**Research Article** **Open Access**

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# A numerical method for solving systems of higher order linear functional differential equations

DOI 10.1515/phys-2015-0052

Received November 20, 2015; accepted November 27, 2015

**Abstract:** Functional differential equations have importance in many areas of science such as mathematical physics. These systems are difficult to solve analytically.In this paper we consider the systems of linear functional differential equations [1–9] including the term $y(\alpha x + \beta)$ and advance-delay in derivatives of $y$ . To obtain the approximate solutions of those systems, we present a matrix-collocation method by using Müntz-Legendre polynomials and the collocation points. For this purpose, to obtain the approximate solutions of those systems, we present a matrix-collocation method by using Müntz-Legendre polynomials and the collocation points. This method transform the problem into a system of linear algebraic equations. The solutions of last system determine unknown coefficients of original problem. Also, an error estimation technique is presented and the approximate solutions are improved by using it. The program of method is written in Matlab and the approximate solutions can be obtained easily. Also some examples are given to illustrate the validity of the method.

**Keywords:** systems of functional differential equations; matrix-collocation method; Müntz-Legendre polynomials; approximate solutions

**PACS:** 02.30.Ks; 02.70.Jn

## 1 Introduction

In physics, chemistry, biology and engineering, a lot of problems are modelled by differential equations, delay differential equations [10–13] and their systems [1–9, 14–17]. Also, solving these equation may be analytical difficult and therefore numerical techniques are needed. Until now many analytical and numerical solution techniques such as method of steps, Euler's method, Runga Kutta method, shooting method, spline method [18–20], variational iteration method [6], Adam's method [8], Adomian decomposition method [21], homotopy perturbation method [14] etc. are used for differential equations and systems of these. For this purpose, in this study we will focus on the numerical solutions of systems of functional differential equations. Some matrix and collocation methods [20, 22–27] which was applied successfully for ordinary differential equations, partial differential equations, integral equations and difference equations previously.

In this paper we will consider the systems of equations in $[0, 1]$ of the form,

$$\sum_{r=0}^{R}\sum_{n=0}^{m}\sum_{j=0}^{k}\left[\mu_{i,j}^{n,r}(x)y_j^{(n)}(\alpha_{i,j}^{n,r}x+\beta_{i,j}^{n,r})\right]=g_i(x) \qquad (1)$$

for $i = 1, 2, \ldots, k$ under the conditions

$$\sum_{n=1}^{m}(\phi_{n,i}y_n(0)+\psi_{n,i}y_n(1))=\lambda_i,\ n=1,2,\ldots,k. \qquad (2)$$

Here a matrix-collocation method will be applied to the problem to get the approximate solutions in the truncated series form

$$y_j(x)=\sum_{n=0}^{N}a_{j,n}L_n(x),\ j=1,2,\ldots,k, \qquad (3)$$

where $L_n(x)$ are the Müntz-Legendre polynomials defined by the formula

$$L_n(x)=\sum_{j=n}^{N}(-1)^{N-j}\binom{N+1+j}{N-n}\binom{N-n}{N-j}x^j. \qquad (4)$$

In the problem $y_j^{(0)}(x) = y_j(x)$ are the unknown functions, $\mu_{i,j}^{n,r}(x)$ and $g_i(x)$ functions defined in the interval $0 \le x \le 1$. On the other hand $\alpha_{i,j}^{n,r}$, $\beta_{i,j}^{n,r}$, $\phi_{n,i}$, $\psi_{n,i}$, $\lambda_i$ are real constants, , $a_{j,n}$, $n = 0, 1, 2, \ldots, N$ are unknown Müntz-Legendre coefficients.

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## 2 Matrix relations required for solution method

At the beginning let us consider the equation (1) and try to construct the matrix form of each term in the equation. The approximate solutions $y_j(x)$ the truncated series of Müntz-Legendre polynomials and its derivatives can be written in the matrix form as,

$$[y_j(x)] = \mathbf{L}(x)\mathbf{A}_j,\ \left[y_j^k(x)\right] = \mathbf{L}^{(k)}(x)\mathbf{A}_j \tag{5}$$

where $\mathbf{L}(x) = \begin{bmatrix} L_0(x) & L_1(x) & \cdots & L_N(x) \end{bmatrix}$,
$\mathbf{A}_j = \begin{bmatrix} a_{j,0} & a_{j,1} & \cdots & a_{j,N} \end{bmatrix}^T, j = 1, 2, \ldots, k.$
On the other hand $\mathbf{L}(x)$ matrix can be represented as,

$$\mathbf{L}(x) = \mathbf{T}(x)\mathbf{F}^T, \mathbf{L}^k(x) = \mathbf{T}^k(x)\mathbf{F}^T, \tag{6}$$

where

$$\mathbf{T}(x) = \begin{bmatrix} 1 & x & \ldots & x^N \end{bmatrix}$$

and the matrix $\mathbf{F}$ is defined by

$$\mathbf{F} = \begin{bmatrix} (-1)^N\binom{N+1}{N} & (-1)^{N-1}\binom{N+2}{N}\binom{N}{N-1} & \cdots \\ 0 & (-1)^{N-1}\binom{N+2}{N-1} & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \ddots \\ \vdots & \vdots & \cdots \\ 0 & 0 & \cdots \end{bmatrix}$$

$$\begin{bmatrix} (-1)^{N-2}\binom{N+3}{N}\binom{N}{N-2} & \cdots \\ (-1)^{N-2}\binom{N+3}{N-1}\binom{N-1}{N-2} & \cdots \\ (-1)^{N-2}\binom{N+3}{N-2} & \cdots \\ 0 & \ddots \\ \vdots & \cdots \\ 0 & \cdots \end{bmatrix}$$

$$\begin{bmatrix} (-1)^1\binom{2N}{N}\binom{N}{1} & (-1)^0\binom{2N+1}{N} \\ (-1)^1\binom{2N}{N-1}\binom{N-1}{1} & (-1)^0\binom{2N+1}{N-1} \\ (-1)^1\binom{2N}{N-2}\binom{N-2}{N-1} & (-1)^0\binom{2N+1}{N-2} \\ \vdots & \vdots \\ (-1)^1\binom{2N}{1} & (-1)^0\binom{2N+1}{1} \\ 0 & (-1)^0\binom{2N+1}{0} \end{bmatrix}.$$

Using the relations (5) and (6), $y_j(x)$ can be written in the form

$$[y_j(x)] = \mathbf{T}(x)\mathbf{F}^T\mathbf{A}_j. \tag{7}$$

Then substituting $\alpha_{i,j}^{n,r}x + \beta_{i,j}^{n,r}$ instead of $x$ in Equation (7) yields for $j = 1, 2, \ldots, k$

$$\left[y_j\left(\alpha_{i,j}^{n,r}x + \beta_{i,j}^{n,r}\right)\right] = \mathbf{T}\left(\alpha_{i,j}^{n,r}x + \beta_{i,j}^{n,r}\right)\mathbf{F}^T\mathbf{A}_j, j = 1, 2, \ldots, k.$$

