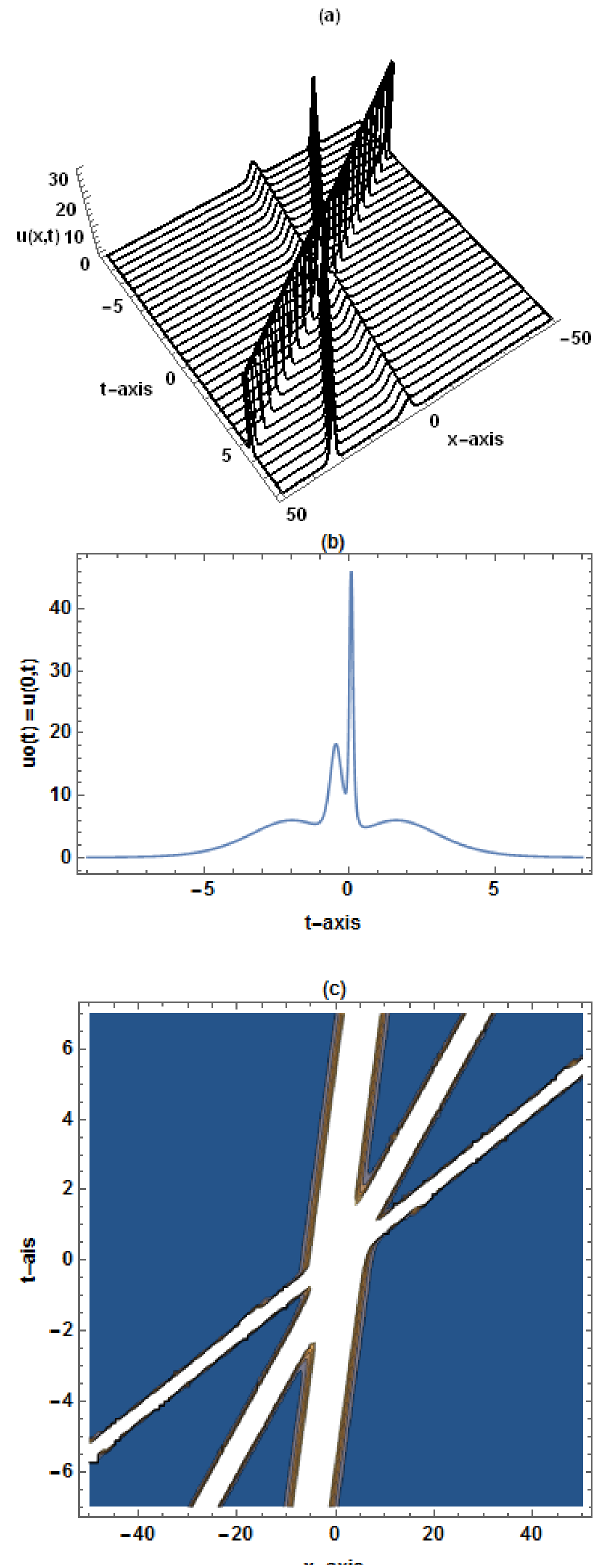

Figure 2: (a) 3D-plot for $u(x, t)$. (b) 2D-plot for $u(x, t)$ when $x = 0$. (c) the contour plot for $u(x, t)$. $\alpha_1 = 2$, $\beta_1 = -1$, $\gamma = 3$, $\lambda = 1$, $\nu = 1/2$, $p_0 = 1/20$, $r_0 = 1$, $r_1 = 3/20$, $r_2 = 1/5$, $r_{2,3} = 1$, and $c_1 = c_2 = c_3 = 1$.

