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Oscillation of fractional order functional differential equations with nonlinear damping

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Abstract: In this paper, we are concerned with the oscillatory behavior of a class of fractional differential equations with functional terms. The fractional derivative is defined in the sense of the modified Riemann-Liouville derivative. Based on a certain variable transformation, by using a generalized Riccati transformation, generalized Philos type kernels, and averaging techniques we establish new interval oscillation criteria. Illustrative examples are also given.

Keywords: fractional derivative; fractional differential equation; Riemann-Liouville derivative; Riccati transformation; oscillation criteria

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1 Introduction

Fractional differential equations are generalizations of classical differential equations of integer order and have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Apart from diverse areas of mathematics, fractional differential equations arise in rheology, viscoelasticity, chemical physics, electrical networks, fluid flows, control, dynamical processes in self-similar and porous structures, *etc.*; see, for example, [1–6]. There have been a number of works in which fractional derivatives are used for a better description of considered material properties; mathematical modeling based on enhanced rheological models naturally leads to differential equations of fractional order and to the necessity of the formulation of initial conditions for such equations. This is due both to the intensive development of the theory of fractional calculus itself and to the applications. The books on the subject of fractional integrals and fractional derivatives by Diethelm [7], Miller and Ross [8], Podlubny [9], and Kilbas *et al.* [10] summarize and organize much of fractional calculus and many of theories and applications of fractional differential equations. Many papers have studied some aspects of fractional differential equations, especially the existence of solutions (or positive solutions) of nonlinear initial (or boundary) value problems for fractional differential equation (or system) by the use of techniques of nonlinear analysis (fixed-point theorems, Leray-Schauder theory), the methods for explicit and numerical solutions and the stability of solutions, we refer to [11–21] and the references cited therein.

Recently, research on oscillation of various equations including differential equations, difference equations, and dynamic equations on time scales has been a hot topic in the literature, and much effort has already been put into establishing new oscillation criteria for these equations; see the monographs [22, 23]. In these investigations, we notice that very little attention is paid to oscillation of fractional differential equations.

In [24], Jumarie proposed a definition for a fractional derivative which is known as the modified Riemann-Liouville derivative in the literature. Since then, many authors have investigated various applications of the modified Riemann-Liouville derivative (e.g., see [25–27]) including various fractional calculus formulae, the fractional variational iteration method, and the fractional sub-equation method for solving fractional partial differential equations.

Recently Feng and Meng [28], Liu, Zheng and Meng [29], Qin and Zheng [30], and Feng [31] have established some new oscillation criteria for the following equations:

$$\begin{aligned}
&D_t^\alpha\left[r(t)\,\psi(x(t))\,D_t^\alpha x(t)\right]+q(t)f(x(t))=e(t),\\
&D_t^\alpha\left[a(t)\left(D_t^\alpha\left(r(t)\,D_t^\alpha x(t)\right)\right)^\gamma\right]+q(t)f(x(t))=e(t),\\
&D_t^\alpha\left[a(t)\,D_t^\alpha\left(r(t)\,D_t^\alpha x(t)\right)\right]+p(t)\,D_t^\alpha\left(r(t)\,D_t^\alpha x(t)\right)\\
&\quad+q(t)f(x(t))=e(t),\\
&D_t^\alpha\left(r(t)\,k_1\left(x(t),D_t^\alpha x(t)\right)\right)+p(t)\,k_2\left(x(t),D_t^\alpha x(t)\right)\\
&D_t^\alpha x(t)+q(t)f(x(t))=0,
\end{aligned}$$

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for $t \geq t_0 > 0$ and $0 < \alpha < 1$, where $D_t^\alpha(\cdot)$ denotes the modified Riemann-Liouville derivative with respect to the variable t.