Also the relation between $\mathbf{T}\left(\alpha_{i,j}^{n,r}x + \beta_{i,j}^{n,r}\right)$ and $\mathbf{T}(x)$ matrices can be defined by

$$\mathbf{T}\left(\alpha_{i,j}^{n,r}x + \beta_{i,j}^{n,r}\right) = \mathbf{T}(x)\mathbf{B}\left(\alpha_{i,j}^{n,r}x + \beta_{i,j}^{n,r}\right)$$

where by using the binomial expansion,

$$\begin{bmatrix} \binom{0}{0}(\alpha_{i,j}^{n,r})^0(\beta_{i,j}^{n,r})^0 & \binom{1}{0}(\alpha_{i,j}^{n,r})^0(\beta_{i,j}^{n,r})^1 & \binom{2}{0}(\alpha_{i,j}^{n,r})^0(\beta_{i,j}^{n,r})^2 & \cdots \\ 0 & \binom{1}{1}(\alpha_{i,j}^{n,r})^1(\beta_{i,j}^{n,r})^0 & \binom{2}{1}(\alpha_{i,j}^{n,r})^1(\beta_{i,j}^{n,r})^1 & \cdots \\ 0 & 0 & \binom{2}{2}(\alpha_{i,j}^{n,r})^2(\beta_{i,j}^{n,r})^0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots \end{bmatrix}$$

$$\begin{bmatrix} \binom{N}{0}(\alpha_{i,j}^{n,r})^0(\beta_{i,j}^{n,r})^N \\ \binom{N}{1}(\alpha_{i,j}^{n,r})^1(\beta_{i,j}^{n,r})^{N-1} \\ \binom{N}{2}(\alpha_{i,j}^{n,r})^2(\beta_{i,j}^{n,r})^{N-2} \\ \vdots \\ \binom{N}{N}(\alpha_{i,j}^{n,r})^N(\beta_{i,j}^{n,r})^0 \end{bmatrix}.$$

Besides the relation $\mathbf{T}(x)$ and its derivative $\mathbf{T}^{(k)}(x)$ can be written as

$$\mathbf{T}^{(k)}(x) = \mathbf{T}(x)\left(\mathbf{B}^T\right)^k$$

where

$$\mathbf{B}^T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus using these relations we obtain the matrix form of

$$\left[y_j^{(k)}\left(\alpha_{i,j}^{n,r}x + \beta_{i,j}^{n,r}\right)\right] = \mathbf{T}(x)\left(\mathbf{B}^T\right)^k\mathbf{B}\left(\alpha_{i,j}^{n,r}, \beta_{i,j}^{n,r}\right)\mathbf{F}^T\mathbf{A}_j. \tag{8}$$

Furthermore, the matrix form of $\left[\mathbf{y}^{(k)}(\alpha_{i,j}^{n,r}x + \beta_{i,j}^{n,r})\right]$ is given below:

$$\left[\mathbf{y}^{(k)}\left(\alpha_{i,j}^{n,r}x + \beta_{i,j}^{n,r}\right)\right] = \bar{\mathbf{T}}(x)\bar{\mathbf{B}}^k\bar{\mathbf{B}}_{\alpha,\beta}\bar{\mathbf{F}}\mathbf{A}. \tag{9}$$

where

$$\mathbf{y}^{(k)}(\alpha_{i,j}^{n,r}x + \beta_{i,j}^{n,r}) = \begin{bmatrix} y_1^{(k)}(\alpha_{i,j}^{n,r}x + \beta_{i,j}^{n,r}) \\ y_2^{(k)}(\alpha_{i,j}^{n,r}x + \beta_{i,j}^{n,r}) \\ \vdots \\ y_k^{(k)}(\alpha_{i,j}^{n,r}x + \beta_{i,j}^{n,r}) \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix},$$

$$\bar{\mathbf{T}}(x) = \begin{bmatrix} \mathbf{T}(x) & 0 & \cdots & 0 \\ 0 & \mathbf{T}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{T}(x) \end{bmatrix},$$

$$\bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B}^T & 0 & \cdots & 0 \\ 0 & \mathbf{B}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B}^T \end{bmatrix},$$

$$\bar{\mathbf{F}} = \begin{bmatrix} \mathbf{F}^T & 0 & \cdots & 0 \\ 0 & \mathbf{F}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{F}^T \end{bmatrix},$$

$$\bar{\mathbf{B}}_{\alpha,\beta} = \begin{bmatrix} \mathbf{B}(\alpha_{i,j}^{n,r}, \beta_{i,j}^{n,r}) & 0 & \cdots & 0 \\ 0 & \mathbf{B}(\alpha_{i,j}^{n,r}, \beta_{i,j}^{n,r}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B}(\alpha_{i,j}^{n,r}, \beta_{i,j}^{n,r}) \end{bmatrix}.$$

## 3 Method of solution

In this Section, the method is constructed by using matrix relations in Section 2, collocation points and matrix operations. Firstly, let us write the matrix equation corresponding to the system (1) as ,

$$\sum_{r=0}^{R}\sum_{n=0}^{m}\left[\mu_{n,r}(x)\mathbf{y}^n\left(\alpha_{i,j}^{n,r}x+\beta_{i,j}^{n,r}right)\right] = \mathbf{g}(x). \quad (10)$$

Here $\mu_{n,r}(x)$ and $\mathbf{g}(x)$ matrices are defined as

$$\mu_{n,r}(x) = \begin{bmatrix} \mu_{1,1}^{n,r}(x) & \mu_{1,2}^{n,r}(x) & \cdots & \mu_{1,k}^{n,r}(x) \\ \mu_{2,1}^{n,r}(x) & \mu_{2,2}^{n,r}(x) & \cdots & \mu_{2,k}^{n,r}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{k,1}^{n,r}(x) & \mu_{k,2}^{n,r}(x) & \cdots & \mu_{k,k}^{n,r}(x) \end{bmatrix}$$

and

$$\mathbf{g}(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_k(x) \end{bmatrix}.$$

Using the collocation points [10–13, 18, 19, 21, 26, 28] defined by the following formula

$$t_s = \frac{1}{N}s,\ s = 0, 1, \ldots, N, \quad (11)$$

in Equation (10) yields,

$$\sum_{r=0}^{R}\sum_{n=0}^{m}\left[\mu_{n,r}(x_s)\mathbf{y}^k\left(\alpha_{i,j}^{n,r}x_s+\beta_{i,j}^{n,r}\right)\right] = \mathbf{g}(x_s).$$