By substituting from (17) into (16) and by using any package in symbolic computations, we get

$$q_3=-\frac{R_-\,H_-\,q_0\,r_1\,r_2\,\alpha^2}{R_1\,R_2\,R_+\,H_+},\quad q_2=-\frac{q_0\,r_2\,\alpha}{R_2},\quad q_1=-\frac{q_0\,r_1\,\alpha}{R_1},$$

$$p_0=\frac{6\,c_2\,\beta_2\,q_0\,r_2\,\beta-p_2\,R_2}{r_2\,\alpha},$$

$$p_1=\frac{r_1\,(p_2\,R_2+6\,r_2\,\beta\,q_0\,H_-)}{r_2\,R_1},$$

$$p_3=-\frac{\alpha\,r_1\,R_-\,H_-\,(p_2\,R_2+6\,c_1\,q_0\,\alpha_2\,\beta\,r_2)}{R_+\,H_+\,R_1\,R_2}$$

$$r_3=-\frac{r_1\,r_2\,R_-\,H_-\,(R_1+R_2+r_0\,\alpha)}{R_1\,R_2\,R_+\,H_+},$$

$$\alpha_3=c_1^2\,\alpha_1^3,\quad \beta_3=c_2^2\,\beta_1^3, \tag{18}$$

where $R_\pm=c_1\,\alpha_1\pm c_2\,\beta_1$, $H_\pm=c_1\,\alpha_2\pm c_2\,\beta_2$, $R_1=6\,c_1\,\alpha_1\,q_0-r_0\,\alpha$ and $R_2=6\,c_2\,\beta_1\,q_0-r_0\,\alpha$.
p_2, q_0, r_0, r_1, r_2, c_1, c_2, α, β, α_1, α_2, $\beta1$ and β_2 are arbitrary constants.

By solving the auxiliary equations $\phi_j'(z_j)=c_j\,\phi_j(z_j)$, $j=1,2$ and substituting together with (18) into (17), we get the solution of Equation (15) namely

$$u(x,y,t)=u_{1x}(x,y,t),\quad u_1(x,y,t)=U(z_1,z_2),$$
$$U(z_1,z_2)=\frac{R_+\,H_+\,R_2\,(\alpha\,p_2\,r_2\,R_1\,e^{c_2\,z_2}+\alpha\,r_1\,e^{c_1\,z_1}\,(p_2\,R_2+6\,q_0\,r_2\,\beta\,H_-))}{q_0\,(R_+\,H_+\,(R_1\,R_2-R_1\,r_2\,\alpha\,e^{c_2\,z_2}-r_1\,\alpha\,R_2\,e^{c_1\,z_1})+R_-\,H_-\,r_1\,r_2\,\alpha^2\,e^{c_1\,z_1+c_2\,z_2})}$$
$$-\frac{R_-\,H_-\,r_1\,\alpha\,(p_2\,R_2+6\,c_1\,\alpha_2\,q_0\,r_2\,\beta)\,e^{c_1\,z_1+c_2\,z_2}}{q_0\,(R_+\,H_+\,(R_1\,R_2-R_1\,r_2\,\alpha\,e^{c_2\,z_2}-r_1\,\alpha\,R_2\,e^{c_1\,z_1})+R_-\,H_-\,r_1\,r_2\,\alpha^2\,e^{c_1\,z_1+c_2\,z_2})}, \tag{19}$$

$$v(x,y,t)=v_{1x}(x,y,t),\quad v_1(x,y,t)=V(z_1,z_2),$$
$$V(z_1,z_2)=\frac{R_+\,H_+\,R_1\,R_2\,\left(r_0+r_1\,e^{c_1\,z_1}+r_2\,e^{c_2\,z_2}\right)-R_-\,H_-\,r_1\,r_2\,\alpha\,(R_1+R_2+r_0\,\alpha)\,e^{c_1\,z_1+c_2\,z_2}}{q_0\,\left(R_+\,H_+(R_1\,R_2-R_1\,r_2\,\alpha\,e^{c_2\,z_2}-r_1\,\alpha\,R_2\,e^{c_1\,z_1})+R_-\,H_-\,r_1\,r_2\,\alpha^2\,e^{c_1\,z_1+c_2\,z_2}\right)}, \tag{20}$$

where $z_1=\alpha_1\,x+\alpha_2\,y+c_1^2\,\alpha_1^3\,t$, $z_2=\beta_1\,x+\beta_2\,y+c_2^2\,\beta_1^3\,t$.

The solution given by Equations (19)-(20) of Equation (15) is shown in Figures 3-4.

Multi-soliton rational solutions (when $N=3$)

In this part, we find a multi-soliton rational solution of Equations (15)-(16) when $N=3$.