In this paper, we are concerned with the oscillation of fractional differential equations with nonlinear damping terms in the form of

$$D_t^\alpha\left(r(t)k_1\left(x(t), D_t^\alpha x(t)\right)\right) + p(t)k_2\left(x(t), D_t^\alpha x(t)\right) D_t^\alpha x(t) + F(t, x(t), x(\tau(t))) = e(t) \tag{1}$$

for $t \geq t_0 > 0$ and $0 < \alpha < 1$, where $D_t^\alpha(\cdot)$ denotes the modified Riemann-Liouville derivative with respect to the variable t, the function $r \in C^\alpha\left([t_0,\infty),(0,\infty)\right)$, which is the set of functions with continuous derivative or fractional order α, the function $p \in C^\alpha\left([t_0,\infty),[0,\infty)\right)$, the function k_1 belongs to $C^\alpha\left(R^2,R^2\right)$ satisfying $k_1^2(u,v) \leq Kvk_1(u,v)$ for some positive constant K, all $u \in R$ and all $v \in R\backslash\{0\}$, the function k_2 belongs to $C\left(R^2,R^2\right)$ satisfying $uvk_2(u,v) > 0$ for all $u \in R$ and all $v \in R\backslash\{0\}$ the function $F \in C\left([t_0,\infty)\times R^2, R\right)$, and the function τ belongs to $C\left([t_0,\infty),(0,\infty)\right)$ with $\lim_{t\to\infty}\tau(t) = \infty$.

A solution of (1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

The definition and some of the key properties of modified Riemann-Liouville derivative are:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha} f(\xi)d\xi, & 0<\alpha<1 \\ \left(f^{(n)}(t)\right)^{(\alpha-n)}, & 1 \leq n \leq \alpha < n+1 \end{cases},$$

where $f(\xi) = (f(\xi) - f(0))$.

$$D_t^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha},$$

$$D_t^\alpha\left(f(t)g(t)\right) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t),$$

$$D_t^\alpha f(g(t)) = f_g'(g(t))D_t^\alpha g(t) = D_g^\alpha f(g(t))\left(g'(t)\right)^\alpha,$$

which do not hold for classical Riemann-Liouville and Caputo derivates. In particular, Leibniz's rule (product rule) and Faà di Bruno's formula (chain rule) are important tools in our proofs.

2 Main results

We will use a transformation technique also used in [28–33] in our proofs. For the sake of convenience, through the rest of this paper, we denote

$$\xi = \xi(t) := \frac{t^\alpha}{\Gamma(1+\alpha)}, \qquad \xi_{t_0} := \xi(t_0) = \frac{t_0^\alpha}{\Gamma(1+\alpha)}$$

and for any function f, we denote $\widetilde{f} = f \circ \xi^{-1}$ *i.e.* $\widetilde{f}(\xi) = f(t)$. We immediately arrive at the conclusion $D_t^\alpha f(t) = \widetilde{f}'(\xi)$, which gives us the ability to build a connection between fractional and integer order derivatives of functions. We will transform fractional order differential inequalities to integer order differential inequalities in our proofs with this connection.

Now we introduce a functional that will be used in the proofs of some results.

Let

$$B(s_i,t_i) = \left\{u \in C^1[s_i,t_i] : u(t) \neq 0 \text{ for } t \in (s_i,t_i),\ u(s_i) = u(t_i) = 0\right\},$$

for $i = 1,2$. We define the functional $A(\cdot;s_i,t_i)$ for $G \in B(s_i,t_i)$ such as;

$$A(g;s_i,t_i) = \int_{s_i}^{t_i} G^2(t)g(t)\,dt, \quad s_i \leq t \leq t_i,\ i = 1,2,$$

where $g \in C\left([t_0,\infty),R\right)$. It is easily seen that the linear functional $A(\cdot;s_i,t_i)$ satisfies

$$A(g';s_i,t_i) = -A(2\frac{G'}{G}g;s_i,t_i) \geq -A(2\left|\frac{G'}{G}\right||g|;s_i,t_i).$$

In proofs of some of our results, we will also use another class of averaging functions $H \in C(D,R)$ which satisfy

(i) $H(t,t) = 0$, $H(t,s) > 0$ for $t > s$,

(ii) H has partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on D such that

$$\frac{\partial H}{\partial t} = h_1(t,s)\sqrt{H(t,s)}, \qquad \frac{\partial H}{\partial s} = -h_2(t,s)\sqrt{H(t,s)}$$

where $D = \{(t,s) : t_0 \leq s \leq t < \infty\}$ and $h_1,h_2 \in L_{loc}\left(D,\mathbf{R}^+\right)$.

We will also use the function class Y to study the oscillatory behavior of (1). We say that a function $\Phi(t,s,l)$ belongs to the function class Y dentoed by $\Phi \in Y$, if $\Phi \in C(E,R)$, where $E = \{(t,s,l) : t_0 \leq l \leq s \leq t < \infty\}$, which satisfies $\Phi(t,t,l) = 0$, $\Phi(t,l,l) = 0$, $\Phi(t,s,l) \neq 0$ for $l < s < t$, and has the partial derivative $\partial\Phi/\partial s$ on E such that $\partial\Phi/\partial s$ is locally integrable with respect to s in E.