So the fundamental matrix [15, 22, 23] equation can be represented as

$$\mu\mathbf{Y}_{\alpha,\beta}^{k} = \mathbf{G}, \quad (12)$$

where

$$\mu = \begin{bmatrix} \mu_{n,r}(x_0) & 0 & \cdots & 0 \\ 0 & \mu_{n,r}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_{n,r}(x_N) \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}(x_0) \\ \mathbf{g}(x_1) \\ \vdots \\ \mathbf{g}(x_N) \end{bmatrix},$$

$$\mathbf{Y}_{\alpha,\beta}^{(k)} = \begin{bmatrix} \mathbf{y}^{(k)}(\alpha_{i,j}^{n,r}x_0+\beta_{i,j}^{n,r}) \\ \mathbf{y}^{(k)}(\alpha_{i,j}^{n,r}x_1+\beta_{i,j}^{n,r}) \\ \vdots \\ \mathbf{y}^{(k)}(\alpha_{i,j}^{n,r}x_N+\beta_{i,j}^{n,r}) \end{bmatrix}.$$

From the relation (9) and the collocation points given by (11) we have

$$\mathbf{y}^k\left(\alpha_{i,j}^{n,r}x_s+\beta_{i,j}^{n,r}\right) = \bar{\mathbf{T}}(x_s)\bar{\mathbf{B}}^k\bar{\mathbf{B}}_{\alpha,\beta}\bar{\mathbf{F}}\mathbf{A},\ s = 0, 1, \ldots, N. \quad (13)$$

where

$$\mathbf{T} = \begin{bmatrix} \bar{\mathbf{T}}(x_0) \\ \bar{\mathbf{T}}(x_1) \\ \vdots \\ \bar{\mathbf{T}}(x_N) \end{bmatrix},$$

$$\bar{\mathbf{T}}(x_s) = \begin{bmatrix} \mathbf{T}(x_s) & 0 & \cdots & 0 \\ 0 & \mathbf{T}(x_s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{T}(x_s) \end{bmatrix}.$$

Hence, fundamental matrix equation is

$$\mu\mathbf{T}\bar{\mathbf{B}}^k\bar{\mathbf{B}}_{\alpha,\beta}\bar{\mathbf{F}}\mathbf{A} = \mathbf{G}. \quad (14)$$

The dimensiones of the matrices $\mu$, $\mathbf{T}$, $\bar{\mathbf{B}}_{\alpha,\beta}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{F}}$ in Equation (14) are $k(N+1)\times k(N+1)$ and the dimensions of $\mathbf{A}$ and $\mathbf{G}$ are $k(N+1)\times 1$ . Thus the fundamental matrix equation corresponding to the Equation (1) can be written briefly in the form

$$\mathbf{WA} = \mathbf{G}\ or\ [\mathbf{W};\mathbf{G}], \quad (15)$$

where

$$\mathbf{W} = \mu\mathbf{T}\bar{\mathbf{B}}^k\bar{\mathbf{B}}_{\alpha,\beta}\bar{\mathbf{F}} = [w_{p,q}], p, q = 1, 2, \ldots, k(N+1).$$

Equation (15) is a system of equations consisting $k(N+1)$ unknown Müntz-Legendre coefficients and $k(N+1)$ linear algebraic equations. On the other hand matrix form that corresponds to the conditions (2) can be written in the form of

$$\left[\bar{\phi}\bar{T}(0) + \bar{\psi}\bar{T}(1)\right]\bar{\mathbf{F}}\mathbf{A} = \lambda, \tag{16}$$

where for $i = 1, 2, \ldots, k$

$$\bar{\phi} = \begin{bmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \vdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_k \end{bmatrix},$$

$$\phi_i = \begin{bmatrix} \phi_{i,1} \\ \phi_{i,2} \\ \vdots \\ \phi_{i,k} \end{bmatrix},$$

$$\bar{\psi} = \begin{bmatrix} \psi_1 & 0 & \cdots & 0 \\ 0 & \psi_2 & \vdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \psi_k \end{bmatrix},$$

$$\psi_i = \begin{bmatrix} \psi_{i,1} \\ \psi_{i,2} \\ \vdots \\ \psi_{i,k} \end{bmatrix},$$

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{bmatrix}.$$

So that the matrix form of the conditions can be represented briefly in the form

$$\mathbf{UA} = \lambda \text{ or } [\mathbf{U}; \lambda], \tag{17}$$

where

$$\mathbf{U} = \left[\bar{\phi}\bar{T}(0) + \bar{\psi}\bar{T}(1)\right]\bar{\mathbf{F}}.$$

Finally if we replace any rows of $[\mathbf{W}; \mathbf{G}]$ by the rows of $[\mathbf{U}; \lambda]$ we have the new augmented matrix [24, 25] equation:

$$\tilde{\mathbf{W}}\mathbf{A} = \tilde{\mathbf{G}} \tag{18}$$

that is

$$\left[\tilde{\mathbf{W}}; \tilde{\mathbf{G}}\right] = \begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,k(N+1)} & ; & g_1(x_0) \\ w_{2,1} & w_{2,2} & \cdots & w_{2,k(N+1)} & ; & g_2(x_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{k,1} & w_{k,2} & \cdots & w_{k,k(N+1)} & ; & g_k(x_0) \\ w_{k+1,1} & w_{k+1,2} & \cdots & w_{k+1,k(N+1)} & ; & g_1(x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{kN,1} & w_{kN,2} & \cdots & w_{kN,k(N+1)} & ; & g_k(x_{N-1}) \\ v_{1,1} & v_{1,2} & \cdots & v_{1,k(N+1)} & ; & \lambda_1 \\ v_{2,1} & v_{2,2} & \cdots & v_{2,k(N+1)} & ; & \lambda_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{k,1} & v_{k,2} & \cdots & v_{k,k(N+1)} & ; & \lambda_k \end{bmatrix}. \tag{19}$$

If $rank\tilde{\mathbf{W}} = rank[\tilde{\mathbf{W}}; \tilde{\mathbf{G}}] = k(N+1)$, we could say

$$\mathbf{A} = \tilde{\mathbf{W}}^{-1}\tilde{\mathbf{G}}. \tag{20}$$

Therefore the unknown Müntz-Legendre coefficients matrix can be find by solving this linear equation system. Finally substituting the coefficients $a_{j,0}, a_{j,1}, \ldots, a_{j,N}$, $j = 1, 2, \ldots, k$ in Equation (3)gives us the approximate solutions:

$$y_{j,N}(x) = \sum_{n=0}^{N} a_{j,N}L_N(x), i = 1, 2, \ldots, k.$$

# 4 Error estimation and improved approximate solutions

In this section an error analysis based on the residual function is given for the method defined in the previous section. When the approximate solutions obtained by our method substituted in Equation (1), the equation will be satisfied approximately, that is;

$$\sum_{r=0}^{R}\sum_{n=0}^{m}\sum_{j=0}^{k}\left[\mu_{i,j}^{n,r}(x)y_{j,N}^{(n)}\left(\alpha_{i,j}^{n,r}x + \beta_{i,j}^{n,r}\right)\right] = g_i(x) + R_{j,N}(x), \tag{21}$$

where $R_{j,N}(x)$ is the residual function [23–26]. Let us write the error function $e_{j,N} = y_j(x) - y_{j,N}(x)$ and to obtain the error problem let us subtract the equations in (21) from the equations (1) side by side. So we could reach the error problem

$$\sum_{r=0}^{R}\sum_{n=0}^{m}\sum_{j=0}^{k}\left[\mu_{i,j}^{n,r}(x)e_{j,N}^{(n)}\left(\alpha_{i,j}^{n,r}x + \beta_{i,j}^{n,r}\right)\right] = -R_{j,N}(x). \tag{22}$$