From equations (4) and (6) when $N=3$, we have

$$u_1(x,y,t)=U(z_1,z_2,z_3)=\frac{p_0+\sum_{s=1}^{3}p_s\,\phi_s(z_s)+\sum_{i,j=1}^{3}p_{i,j}\,\phi_i(z_i)\,\phi_j(z_j)+p_4\,\phi_1(z_1)\,\phi_2(z_2)\,\phi_3(z_3)}{q_0+\sum_{s=1}^{3}q_s\,\phi_s(z_s)+\sum_{i,j=1}^{3}q_{i,j}\,\phi_i(z_i)\,\phi_j(z_j)+q_4\,\phi_1(z_1)\,\phi_2(z_2)\,\phi_3(z_3)}, \tag{21}$$

$$v_1(x,y,t)=V(z_1,z_2,z_3)=\frac{r_0+\sum_{s=1}^{3}r_s\,\phi_s(z_s)+\sum_{i,j=1}^{3}r_{i,j}\,\phi_i(z_i)\,\phi_j(z_j)+r_4\,\phi_1(z_1)\,\phi_2(z_2)\,\phi_3(z_3)}{q_0+\sum_{s=1}^{3}q_s\,\phi_s(z_s)+\sum_{i,j=1}^{3}q_{i,j}\,\phi_i(z_i)\,\phi_j(z_j)+q_4\,\phi_1(z_1)\,\phi_2(z_2)\,\phi_3(z_3)},$$

where $i<j$, $z_1=\alpha_1\,x+\alpha_2\,y+\alpha_3\,t$, $z_2=\beta_1\,x+\beta_2\,y+\beta_3\,t$, $z_3=\gamma_1\,x+\gamma_2\,y+\gamma_3\,t$ and α_k, β_k, γ_k, p_0, r_0, p_s, q_s, r_s, $p_{i,j}$, $q_{i,j}$, $r_{i,j}$, $i,j=1,2,3$, $s=1,2,3,4$, $k=1,2,3$ are arbitrary constants. The auxiliary functions $\phi_l(z_l)$ satisfy the auxiliary equations $\phi_l'(z_l)=c_l\,\phi_l(z_l)$, where c_l are arbitrary constants, $l=1,2,3$.

By substituting from (21) into (16) and by a similar way as we did in the last case (when $N=2$), we get

$$p_0=\frac{6\,q_0\,(R_++\gamma_2\,c_3)\,\beta}{\alpha},\quad p_1=\frac{6\,q_1\,\beta\,L_+}{\alpha},$$

$$p_2=\frac{6\,q_2\,\beta\,N_+}{\alpha},\quad p_3=\frac{6\,q_3\,\beta\,H_+}{\alpha},$$

$$p_{1,2}=\frac{6\,\gamma_2\,q_1\,q_2\,c_3\,\beta\,H_-\,R_-}{R_+\,H_-\,q_0\,\alpha},\quad p_{1,3}=\frac{6\,c_2\,\beta_2\,q_1\,q_2\,M_-\,N_-}{q_0\,\alpha\,M_+\,N_+},$$

$$p_{2,3}=\frac{6\,c_1\,\alpha_2\,q_2\,q_3\,\beta\,Q_-\,L_-}{q_0\,Q_+\,L_+\,\alpha},\quad p_4=0,$$

$$r_1=q_1\,(\frac{r_0}{q_0}-\frac{6\,c_1\,\alpha_1}{\alpha}),\quad r_2=q_2\,(\frac{r_0}{q_0}-\frac{6\,c_2\,\beta_1}{\alpha})$$

$$r_3=q_3\,(\frac{r_0}{q_0}-\frac{6\,c_3\,\gamma_1}{\alpha}),$$

$$r_{1,2}=-\frac{R_-\,H_-\,q_1\,q_2\,(R_1+R_2+r_0\,\alpha)}{R_+\,H_+\,q_0^2\,\alpha}, \tag{22}$$