In the proof of some of our results we define below a useful operator, introduced in [34], as an important tool.

For $\Phi \in Y$, we define the operator

$$T[g;l,t] = \int_l^t \Phi^2(t,s,l)g(s)\,ds \qquad t \geq s \geq l \geq t_0, \tag{2}$$

where $g \in C\left([t_0,\infty), R\right)$. It is easy to verify that the linear operator $T[\cdot; l, t]$ satisfies

$$T\left[g^{'}; l, t\right] = -2T[\phi g; l, t] \quad for\ g \in C^1\left([t_0,\infty), R\right), \quad (3)$$

where the function $\phi = \phi(t, s, l)$ is defined by

$$\frac{\partial \Phi(t,s,l)}{\partial s} = \phi(t,s,l)\,\Phi(t,s,l). \quad (4)$$

Before our main results, now we state a useful lemma [35] (Young's inequality).

Lemma 1. *If A and B are nonnegative constants and $m, n \in R$ such that $\frac{1}{m} + \frac{1}{n} = 1$, then*

$$\frac{1}{m}A + \frac{1}{n}B \geq A^{\frac{1}{m}} B^{\frac{1}{n}}.$$

We are ready to state our main results concerning oscillatory behavior of Equation (1) now. We shall make use of the following conditions in our results:

(C_1)For any $T \geq t_0$, there exists $T \leq s_1 < t_1 \leq s_2 < t_2$ such that

$$\begin{aligned} e(t) &\leq 0 \text{ for } t \in [s_1, t_1], \\ e(t) &\geq 0 \text{ for } t \in [s_2, t_2], \end{aligned}$$

(C_2)there exists a function $q(t) > 0$ and a constant $\gamma \geq 1$ such that $F(t,x,u)/x \geq q(t)|x|^{\gamma-1}$ holds for $t \in [s_1,t_1] \cup [s_2,t_2]$ and $x \neq 0$, $u \in \mathbf{R}$.

Theorem 1. *Suppose the conditions $(C_1 - C_2)$ hold. If there exist some $\delta_i \in (s_i, t_i)$, for $i = 1, 2$; $H \in C(D, R)$ satisfying (i)-(ii) and a positive function $\rho \in C^1\left([t_0,\infty), \mathbf{R}^+\right)$ such that*

$$\begin{aligned} &\frac{1}{H(\delta_i,\xi_{s_i})}\int_{\xi_{s_i}}^{\delta_i} \widetilde{\rho}(s)\left[H(s,\xi_{s_i})\widetilde{Q}(s) - \frac{K\widetilde{r}(s)}{4}H_1(s,\xi_{s_i})\right]ds \\ &+\frac{1}{H(\xi_{t_i},\delta_i)}\int_{\delta_i}^{\xi_{t_i}} \widetilde{\rho}(s)\left[H(\xi_{t_i},s)\widetilde{Q}(s) - \frac{K\widetilde{r}(s)}{4}H_2(s,\xi_{t_i})\right]ds > 0 \end{aligned} \quad (5)$$

for i = 1, 2, where $H_i(t,s) := h_i(t,s) - \frac{\widetilde{\rho}^{'}(t)}{\widetilde{\rho}(t)}\sqrt{H(t,s)}$ and $Q(t) = \gamma(\gamma-1)^{(1-\gamma)/\gamma}[q(t)]^{1/\gamma}|e(t)|^{(\gamma-1)\gamma}$ with the convention $0^0 = 1$, then Equation (1) is oscillatory.