Applying the same procedure for the conditions gives us the new homogenous conditions of the form,

$$\sum_{n=1}^{m}\left(\phi_{n,i}e_n(0) + \psi_{n,i}e_n(1)\right) = 0,\ n = 1, 2, \ldots, k. \tag{23}$$

And solving the error problem (22)-(23) by the same method defined in section 3, an approximation $e_{j,N,M}$ can be find to the error function $e_{j,N}$ , $j = 1, 2, \ldots, k$ . When solving this error problem it is better to choose $M$ and replacing the matrix

$$\mathbf{R}_N(x) = \begin{bmatrix} -R_{1,N}(x) \\ -R_{1,N}(x) \\ \vdots \\ -R_{k,N}(x) \end{bmatrix}$$

instead of the matrix $\mathbf{G}$ . Therefore the approximate solution $y_{j,N}(x)$ can be improved as $y_{j,N,M}(x) = y_{j,N}(x) + e_{i,N,M}(x)$. So a new error function can be defined as improved absolute error function by the relation $\left|E_{j,N,M}(x)\right| = \left|y_j(x) - y_{j,N,M}(x)\right|$.

# 5 Numerical examples

In this section some examples will be given to explain the method in details and to show the numerical results. All the computations and graphs are performed by a code written in MATLAB R2007b.

**Example 1.** *First of all let us consider the system of equations of order two with two unknown functions in* $0 \le x \le 1$ ,

$$\begin{aligned} y_1^{(2)}(0.3x - 0.1) &+ y_2^{(2)}(0.3x - 0.1) + 2y_1^{(1)}(x - 0.2) \\ &+ 3y_2^{(1)}(x - 0.3) = g_1(x) \\ y_1^{(2)}(0.2x - 0.1) &- y_2^{(2)}(0.1x - 0.3) + y_1^{(1)}(x - 0.3) \\ &+ 4y_2^{(1)}(x - 0.2) = g_2(x) \end{aligned} \quad (24)$$

under the conditions $y_1(0) = 1, y_1^{(1)}(0) = 1, y_2(0) = 1, y_2^{(1)}(0) = -1$.The exact solutions of the problem are $y_1(x) = e^x$ and $y_2(x) = e^{-x}$. Here $k = 2$, $m = 2$,$R = 1$, $\mu_{1,1}^{2,1}(x) = 1$, $\mu_{1,2}^{2,1}(x) = 1$, $\mu_{1,1}^{1,1}(x) = 2$, $\mu_{1,2}^{1,1}(x) = 3$, $\mu_{1,1}^{0,1}(x) = 0$, $\mu_{1,2}^{0,1}(x) = 0$, $\mu_{2,1}^{2,1}(x) = 1$, $\mu_{2,2}^{2,1}(x) = -1$, $\mu_{2,1}^{1,1}(x) = 1$, $\mu_{2,2}^{1,1}(x) = 4$, $\alpha_{1,1}^{1,1} = \alpha_{1,2}^{1,1} = 1$, $\alpha_{1,1}^{2,1} = 0$, $\alpha_{1,2}^{2,1} = 0.3$, $\beta_{1,1}^{1,1} = -0.2$, $\beta_{1,2}^{1,1} = -0.3$, $\beta_{1,1}^{2,1} = -0.1$, $\beta_{1,2}^{2,1} = -0.1$, $\alpha_{2,1}^{1,1} = \alpha_{2,2}^{1,1} = 1$, $\alpha_{2,1}^{2,1} = 0.2$, $\alpha_{2,2}^{2,1} = 0.1$, $\beta_{2,1}^{1,1} = -0.3$, $\beta_{2,2}^{1,1} = -0.2$,$\beta_{2,1}^{2,1} = -0.1$, $\beta_{2,2}^{2,1} = -0.3$, $g_1(x) = e^{3x/10-1/10} + e^{-3x/10+1/10} + 2e^{x-1/5} - 3e^{-x+3/10}$ and $g_2(x) = e^{x/5-1/10} - e^{-x/10+3/10} + e^{x-3/10} - 4e^{-x+1/5}$.
Now by taking $N = 2$ for the problem we search for the approximate solutions of the form

$$y_j(x) = \sum_{n=0}^{2} a_{j,n} L_n(x).$$

The set of the collocation points for $N = 2$ is calculated as

$$\left\{ x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1 \right\}.$$

The fundamental matrix equation corresponding to the problem is given by $\left(\mu_1 \mathbf{T}\bar{\mathbf{B}}\bar{\mathbf{B}}_{1,0.2} + \mu_2 \mathbf{T}\bar{\mathbf{B}}\bar{\mathbf{B}}_{1,-0.3} + \mu_3 \mathbf{T}\bar{\mathbf{B}}^2\bar{\mathbf{B}}_{0.3,-0.1} + \mu_4 \mathbf{T}\bar{\mathbf{B}}^2\bar{\mathbf{B}}_{0.2,-0.1} + \mu_5 \mathbf{T}\bar{\mathbf{B}}^2\bar{\mathbf{B}}_{0.1,-0.3}\right) \bar{\mathbf{F}}\mathbf{A} = \mathbf{G}$ and the matrices in this equation is defined by

$$\mu_1(x) = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \ \mu_2(x) = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix},$$

$$\mu_3(x) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \ \mu_4(x) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$\mu_5(x) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \ \mu_1 = \begin{bmatrix} \mu_1(0) & 0 & 0 \\ 0 & \mu_1(1/2) & 0 \\ 0 & 0 & \mu_1(1) \end{bmatrix},$$

$$\mu_2 = \begin{bmatrix} \mu_2(0) & 0 & 0 \\ 0 & \mu_2(1/2) & 0 \\ 0 & 0 & \mu_2(1) \end{bmatrix},$$

$$\mu_3 = \begin{bmatrix} \mu_3(0) & 0 & 0 \\ 0 & \mu_3(1/2) & 0 \\ 0 & 0 & \mu_3(1) \end{bmatrix},$$

$$\mu_4 = \begin{bmatrix} \mu_4(0) & 0 & 0 \\ 0 & \mu_4(1/2) & 0 \\ 0 & 0 & \mu_4(1) \end{bmatrix},$$

$$\mu_5 = \begin{bmatrix} \mu_5(0) & 0 & 0 \\ 0 & \mu_5(1/2) & 0 \\ 0 & 0 & \mu_5(1) \end{bmatrix},$$

$$\bar{\mathbf{B}}_{1,-0.2} = \begin{bmatrix} \mathbf{B}(1, -0.2) & 0 \\ 0 & \mathbf{B}(1, -0.2) \end{bmatrix},$$