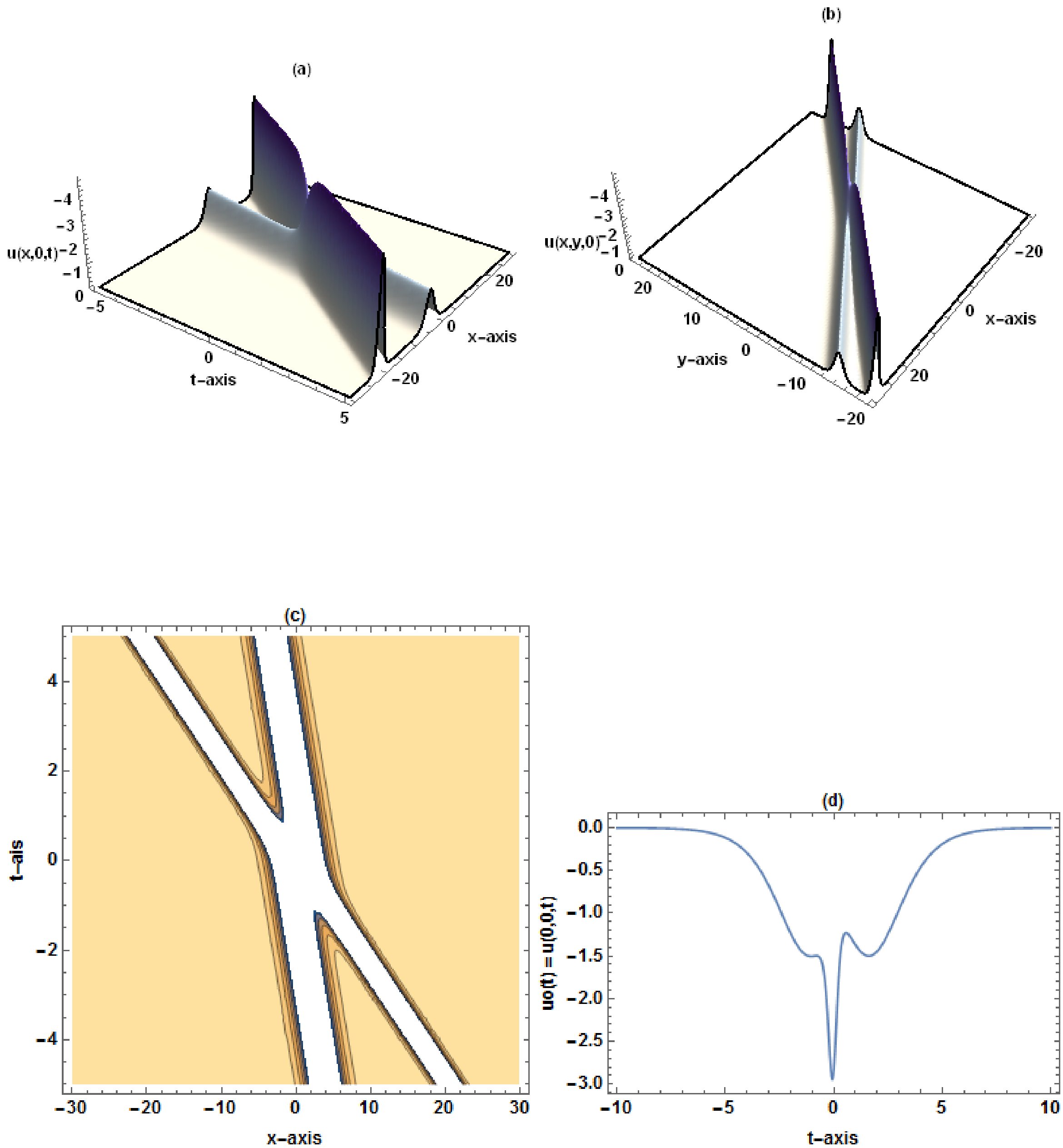

Figure 3: (a) 3D-plot for $u(x, y, t)$ when $y = 0$. (b) 3D-plot for $u(x, y, t)$ when $t = 0$. (c) the contour plot for $u(x, y, t)$ when $y = 0$. (d) 2D-plot for $u(x, y, t)$ when $x = y = 0$. $\alpha_1 = 2$, $\alpha_2 = 3$, $\beta_1 = 1$, $\beta_2 = -2$, $\alpha = 2$, $\beta = 1$, $r_0 = 0$, $r_1 = -2$, $r_2 = 1$, $q_0 = 1$, $p_2 = -1$ and $c_1 = c_2 = 1$.