Proof. On the contrary, suppose that Equation (1) has a nonoscillatory solution $x(t)$. Then $x(t)$ eventually must have one sign, i.e. $x(t) \neq 0$ on $[T_0,\infty)$ for some large $T_0 \geq t_0$. Define

$$w(t) := \rho(t)\frac{r(t)k_1\left(x(t), D_t^\alpha x(t)\right)}{x(t)} \quad (6)$$

for $t \geq T_0$. Then we deduce

$$\begin{aligned} D_t^\alpha w(t) &= \rho(t)\frac{e(t)}{x(t)} - \rho(t)\frac{F(t,x(t),x(\tau(t)))}{x(t)} \\ &-\rho(t)p(t)\frac{k_2\left(x(t), D_t^\alpha x(t)\right)D_t^\alpha x(t)}{x(t)} \\ &-\rho(t)r(t)\frac{k_1\left(x(t), D_t^\alpha x(t)\right)D_t^\alpha x(t)}{x^2(t)} \\ &+\frac{D_t^\alpha \rho(t)}{\rho(t)}w(t) \end{aligned}$$

for $t \geq T_0$. By assuming (C_2) and the condition on k_2, we obtain

$$\begin{aligned} D_t^\alpha w(t) &\leq \rho(t)\left(\frac{e(t)}{x(t)} - q(t)|x(t)|^{\gamma-1}\right) \\ &+\frac{D_t^\alpha \rho(t)}{\rho(t)}w(t) - \frac{1}{K\rho(t)r(t)}w^2(t) \end{aligned} \quad (7)$$

for $t \geq T_0$. By assuming (C_1), if $x(t) > 0$, then we can choose $s_1, t_1 \geq T_0$ such that $e(t) \leq 0$ for $t \in [s_1,t_1]$. Similarly if $x(t) < 0$, then we can choose $s_2, t_2 \geq T_0$ such that $e(t) \geq 0$ for $t \in [s_2,t_2]$. So $\frac{e(t)}{x(t)} \leq 0$ (i.e. $-\frac{e(t)}{x(t)} = \left|\frac{e(t)}{x(t)}\right|$) for $t \in [s_i,t_i]$, $i = 1$ or 2 and from (7) one can deduce

$$\begin{aligned} D_t^\alpha w(t) &\leq \rho(t)\left(-\left|\frac{e(t)}{x(t)}\right| - q(t)|x(t)|^{\gamma-1}\right) \\ &+\frac{D_t^\alpha \rho(t)}{\rho(t)}w(t) - \frac{1}{K\rho(t)r(t)}w^2(t). \end{aligned} \quad (8)$$

For $\gamma > 1$, by setting $m = \gamma$, $n = \frac{\gamma}{\gamma-1}$, $A = \gamma q(t)|x(t)|^{\gamma-1}$, $B = \frac{\gamma}{\gamma-1}\left|\frac{e(t)}{x(t)}\right|$ and using Lemma 1, we obtain

$$q(t)|x(t)|^{\gamma-1} + \left|\frac{e(t)}{x(t)}\right| \geq Q(t). \quad (9)$$

Note that inequality (9) trivially holds for $\gamma = 1$. So, by combining (8) and (9), we obtain

$$D_t^\alpha w(t) \leq \rho(t)Q(t) + \frac{D_t^\alpha \rho(t)}{\rho(t)}w(t) - \frac{1}{K\rho(t)r(t)}w^2(t) \quad (10)$$

for $\gamma \geq 1$ and $t \in [s_i,t_i]$, $i = 1$ or 2.

Now let $w(t) = \widetilde{w}(\xi)$. Then we have $D_t^\alpha w(t) = \widetilde{w}^{'}(\xi)$ and $D_t^\alpha \rho(t) = \widetilde{\rho}^{'}(\xi)$ So (10) is transformed to

$$\widetilde{w}^{'}(\xi) \leq \widetilde{\rho}(\xi)\widetilde{Q}(\xi) + \frac{\widetilde{\rho}^{'}(\xi)}{\widetilde{\rho}(\xi)}\widetilde{w}(\xi) - \frac{1}{K\widetilde{\rho}(\xi)\widetilde{r}(\xi)}\widetilde{w}^2(\xi) \quad (11)$$

for $\xi \in [\xi_{s_i}, \xi_{t_i}]$, $i = 1$ or 2.