$$\mathbf{B} = (1, -0.2) = \begin{bmatrix} 1 & -1/5 & 1/25 \\ 0 & 1 & -2/5 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\bar{\mathbf{B}}_{1,-0.3} = \begin{bmatrix} \mathbf{B}(1, -0.3) & 0 \\ 0 & \mathbf{B}(1, -0.3) \end{bmatrix},$$

$$\mathbf{B} = (1, -0.3) = \begin{bmatrix} 1 & -3/10 & 9/100 \\ 0 & 1 & -3/5 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\bar{\mathbf{B}}_{0.3,-0.1} = \begin{bmatrix} \mathbf{B}(0.3, -0.1) & 0 \\ 0 & \mathbf{B}(0.3, -0.1) \end{bmatrix},$$

$$\mathbf{B} = (0.3, -0.1) = \begin{bmatrix} 1 & -1/10 & 1/100 \\ 0 & 3/10 & -3/50 \\ 0 & 0 & 9/100 \end{bmatrix},$$

$$\bar{\mathbf{B}}_{0.2,-0.1} = \begin{bmatrix} \mathbf{B}(0.2,-0.1) & 0 \\ 0 & \mathbf{B}(0.2,-0.1) \end{bmatrix},$$

$$\mathbf{B} = (0.2, -0.1) = \begin{bmatrix} 1 & -1/10 & 1/100 \\ 0 & 1/5 & -1/25 \\ 0 & 0 & 1/25 \end{bmatrix},$$

$$\bar{\mathbf{B}}_{0.1,-0.3} = \begin{bmatrix} \mathbf{B}(0.1,-0.3) & 0 \\ 0 & \mathbf{B}(0.1,-0.3) \end{bmatrix},$$

$$\mathbf{B} = (0.1, -0.3) = \begin{bmatrix} 1 & -3/10 & 9/100 \\ 0 & 1/10 & -3/50 \\ 0 & 0 & 1/100 \end{bmatrix},$$

$$\mathbf{T} = \begin{bmatrix} \bar{\mathbf{T}}(0) \\ \bar{\mathbf{T}}(1/2) \\ \bar{\mathbf{T}}(1) \end{bmatrix},$$

$$(0) = \begin{bmatrix} \mathbf{T}(0) & 0 \\ 0 & \mathbf{T}(0) \end{bmatrix},$$

$$\bar{\mathbf{T}}(1/2) = \begin{bmatrix} \mathbf{T}(1/2) & 0 \\ 0 & \mathbf{T}(1/2) \end{bmatrix},$$

$$\bar{\mathbf{T}}(1) = \begin{bmatrix} \mathbf{T}(1) & 0 \\ 0 & \mathbf{T}(1) \end{bmatrix},$$

$$\mathbf{T}(0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

$$\mathbf{T}(1/2) = \begin{bmatrix} 1 & 1/2 & 1/4 \end{bmatrix},$$

$$\mathbf{T}(1) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},$$

$$\bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B}^T & 0 \\ 0 & \mathbf{B}^T \end{bmatrix},$$

$$\bar{\mathbf{F}} = \begin{bmatrix} \mathbf{F}^T & 0 \\ 0 & \mathbf{F}^T \end{bmatrix},$$

$$\mathbf{B}^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{F}^T = \begin{bmatrix} 3 & 0 & 0 \\ -12 & -4 & 0 \\ 10 & 5 & 1 \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}(0) \\ \mathbf{g}(1/2) \\ \mathbf{g}(1) \end{bmatrix},$$

$$\mathbf{g}(0) = \begin{bmatrix} -649/1614 \\ -1712/373 \end{bmatrix},$$

$$\mathbf{g}(1/2) = \begin{bmatrix} 2967/1321 \\ -1017/502 \end{bmatrix},$$

$$\mathbf{g}(1) = \begin{bmatrix} 3426/685 \\ 391/3902 \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix},$$

$$\mathbf{A}_1 = \begin{bmatrix} a_{1,0} \\ a_{1,1} \\ a_{1,2} \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} a_{2,0} \\ a_{2,1} \\ a_{2,2} \end{bmatrix}.$$

The matrix that corresponding to this fundamental matrix equation can be expressed as

$$[\mathbf{W};\mathbf{G}] = \left[\begin{array}{cccccc;{2pt/2pt}c} -12 & -2 & 6/5 & -34 & -11 & 1/5 & -649/1614 \\ 2 & 3 & 7/5 & -84 & -34 & -18/5 & -1712/373 \\ 8 & 8 & 16/5 & -4 & 4 & 16/5 & 2967/1321 \\ 12 & 8 & 12/5 & -44 & -14 & 2/5 & -1017/502 \\ 28 & 18 & 26/5 & 26 & 19 & 31/5 & 3426/685 \\ 22 & 13 & 17/5 & -4 & 6 & 22/5 & 391/3902 \end{array}\right].$$

On the other hand the matrix corresponding to the conditions can be expressed as

$$[\mathbf{U};\lambda] = \left[\begin{array}{cccccc;{2pt/2pt}c} 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ -12 & -4 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & -12 & -4 & 0 & -1 \end{array}\right].$$

Thus the new augmented matrix is as follows

$$[\tilde{\mathbf{W}};\tilde{\mathbf{G}}] = \left[\begin{array}{cccccc;{2pt/2pt}c} -12 & -2 & 6/5 & -34 & -11 & 1/5 & -649/1614 \\ 2 & 3 & 7/5 & -84 & -34 & -18/5 & -1712/373 \\ 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ -12 & -4 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & -12 & -4 & 0 & -1 \end{array}\right].$$

By solving the linear equations system we obtain the unknown coefficient matrix as

$$\mathbf{A} = \begin{bmatrix} 1/3 & -5/4 & 2659/802 & 1/3 & -3/4 & 379/374 \end{bmatrix}^T$$

Thus substituting these coefficients in the Equation (3), gives us the approximate solutions $y_1(x) = 1 + x + 0.398794232618x^2$ and $y_2(x) = 1 - x + 0.596701701990x^2$. Now by taking $(N, M) =$