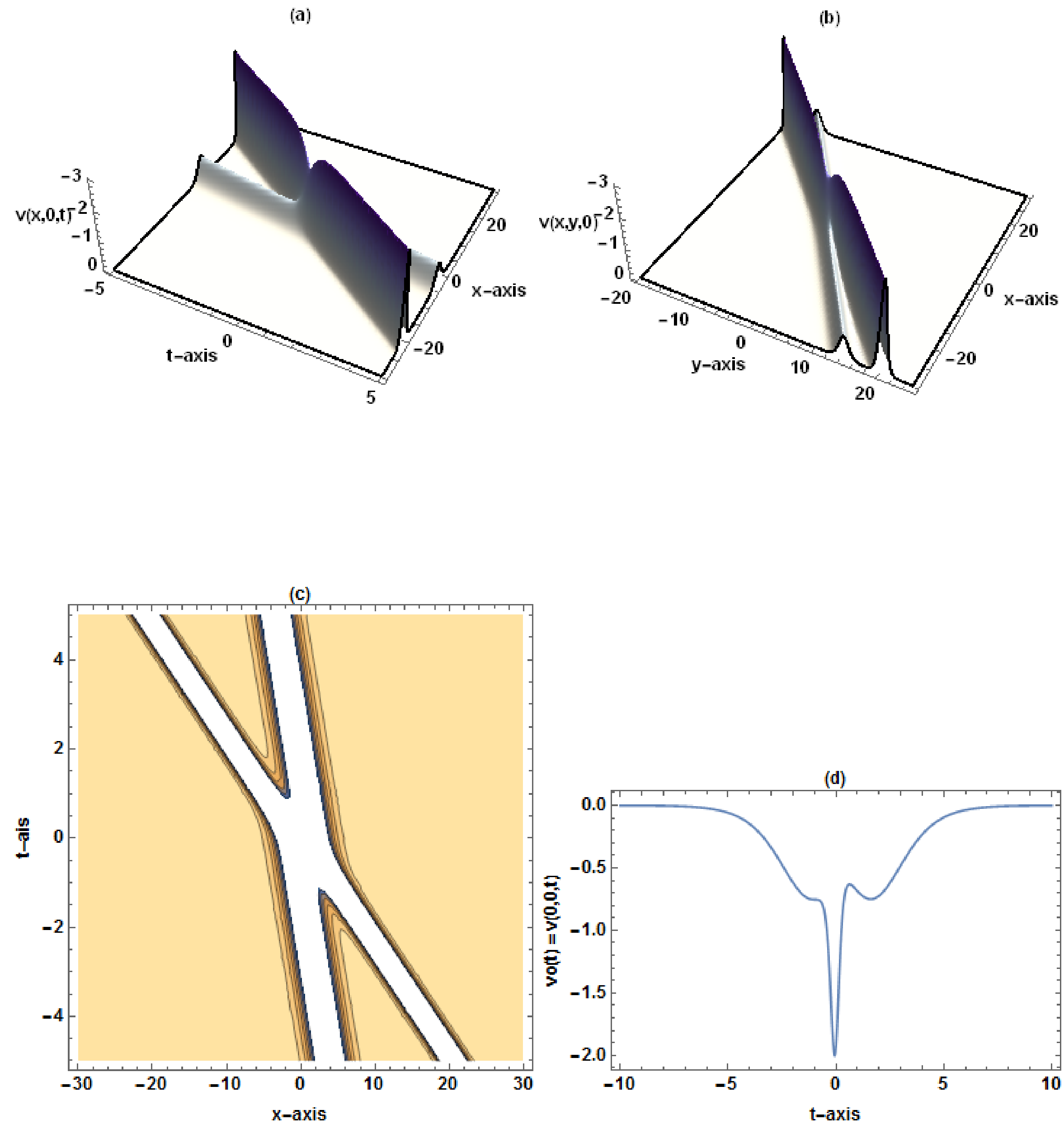

Figure 4: (a) 3D-plot for $v(x, y, t)$ when $y = 0$. (b) 3D-plot for $v(x, y, t)$ when $t = 0$. (c) the contour plot for $v(x, y, t)$ when $y = 0$. (d) 2D-plot for $v(x, y, t)$ when $x = y = 0$. The same parameters as in Figure 3.