Let δ_i an arbitrary point in (ξ_{s_i}, ξ_{t_i}). Substituting ξ with s, multiplying (11) with $H(\xi, s)$ and integrating it over

$[\delta_i, \xi)$ for $\xi \in [\delta_i, \xi_{t_i})$, $i = 1$ or 2, we obtain

$$\begin{aligned}
&\int_{\delta_i}^{\xi} H(\xi,s)\widetilde{\rho}(s)\widetilde{Q}(s)\,ds \le -\int_{\delta_i}^{\xi} H(\xi,s)\widetilde{w}'(s)\,ds \\
&+\int_{\delta_i}^{\xi} H(\xi,s)\left[\tfrac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\widetilde{w}(s)-\tfrac{1}{K\widetilde{\rho}(s)\widetilde{r}(s)}\widetilde{w}^2(s)\right]ds \\
&= H(\xi,\delta_i)\widetilde{w}(\delta_i)-\int_{\delta_i}^{\xi}\widetilde{w}'(s)h_2(\xi,s)\sqrt{H(\xi,s)}ds \\
&+\int_{\delta_i}^{\xi} H(\xi,s)\left[\tfrac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\widetilde{w}(s)-\tfrac{1}{K\widetilde{\rho}(s)\widetilde{r}(s)}\widetilde{w}^2(s)\right]ds \\
&= H(\xi,\delta_i)\widetilde{w}(\delta_i)-\int_{\delta_i}^{\xi}\left[A_1\widetilde{w}(s)-\tfrac{1}{2}A_2\right]^2 ds \\
&+\int_{\delta_i}^{\xi}\tfrac{K\widetilde{\rho}(s)\widetilde{r}(s)}{4}H_2^2(\xi,s)\,ds \\
&\le H(\xi,\delta_i)\widetilde{w}(\delta_i)+\int_{\delta_i}^{\xi}\tfrac{K\widetilde{\rho}(s)\widetilde{r}(s)}{4}H_2^2(\xi,s)\,ds.
\end{aligned}$$

where

$$A_1 = \left(\frac{H(\xi,s)}{K\widetilde{\rho}(s)\widetilde{r}(s)}\right)^{1/2}$$

and

$$A_2 = \left(K\widetilde{\rho}(s)\widetilde{r}(s)\right)^{1/2} H_2(\xi,s).$$

Now letting $\xi \to \xi_{t_i}^-$ and dividing it by $H(\xi_{t_i}, \delta_i)$, we obtain

$$\frac{1}{H(\xi_{t_i},\delta_i)}\int_{\delta_i}^{\xi_{t_i}} H(\xi_{t_i},s)\widetilde{\rho}(s)\widetilde{Q}(s)\,ds \le \widetilde{w}(\delta_i) + \frac{1}{H(\xi_{t_i},\delta_i)}\int_{\delta_i}^{\xi_{t_i}} \frac{K\widetilde{\rho}(s)\widetilde{r}(s)}{4}H_2^2(\xi_{t_i},s)\,ds. \tag{12}$$

On the other hand, substituting ξ with s, multiplying (11) with $H(s,\xi)$ and integrating it over (ξ, δ_i) for $\xi \in [\xi_{s_i}, \delta_i)$, $i = 1$ or 2, with similar calculations, we obtain

$$\begin{aligned}
&\int_{\xi}^{\delta_i} H(s,\xi)\widetilde{\rho}(s)\widetilde{Q}(s)\,ds \le -\int_{\xi}^{\delta_i} H(s,\xi)\widetilde{w}'(s)\,ds \\
&+\int_{\xi}^{\delta_i} H(s,\xi)\left[\tfrac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\widetilde{w}(s)-\tfrac{1}{K\widetilde{\rho}(s)\widetilde{r}(s)}\widetilde{w}^2(s)\right]ds \\
&= -H(\delta_i,\xi)\widetilde{w}(\delta_i)-\int_{\xi}^{\delta_i}\widetilde{w}(s)h_1(s,\xi)\sqrt{H(s,\xi)}ds \\
&+\int_{\xi}^{\delta_i} H(s,\xi)\left[\tfrac{\widetilde{\rho}'(s)}{\widetilde{\rho}(s)}\widetilde{w}(s)-\tfrac{1}{K\widetilde{\rho}(s)\widetilde{r}(s)}\widetilde{w}^2(s)\right]ds \\
&\le -H(\delta_i,\xi)\widetilde{w}(\delta_i)+\int_{\xi}^{\delta_i}\tfrac{K\widetilde{\rho}(s)\widetilde{r}(s)}{4}H_1^2(s,\xi)\,ds.
\end{aligned}$$

Now letting $\xi \to \xi_{s_i}^+$ and dividing it by $H(\delta_i, \xi_{s_i})$, we obtain

$$\frac{1}{H(\delta_i,\xi_{s_i})}\int_{\xi_{s_i}}^{\delta_i} H(s,\xi_{s_i})\widetilde{\rho}(s)\widetilde{Q}(s)\,ds \le -\widetilde{w}(\delta_i) + \frac{1}{H(\delta_i,\xi_{s_i})}\int_{\xi_{s_i}}^{\delta_i} \frac{K\widetilde{\rho}(s)\widetilde{r}(s)}{4}H_1^2(s,\xi_{s_i})\,ds. \tag{13}$$