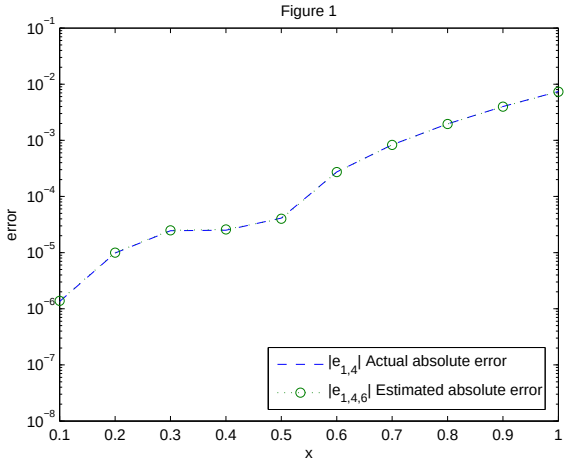


**Figure 1:** Comparison of the absolute errors of $y_1(x)$ in Example 1.

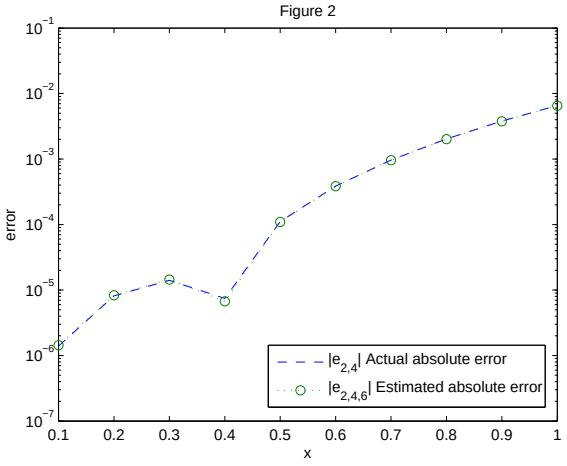


**Figure 2:** Comparison of the absolute errors of $y_2(x)$ in Example 1.

(4, 6), (8, 10) we can compute better results for our problem. After completing the computations by using MATLAB, we can give the numerical results and figures. In Table 1 and Table 2 actual absolute errors(Act.Abs.Err.) and estimated absolute errors(Est.Abs.Err.) are compared for $y_1(x)$ and $y_2(x)$ for $(N, M) = (4, 6), (8, 10)$. In Table 3 and Table 4 actual absolute errors and improved absolute errors are compared for $y_1(x)$ and $y_2(x)$ for $(N, M) = (4, 6), (8, 10)$.

After comparing the numerical results of the errors, some figures are given below to illustrate these results. In Figure 1 actual absolute errors and estimated absolute errors are drawn for $y_1(x)$ for $(N, M) = (4, 6)$ and in Figure 2,actual absolute errors and estimated absolute errors are drawn for $y_2(x)$ for $(N, M) = (4, 6)$. In Figure 3 actual absolute errors and improved absolute errors are drawn for $y_1(x)$ for $(N, M) = (4, 6)$ and in Figure 4,actual absolute errors and improved absolute errors are drawn for $y_2(x)$ for $(N, M) = (4, 6)$.

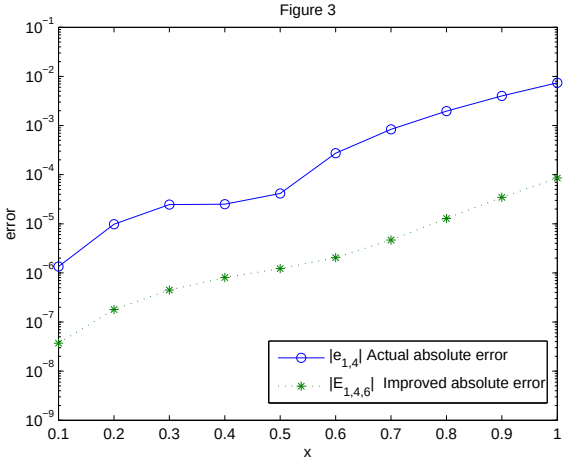


**Figure 3:** Comparison of the absolute errors of $y_1(x)$ in Example 1.

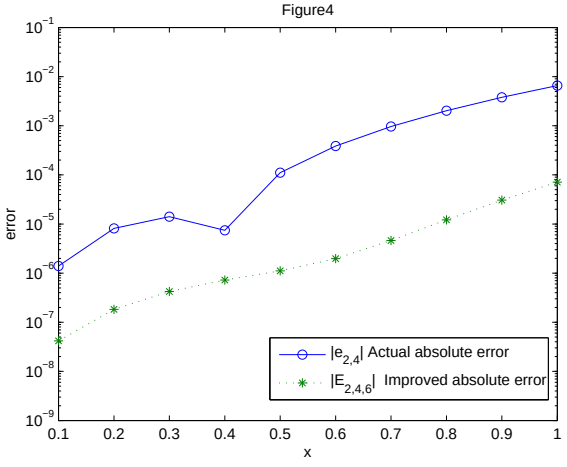


**Figure 4:** Comparison of the absolute errors of $y_2(x)$ in Example 1.

**Example 2.** *As a second example let us consider the system of equations*

$$y_1^{(1)}(x-0.5) + 2y_2^{(1)}(x-1) + 3xy_1(x-0.8) = g_1(x)$$
$$xy_1^{(1)}(x-2) - 3x^2y_2^{(1)}(x-0.1) + y_2(x-1) = g_2(x) \quad (25)$$

under the conditions $y_1(0) = 2$, $y_1(1) = e + 1$, $y_2(0) = 1$, $y_2(1) = e^{-1}$. The exact solutions of the problem are $y_1(x) = e^x + 1$ and $y_2(x) = e^{-x}$. Here $g_1(x) = e^{x-1/2} - 2e^{-x+1} + 3x(e^{x-4/5} + 1)$ and $g_2(x) = xe^{x-2} + 3x^2e^{-x+1/10} + e^{-x+1}$ . For this problem taking $(N, M) = (4, 6), (8, 10)$, in Table 5 and Table 6 actual absolute errors and estimated absolute errors are compared for $y_1(x)$ and $y_2(x)$. And in Table 7 and

**Table 1:** Comparison of absolute errors of $y_1(x)$ in Equation (24).

| | Act.Abs.Err. $N = 4$ | Est.Abs.Err. $N = 4$ $M = 6$ | Act.Abs.Err. $N = 8$ | Est.Abs.Err. $N = 8$ $M = 10$ |
|---|---|---|---|---|
| $x_i$ | $\lvert e_{1,4}(x_i)\rvert$ | $\lvert e_{1,4,6}(x_i)\rvert$ | $\lvert e_{1,8}(x_i)\rvert$ | $\lvert e_{1,8,10}(x_i)\rvert$ |
| 0 | 0 | 4.22e-19 | 4.00e-12 | 4.00e-12 |
| 0.2 | 9.81e-6 | 9.99e-6 | 4.74e-10 | 4.77e-10 |
| 0.4 | 2.50e-5 | 2.58e-5 | 1.69e-09 | 1.71e-9 |
| 0.6 | 2.73e-4 | 2.71e-4 | 3.52e-09 | 3.56e-9 |
| 0.8 | 1.96e-3 | 1.95e-3 | 1.95e-08 | 1.96e-8 |
| 1 | 7.41e-3 | 7.32e-3 | 3.33e-7 | 3.34e-7 |