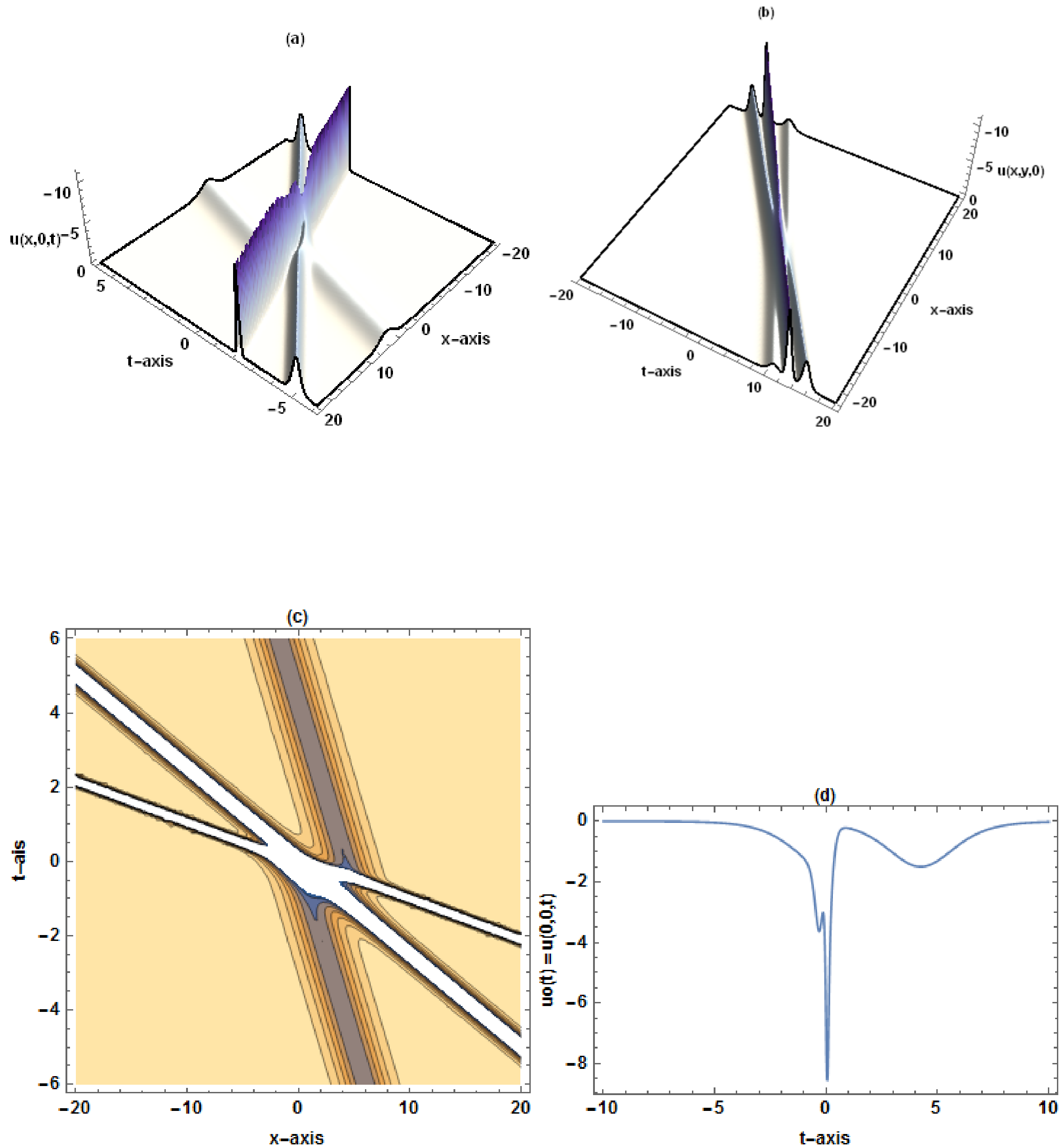

Figure 5: (a) 3D-plot for $u(x, y, t)$ when $y = 0$. (b) 3D-plot for $u(x, y, t)$ when $t = 0$. (c) the contour plot for $u(x, y, t)$ when $y = 0$. (d) 2D-plot for $u(x, y, t)$ when $x = y = 0$. $\alpha_1 = 2, \alpha_2 = 3, \beta_1 = 1, \beta_2 = -2, \alpha = 2, \beta = 1, r_0 = 0,\ r_1 = -2,\ r_2 = 1,\ q_0 = 1,\ p_2 = -1$ and $c_1 = c_2 = 1$.