By combining (12) and (13), we obtain

$$\begin{aligned}
&\tfrac{1}{H(\delta_i,\xi_{s_i})}\int_{\xi_{s_i}}^{\delta_i} H(s,\xi_{s_i})\widetilde{\rho}(s)\widetilde{Q}(s)\,ds \\
&+\tfrac{1}{H(\xi_{t_i},\delta_i)}\int_{\delta_i}^{\xi_{t_i}} H(\xi_{t_i},s)\widetilde{\rho}(s)\widetilde{Q}(s)\,ds \\
&\le \tfrac{1}{H(\delta_i,\xi_{s_i})}\int_{\xi_{s_i}}^{\delta_i}\tfrac{K\widetilde{\rho}(s)\widetilde{r}(s)}{4}H_1^2(s,\xi_{s_i})\,ds \\
&+\tfrac{1}{H(\xi_{t_i},\delta_i)}\int_{\delta_i}^{\xi_{t_i}}\tfrac{K\widetilde{\rho}(s)\widetilde{r}(s)}{4}H_2^2(\xi_{t_i},s)\,ds
\end{aligned}$$

which contradicts to (5). The proof is complete. □

Theorem 2. *Suppose the conditions $(C_1 - C_2)$ hold. If there exists a $G \in B(s_i, t_i)$ such that the inequality*

$$A(\widetilde{Q};\xi_{s_i},\xi_{t_i}) > A\Big(K\frac{\left(G'\right)^2}{G^2}\widetilde{r};\xi_{s_i},\xi_{t_i}\Big) \tag{14}$$

holds for i = 1, 2, then Equation (1) is oscillatory.

Proof. On the contrary, suppose that Equation (1) has a nonoscillatory solution $x(t)$. Then $x(t)$ eventually must have one sign, i.e. $x(t) \neq 0$ on $[T_0, \infty)$ for some large $T_0 \ge t_0$. Define

$$w(t) := \frac{r(t)\,k_1\left(x(t), D_t^\alpha x(t)\right)}{x(t)} \tag{15}$$

for $t \ge T_0$. Then we deduce

$$\begin{aligned}
D_t^\alpha w(t) \;=\; & \frac{e(t)}{x(t)} - \frac{F(t,x(t),x(\tau(t)))}{x(t)} \\
& -p(t)\frac{k_2\left(x(t),D_t^\alpha x(t)\right)D_t^\alpha x(t)}{x(t)} \\
& -r(t)\frac{k_1\left(x(t),D_t^\alpha x(t)\right)D_t^\alpha x(t)}{x^2(t)}
\end{aligned}$$

for $t \ge T_0$. By assuming (C_2) and condition on k_2, we obtain

$$D_t^\alpha w(t) \le \frac{e(t)}{x(t)} - q(t)\,|x(t)|^{\gamma-1} - \frac{1}{Kr(t)}w^2(t) \tag{16}$$

for $t \ge T_0$. By assuming (C_1), if $x(t) > 0$, then we can choose $s_1, t_1 \ge T_0$ such that $e(t) \le 0$ for $t \in [s_1, t_1]$. Similarly if $x(t) < 0$, then we can choose $s_2, t_2 \ge T_0$ such that

$e(t) \geq 0$ for $t \in [s_2, t_2]$. So $\frac{e(t)}{x(t)} \leq 0$ (i.e. $-\frac{e(t)}{x(t)} = \left|\frac{e(t)}{x(t)}\right|$) for $t \in [s_i, t_i]$, $i = 1$ or 2 and from (16) one can deduce

$$D_t^\alpha w(t) \leq -\left|\frac{e(t)}{x(t)}\right| - q(t)|x(t)|^{\gamma-1} + \frac{1}{Kr(t)} w^2(t).$$

As in the proof of the previous result, using Young's inequality, we obtain

$$D_t^\alpha w(t) \leq -Q(t) - \frac{1}{Kr(t)} w^2(t) \tag{17}$$

which holds for $t \in [s_i, t_i]$, $i = 1$ or 2.