**Table 2:** Comparison of absolute errors of $y_2(x)$ in Equation (24).

| | Act.Abs.Err. $N = 4$ | Est.Abs.Err. $N = 4$ $M = 6$ | Act.Abs.Err. $N = 8$ | Est.Abs.Err. $N = 8$ $M = 10$ |
|---|---|---|---|---|
| $x_i$ | $\lvert e_{2,4}(x_i)\rvert$ | $\lvert e_{2,4,6}(x_i)\rvert$ | $\lvert e_{2,8}(x_i)\rvert$ | $\lvert e_{2,8,10}(x_i)\rvert$ |
| 0 | 0 | 1.32e-20 | 0 | 2.19e-24 |
| 0.2 | 8.16e-6 | 8.34e-6 | 4.55e-10 | 4.58e-10 |
| 0.4 | 7.42e-6 | 6.70e-6 | 1.67e-9 | 1.69e-9 |
| 0.6 | 3.86e-4 | 3.84e-4 | 3.48e-9 | 3.51e-9 |
| 0.8 | 2.02e-3 | 2.01e-3 | 3.63e-8 | 3.63e-8 |
| 1 | 6.57e-3 | 6.49e-3 | 4.94e-7 | 4.92e-7 |

**Table 3:** Absolute errors of $y_1(x)$ in Equation (24).

| | Act.Abs.Err. $N = 4$ | Imp.Abs.Err. $N = 4$ $M = 6$ | Act.Abs.Err. $N = 8$ | Imp.Abs.Err. $N = 8$ $M = 10$ |
|---|---|---|---|---|
| $x_i$ | $\lvert e_{1,4}(x_i)\rvert$ | $\lvert E_{1,4,6}(x_i)\rvert$ | $\lvert e_{1,8}(x_i)\rvert$ | $\lvert E_{1,8,10}(x_i)\rvert$ |
| 0 | 0 | 0 | 4.00e-12 | 4.00e-12 |
| 0.2 | 9.81e-6 | 1.79e-7 | 4.74e-10 | 3.17e-12 |
| 0.4 | 2.50e-5 | 8.03e-7 | 1.69e-9 | 1.73e-11 |
| 0.6 | 2.73e-4 | 2.04e-6 | 3.52e-9 | 3.66e-11 |
| 0.8 | 1.96e-3 | 1.28e-5 | 1.95e-8 | 1.05e-10 |
| 1 | 7.41e-3 | 8.52e-5 | 3.33e-7 | 9.61e-10 |

**Table 4:** Absolute errors of $y_2(x)$ in Equation (24).

| | Act.Abs.Err. $N = 4$ | Imp.Abs.Err. $N = 4$ $M = 6$ | Act.Abs.Err. $N = 8$ | Imp.Abs.Err. $N = 8$ $M = 10$ |
|---|---|---|---|---|
| $x_i$ | $\lvert e_{2,4}(x_i)\rvert$ | $\lvert E_{2,4,6}(x_i)\rvert$ | $\lvert e_{2,8}(x_i)\rvert$ | $\lvert E_{2,8,10}(x_i)\rvert$ |
| 0 | 0 | 0 | 0 | 0 |
| 0.2 | 8.16e-6 | 1.83e-7 | 4.55e-10 | 3.66e-12 |
| 0.4 | 7.42e-6 | 7.22e-7 | 1.67e-9 | 1.40e-11 |
| 0.6 | 3.86e-4 | 1.97e-6 | 3.48e-9 | 2.96e-11 |
| 0.8 | 2.02e-3 | 1.21e-5 | 3.63e-8 | 3.19e-11 |
| 1 | 6.57e-3 | 7.13e-5 | 4.94e-7 | 2.78e-9 |

**Table 5:** Absolute errors of $y_1(x)$ in Equation (25).

| | Act.Abs.Err. $N = 4$ | Est.Abs.Err. $N = 4$ $M = 6$ | Act.Abs.Err. $N = 8$ | Est.Abs.Err. $N = 8$ $M = 10$ |
|---|---|---|---|---|
| $x_i$ | $\lvert e_{1,4}(x_i)\rvert$ | $\lvert e_{1,4,6}(x_i)\rvert$ | $\lvert e_{1,8}(x_i)\rvert$ | $\lvert e_{1,8,10}(x_i)\rvert$ |
| 0 | 0 | 7.91e-16 | 1.0e-11 | 2.48e-16 |
| 0.2 | 2.84e-4 | 3.29e-4 | 2.90e-6 | 3.55e-6 |
| 0.4 | 2.26e-3 | 2.20e-3 | 1.67e-5 | 2.04e-5 |
| 0.6 | 5.80e-3 | 5.78e-3 | 2.89e-5 | 3.49e-5 |
| 0.8 | 6.91e-3 | 6.92e-3 | 2.38e-5 | 2.88e-5 |
| 1 | 4.45e-13 | 1.92e-14 | 3.30e-12 | 5.35e-15 |

**Table 6:** Absolute errors of $y_2(x)$ in Equation (25).

| | Act.Abs.Err. $N = 4$ | Est.Abs.Err. $N = 4$ $M = 6$ | Act.Abs.Err. $N = 8$ | Est.Abs.Err. $N = 8$ $M = 10$ |
|---|---|---|---|---|
| $x_i$ | $\lvert e_{2,4}(x_i)\rvert$ | $\lvert e_{2,4,6}(x_i)\rvert$ | $\lvert e_{2,8}(x_i)\rvert$ | $\lvert e_{2,8,10}(x_i)\rvert$ |
| 0 | 0 | 3.96e-18 | 0 | 2.37e-18 |
| 0.2 | 4.62e-3 | 4.56e-3 | 6.19e-6 | 7.34e-6 |
| 0.4 | 9.63e-3 | 9.40e-3 | 8.69e-6 | 1.01e-5 |
| 0.6 | 1.29e-2 | 1.24e-2 | 7.99e-6 | 8.97e-6 |
| 0.8 | 1.13e-2 | 1.07e-2 | 6.34e-6 | 6.63e-6 |
| 1 | 3.65e-12 | 6.0e-14 | 1.50e-12 | 2.16e-17 |

**Table 7:** Absolute errors of $y_1(x)$ in Equation (25).

| | Act.Abs. Err. $N = 4$ | Imp.Abs.Err. $N = 4$ $M = 6$ | Act.Abs.Err. $N = 8$ | Imp.Abs.Err. $N = 8$ $M = 10$ |
|---|---|---|---|---|
| $x_i$ | $\lvert e_{1,4}(x_i)\rvert$ | $\lvert E_{1,4,6}(x_i)\rvert$ | $\lvert e_{1,8}(x_i)\rvert$ | $\lvert E_{1,8,10}(x_i)\rvert$ |
| 0 | 0 | 0 | 1.0e-11 | 1.0e-11 |
| 0.2 | 2.84e-4 | 4.46e-5 | 2.90e-6 | 6.48e-7 |
| 0.4 | 2.26e-3 | 5.14e-5 | 1.67e-5 | 3.62e-6 |
| 0.6 | 5.80e-3 | 1.63e-5 | 2.89e-5 | 6.06e-6 |
| 0.8 | 6.91e-3 | 1.69e-5 | 2.38e-5 | 4.98e-6 |
| 1 | 4.45e-13 | 4.65e-13 | 3.30e-12 | 3.31e-12 |