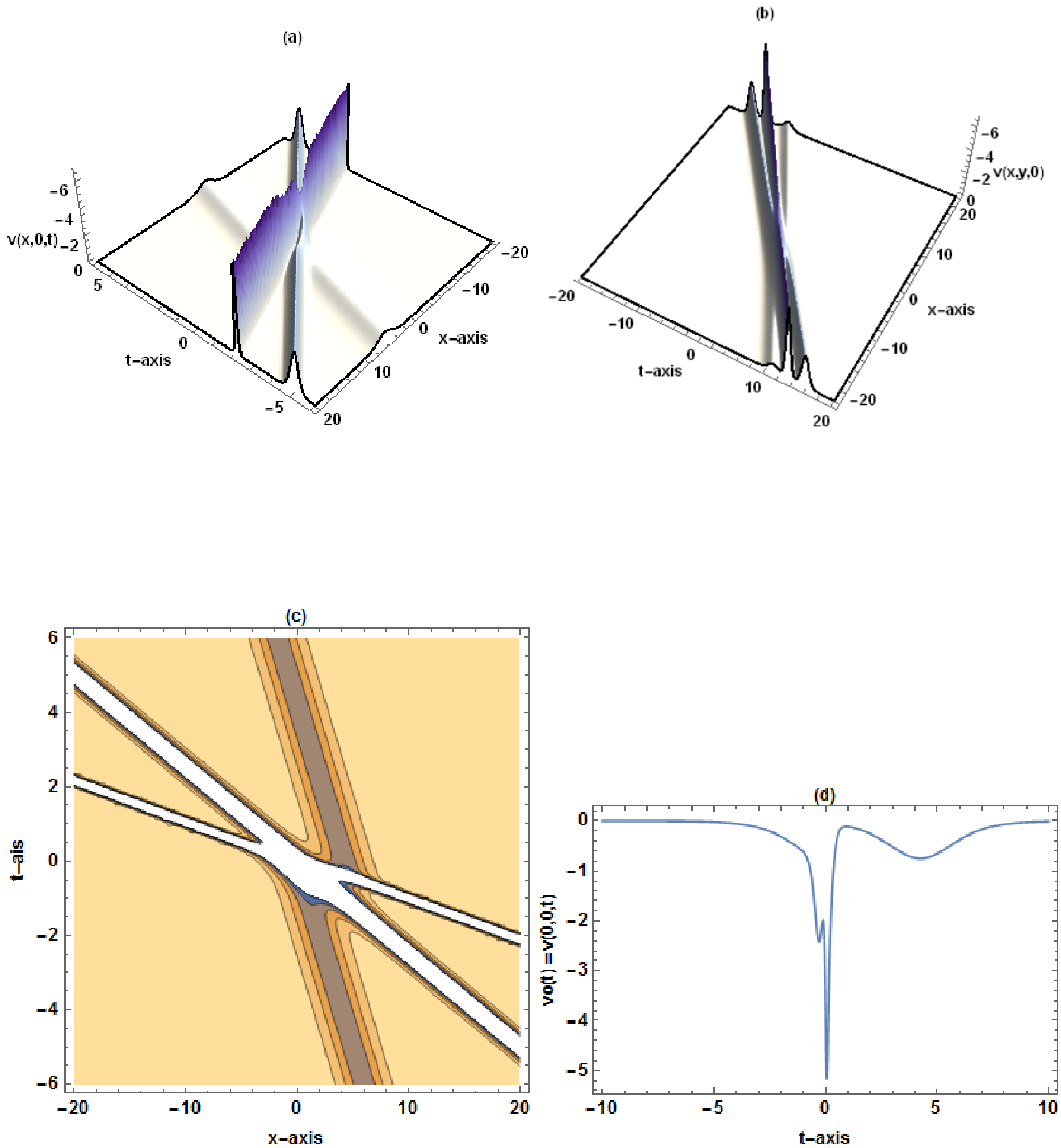

Figure 6: (a) 3D-plot for $v(x, y, t)$ when $y = 0$. (b) 3D-plot for $v(x, y, t)$ when $t = 0$. (c) the contour plot for $v(x, y, t)$ when $y = 0$. (d) 2D-plot for $v(x, y, t)$ when $x = y = 0$. The same parameters as in Figure 5.

$$r_{1,3} = -\frac{M_- N_- q_1 q_3 (R_1 + R_3 + r_0 \alpha)}{M_+ N_+ q_0^2 \alpha},$$

$$r_{2,3} = -\frac{Q_- L_- q_2 q_3 (R_2 + R_3 + r_0 \alpha)}{Q_+ L_+ q_0^2 \alpha},$$

$$r_4 = -\frac{R_- H_- Q_- L_- M_- N_- q_1 q_2 q_3 (R_1 + R_2 + R_3 + 2 r_0 \alpha)}{R_+ H_+ Q_+ L_+ M_+ N_+ q_0^3 \alpha}, \tag{22}$$

$$q_{1,2} = \frac{R_- H_- q_1 q_2}{R_+ H_+ q_0}, \quad q_{1,3} = \frac{M_- N_- q_1 q_3}{M_+ N_+ q_0},$$

$$q_{2,3} = \frac{Q_- L_- q_2 q_3}{Q_+ L_+ q_0}, \quad q_4 = \frac{R_- H_- Q_- L_- M_- N_- q_1 q_2 q_3}{R_+ H_+ Q_+ L_+ M_+ N_+ q_0^2},$$

$$\alpha_3 = c_1^2 \alpha_1^3, \quad \beta_3 = c_2^2 \beta_1^3, \quad \gamma_3 = c_3^2 \gamma_1^3, \tag{23}$$

where $R_\pm = c_1 \alpha_1 \pm c_2 \beta_1$, $H_\pm = c_1 \alpha_2 \pm c_2 \beta_2$, $M_\pm = c_2 \alpha_1 \pm c_3 \gamma_1$, $N_\pm = c_1 \alpha_2 \pm c_3 \gamma_2$, $Q_\pm = c_2 \beta_1 \pm c_3 \gamma_1$, $L_\pm = c_2 \beta_2 \pm c_3 \gamma_2$, $R_1 = 6 c_1 \alpha_1 q_0 - r_0 \alpha$, $R_2 = 6 c_2 \beta_1 q_0 - r_0 \alpha$ and $R_3 = 6 c_2 \beta_1 \pm r_0 \alpha$.
p_2, q_i, r_0, c_1, c_2, c_3 α, β, α_1, α_2, $\beta 1$, β_2, γ_1 and γ_2, $i = 0, 1, 2, 3$ are arbitrary constants.

By solving the auxiliary equations $\phi_j^{'}(z_j) = c_j \phi_j(z_j)$, $j = 1, 2, 3$ and substituting together with (22)-(23) into (21), we get the solution of Equation (15) which is very lengthy to be written here.

The solution of (15) when $N = 3$ is shown in Figures 5-6.

4 Conclusions

In summary, via the generalized unified method and symbolic computation, we construct multi-soliton rational solutions for the KdV and NNV. This method can not only give a unified formulation to uniformly construct multi-wave solutions, but also can provide us a guideline to classify the types of these solutions according to the given parameters. This method can be applied to other kinds of nonlinear partial differential equations with the aid of computer systems like Mathematica or Maple to facilitate the algebraic calculations. Also, it is valuable to learn more about these multi-soliton rational solutions and their related evolutional properties, we expect that these solutions may be useful in future studies for the intricate natural world.

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