Now let $w(t) = \widetilde{w}(\xi)$. Then we have $D_t^\alpha w(t) = \widetilde{w}'(\xi)$. So (17) is transformed to

$$\widetilde{w}'(\xi) \leq -\widetilde{Q}(\xi) - \frac{1}{K\widetilde{r}(\xi)} \widetilde{w}^2(\xi) \tag{18}$$

for $\xi \in [\xi_{s_i}, \xi_{t_i}]$, $i = 1$ or 2.

Now multiplying $G^2(\xi)$ throughout inequaity (18) and integrating from ξ_{s_i} to ξ_{t_i} for $i = 1$ or 2 we obtain

$$A(\widetilde{Q}; \xi_{s_i}, \xi_{t_i}) \leq A\left(2\frac{\left|G'\right|}{|G|}|\widetilde{w}| - \frac{1}{K\widetilde{r}(\xi)}\widetilde{w}^2; \xi_{s_i}, \xi_{t_i}\right). \tag{19}$$

Setting

$$m(v) = 2\frac{\left|G'\right|}{|G|}v - \frac{1}{K\widetilde{r}}v^2, \qquad v > 0$$

we have $m'(v^\star) = 0$ and $m''(v^\star) < 0$ where $v^\star = K\frac{|G'|}{|G|}\widetilde{r}$, which implies $m(v)$ obtains its maximum at $v^\star$. So we have

$$m(v) \leq m\left(v^\star\right) = K\frac{\left(G'\right)^2}{G^2}\widetilde{r}. \tag{20}$$

Then, by using (20) in (19), we obtain

$$A(\widetilde{Q}; \xi_{s_i}, \xi_{t_i}) \leq A\left(K\frac{\left(G'\right)^2}{G^2}\widetilde{r}; \xi_{s_i}, \xi_{t_i}\right)$$

which contradicts (14). The proof is complete. □

Theorem 3. *Suppose the conditions* (C_1) *and* (C_2) *with* $\gamma = 1$ *hold. If there exists a* $\Phi \in Y$ *such that the inequality*

$$T\left[\widetilde{q}(s) - K\phi^2\widetilde{r}(s); \xi_{s_i}, \xi_{t_i}\right] > 0 \tag{21}$$

holds for $i = 1, 2$, *where the operator* T *is defined by (2) and the function* $\phi = \phi(t, s, l)$ *is defined by (4), then Equation (1) is oscillatory.*

Proof. On the contrary, suppose that Equation (1) has a nonoscillatory solution $x(t)$. Then $x(t)$ eventually must have one sign, i.e. $x(t) /= 0$ on $[T_0, \infty)$ for some large $T_0 \geq t_0$. By defining (15) and similar calculations with the previous result, we have

$$\widetilde{w}'(\xi) \leq -\widetilde{q}(\xi) - \frac{1}{K\widetilde{r}(\xi)}\widetilde{w}^2(\xi) \tag{22}$$

for $\xi \in [\xi_{s_i}, \xi_{t_i}]$, $i = 1$ or 2. Applying $T[\cdot; \xi_{s_i}, \xi_{t_i}]$ to (22) and using (3), we have

$$T\left[\widetilde{q}(s); \xi_{s_i}, \xi_{t_i}\right] \leq T\left[2|\phi|\left|\widetilde{w}(s)\right| - \frac{1}{K\widetilde{r}(s)}\widetilde{w}^2(s); \xi_{s_i}, \xi_{t_i}\right] \tag{23}$$

for $\xi \geq \xi_\star$ Set

$$F(v) = 2|\phi|v - \frac{1}{K\widetilde{r}}v^2, \qquad v > 0.$$

We have $F'(v^\star) = 0$ and $F''(v^\star) < 0$ where $v^\star = K|\phi|\widetilde{r}$, which implies that F obtains its maximum at $v^\star$. So we have

$$F(v) \leq F\left(v^\star\right) = K\phi^2\widetilde{r}. \tag{24}$$

Then, by using (24) in (23), we obtain

$$T\left[\widetilde{q}(s); \xi_{s_i}, \xi_{t_i}\right] \leq T\left[K\phi^2\widetilde{r}(s); \xi_{s_i}, \xi_{t_i}\right]$$