**Table 8:** Absolute errors of $y_2(x)$ in Equation (25).

| | Act.Abs.Err. $N = 4$ | Imp.Abs.Err. $N = 4$ $M = 6$ | Act.Abs.Err. $N = 8$ | Imp.Abs.Err. $N = 8$ $M = 10$ |
|---|---|---|---|---|
| $x_i$ | $\lvert e_{2,4}(x_i)\rvert$ | $\lvert E_{2,4,6}(x_i)\rvert$ | $\lvert e_{2,8}(x_i)\rvert$ | $\lvert E_{2,8,10}(x_i)\rvert$ |
| 0 | 0 | 0 | 0 | 0 |
| 0.2 | 4.62e-3 | 6.44e-5 | 6.19e-6 | 1.15e-6 |
| 0.4 | 9.63e-3 | 9.63e-3 | 8.69e-6 | 1.44e-6 |
| 0.6 | 1.29e-2 | 4.77e-4 | 7.99e-6 | 9.82e-7 |
| 0.8 | 1.13e-2 | 5.98e-4 | 6.34e-6 | 2.92e-7 |
| 1 | 3.65e-12 | 3.59e-12 | 1.50e-12 | 1.50 e-12 |

**Table 9:** Absolute errors of $y_1(x)$ in Equation (25).

| $x_i$ | PM N=10 | SFM N=10 | BPA N=5 | BPA N=10 | PM N=5 |
|---|---|---|---|---|---|
| 0.1 | 8.13E-014 | 4.30E-006 | 2.34E-007 | 3.33E-015 | 2.34E-007 |
| 0.2 | 2.40E-013 | 2.80E-005 | 1.12E-006 | 8.88E-015 | 1.12E-006 |
| 0.3 | 3.66E-013 | 8.10E-005 | 2.05E-006 | 2.73E-014 | 2.05E-006 |
| 0.4 | 3.79E-013 | 3.00E-004 | 2.57E-006 | 5.39E-014 | 2.57E-006 |
| 0.5 | 2.17E-013 | 7.30E-004 | 3.15E-006 | 9.17E-014 | 3.15E-006 |

**Table 10:** Absolute errors of $y_2(x)$ in Equation (25).

| $x_i$ | PM N=10 | SFM N=10 | BPA N=5 | BPA N=10 | PM N=5 |
|---|---|---|---|---|---|
| 0.1 | 5.97E-014 | 4.10E-006 | 1.40E-007 | 1.40E-007 | 1.40E-007 |
| 0.2 | 1.70E-013 | 2.20E-005 | 6.67E-007 | 4.44E-015 | 6.67E-007 |
| 0.3 | 2.57E-013 | 4.50E-005 | 1.21E-006 | 9.32E-015 | 1.21E-006 |
| 0.4 | 2.26E-013 | 1.40E-006 | 1.52E-006 | 1.23E-014 | 1.52E-006 |
| 0.5 | 5.26E-014 | 2.60E-004 | 1.84E-006 | 1.31E-014 | 1.84E-006 |

Table 8 actual absolute errors and improved absolute errors are compared for $y_1(x)$ and $y_2(x)$.

**Example 3.** *As a second example let us consider the system of equations [20],*

$$\begin{aligned} y_1^{(2)}(x) - y_1(x) + y_2(x) - y_1(x-0.2) &= -e^{x-0.2} + e^{-x} \\ y_2^{(2)}(x) + y_1(x) - y_2(x) - y_2(x-0.2) &= -e^{-x+0.2} + e^{x} \end{aligned} \quad (26)$$

under the conditions $y_1(0) = 1$, $y_1^{(1)}(0) = 1$, $y_2(0) = 1$, $y_2^{(1)}(0) = -1$.For this example in the Table 9 and Table 10 a comparison is given for actual absolute errors between Spline function method (SFM), Bessel polynomial approximation(BPA) method and present method(PM).
The computations in Table 9 and Table 10 show that both Bessel polynomial approximation and our method give so close values at each $x_i$ points. Moreover as seen above when the truncation limit is chosen $N = 10$ the results are far more better than the results of spline function method.

**Example 4.**

$$\begin{aligned} y_1^{(2)}(0.3x-0.1) &+ y_2^{(2)}(0.2x-0.1) + y_3^{(2)}(x) \\ &+ 2y_1^{(1)}(0.5x-0.2) = g_1(x) \\ y_1^{(2)}(0.2x-0.1) &+ y_2^{(2)}(x) + y_3^{(2)}(0.3x-0.1) \\ &+ y_2^{(1)}(0.5x-0.2) = g_2(x) \\ y_1^{(2)}(x) &- 2y_2^{(2)}(0.3x-0.1) + y_3^{(2)}(0.3x-0.1) \\ &+ 3y_3^{(1)}(0.5x-0.2) = g_3(x) \end{aligned} \quad (27)$$

under the conditions $y_1(0) = 1$, $y_1^{(1)}(0) = -1$, $y_2(0) = 2$, $y_2^{(1)}(0) = 2$, $y_3(0) = 1$, $y_3^{(1)}(0) = 3$. Here $g_1(x) = 3x^2/2 + 77x/5 + 16/25$, $g_2(x) = -3x^2/4 + 2x/5 + 92/25$, $g_3(x) = 9x^2/2+33x/5+163/25$.For this problem by taking $N = 3$ gives us the unknown coeffient matrix as

$$\mathbf{A} = \begin{bmatrix} -1/4 & 13/20 & -13/12 & 221/60 & -1/2 & 17/10 \\ -11/3 & 97/15 & -1/4 & 21/20 & -31/12 & 407/60 \end{bmatrix}.$$

Substituting these coefficients in the Equation (3)yields the solutions $y_1(x) = x^3+2x^2-x+1$ ,$y_2(x) = -x^3+x^2+2x+2$ and $y_3(x) = 2x^3 - x^2 + 3x + 1$ which are the exact solutions of the problem.

# 6 Conclusion

In this study a numerical technique is applied to obtain the approximate solutions of system of linear functional differential equations. By using this method, the problem is reduced to a system of algebraic equations. The solutions of last system give coefficients of assumed solutions. An error analysis and residual correction is done for these solutions. Numerical examples are given to explain the method. In the first example the theoretical matrix structures mentioned in Section 2 and Section 3 are shown in details. Besides in the first and second example, the actual absolute errors and the estimated absolute errors are compared and it is seen that they are very close. Therefore, the reliability of results can be test by the error estimation technique when the exact solution of the problem is not known. Also the absolute errors of improved solutions are compared with the absolute errors of standard solution. In

the third example, a comparison is done between spline function method, Bessel polynomial approximation and our method. It is seen that our results are almost the same with the Bessel polynomial approximation and also our results are better than the results of spline function method. And in the last example, a problem which has polynomial solutions is considered and it is seen that the method give the exact solutions. In all examples, the numerical values and comparisons show that the method gives good results.

**Acknowledgement:** The first author is supported by the *Scientific Research Project Administration* of Akdeniz University.

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