which contradicts (21). Thus, the proof is complete. □

3 Applications

Example 1. Consider the fractional differential equation

$$\begin{aligned} &D_t^\alpha\left(D_t^\alpha x(t)\right) + p(t)x(t)D_t^\alpha x(t) + x(t)\left[1 + x^2(\tau(t))\right] \\ &= \sin\left(t^\alpha/\Gamma(1+\alpha)\right) \end{aligned} \tag{25}$$

for $t \geq 0$, $p(t) \geq 0$ and $0 < \alpha < 1$. This corresponds to Equation (1) with $r(t) = 1$, $e(t) = \sin\left(t^\alpha/\Gamma(1+\alpha)\right)$, $k_1(u, v) = v$, $k_2(u, v) = uv$. So we have $K = \gamma = q = Q = 1$. Now, by choosing $H(t, s) = (t-s)^2$, $\xi_{s_1} = 2k\pi$, $\xi_{t_1} = \xi_{s_2} = (2k+2)\pi$, $\xi_{t_2} = (2k+4)\pi$, $\delta_1 = (2k+1)\pi$, $\delta_2 = (2k+3)\pi$ for some sufficiently large k and $\rho(t) = 1$, we have $H_i(t, s) = h_i(t, s) = 2$ for $i = 1, 2$. Since

$$\begin{aligned} &\int_{2k\pi}^{(2k+1)\pi}\left[(s-2k\pi)^2 - \frac{1}{2}\right]ds \\ &+ \int_{(2k+3)\pi}^{(2k+4)\pi}\left[((2k+3)\pi - s)^2 - \frac{1}{2}\right] = \frac{2}{3}\pi^3 - \pi > 0, \end{aligned}$$

inequality (5) clearly holds. Thus, Equation (25) is oscillatory from Theorem 1.

Example 2. Choose $\Phi(t,s,l) = \sin(t-s)\sin(s-l)$, $\xi_{s_1} = 2k\pi$, $\xi_{t_1} = \xi_{s_2} = (2k+1)\pi$, $\xi_{t_2} = (2k+2)\pi$ and consider the Equation (25). We have $\phi(t,s,l) = \sin(t-2s+l)$ and it s easy to verify

$$T\left[\widetilde{q}(s) - K\phi^2\widetilde{r}(s); \xi_{s_i}, \xi_{t_i}\right] = \int_{2k\pi}^{(2k+1)\pi} \sin^2((2k+1)\pi - s)\sin^2(s-2k\pi) \times \cos^2((2k+1)\pi - 2s + 2k\pi) > 0$$

which implies that opscillation of Equation (25) follows from Theorem 3.

Example 3. Consider the fractional differential equation

$$D_t^\alpha\left(\sin^2\left(t^\alpha/\Gamma(1+\alpha)\right)D_t^\alpha x(t)\right) + x(t)\left[1+x^2(\tau(t))\right] = \sin\left(t^\alpha/\Gamma(1+\alpha)\right) \qquad (26)$$

\$for $t \geq 0$ and $0 < \alpha < 1$. This corresponds to Equation (1) with $r(t) = \sin^2\left(t^\alpha/\Gamma(1+\alpha)\right)$, $e(t) = \sin\left(t^\alpha/\Gamma(1+\alpha)\right)$, $k_1(u,v) = v$, $k_2(u,v) = uv$. Therefore $\widetilde{r}(\xi) = \sin^2(\xi)$, $K = \gamma = q = Q = 1$. Now choose $G(t) = \sin^2 t$, $\xi_{s_1} = 2k\pi$, $\xi_{t_1} = \xi_{s_2} = (2k+1)\pi$, $\xi_{t_2} = (2k+2)\pi$. It is easy to verify that

$$\frac{3}{8}\pi = A(\widetilde{Q}; \xi_{s_i}, \xi_{t_i}) > A(K\frac{\left(G'\right)^2}{G^2}\widetilde{r}; \xi_{s_i}, \xi_{t_i}) = \frac{\pi}{4}.$$

So, Equation (26) is oscillatory from Theorem 2.

4 Conclusion

In this paper, we are concerned with the oscillation of solution to fractional order functional differential equations with nonlinear damping. As one can see, the variable transformation used in (ξ) is very important, transforms a fractional differential equation into an ordinary differential equation of integer order, whose oscillation criteria can be established using generalized Riccati transformation, Generalized Philos type kernels and averagaging technique. We note that the approach in establishing the main theorems above can be generalized to research oscillation of fractional differential equations with more complicated forms, which are expected to research further.

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