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Sertan Alkan* and Aydin Secer

Solving nonlinear boundary value problems by the Galerkin method with sinc functions

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Abstract: In this paper, the sinc-Galerkin method is used for numerically solving a class of nonlinear differential equations with boundary conditions. The importance of this study is that sinc approximation of the nonlinear term is stated as a new theorem. The method introduced here is tested on some nonlinear problems and is shown to be a very efficient and powerful tool for obtaining approximate solutions of nonlinear ordinary differential equations.

Keywords: nonlinearity; sinc-Galerkin method; boundary value problems

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1 Introduction

Since nonlinear differential equations are often used to model scientific phenomena, finding the solution of these equations is quite important. Many methods havebeen developed for solving these equations numerically since most of them either have no analytical solution or are quite difficult to solve. Some numerical methods in the literature include the following: the variational iteration method [1–3], the homotopy analysis method [4–7], the Adomian decomposition method [8–10], the homotopy perturbation method [11, 12], the Haar wavelet method [13, 14], the Chebyshev wavelets method [15] and the Legendre wavelets method [16, 17].

One of the numerical techniques that is frequently used in the literature is the sinc-Galerkin method, which is used as a base for translated sinc functions. Some studies involving the sinc-Galerkin include [20–26].

The aim of this paper is use the sinc-Galerkin method to find the approximate solution of the following class of boundary value problems for nonlinear ordinary differential equations:

$$\mu_2(x)y'' + \mu_1(x)y' + \mu_0(x)y + n_1(x)y^s y' + n_2(x)y^r = f(x) \quad (1)$$

subject to homogeneous boundary conditions

$$y(a) = 0, y(b) = 0. \quad (2)$$

The originality of this study is to present a new theorem for sinc approximation of the nonlinear term when $s \geq 2$ in Equation 1.

The rest of this paper is organized as follows: In Section 2, some preliminaries and basic definitions related to sinc functions are recalled. In Section 3, the sinc-Galerkin method is constructed for solving a class of nonlinear ordinary differential equations through a new theorem. In Section 4, numerical examples are presented. Finally, conclusions and remarks are made in Section 5.

2 Preliminaries and notation

In this section, some preliminaries and notations related to sinc basic functions are given. For more details, see [18, 19, 21, 23, 27, 28].

Definition 1. *The sinc function is defined on the whole real line $-\infty < x < \infty$ by*

$$sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0. \end{cases}$$

Definition 2. *For $h > 0$ and $k = 0, \pm1, \pm2, \ldots$ the translated sinc functions with space node are given by:*

$$S(k,h)(x) = sinc\left(\frac{x-kh}{h}\right) = \begin{cases} \frac{\sin\left(\pi\frac{x-kh}{h}\right)}{\pi\frac{x-kh}{h}} & x \neq kh \\ 1 & x = kh. \end{cases}$$

Definition 3. *If $f(x)$ is defined on the real line, then for $h > 0$ the series*

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh) sinc\left(\frac{x-kh}{h}\right)$$

is called the Whittaker cardinal expansion of f whenever this series converges.

***Corresponding Author: Sertan Alkan:** Department of Mathematics, Mustafa Kemal University, Hatay, Turkey, E-mail: drsertanalkan@gmail.com

Aydin Secer: Department of Mathematical Engineering, Yildiz Technical University, Istanbul, Turkey

In general, approximations can be constructed for infinite, semi-infinite and finite intervals. To construct an approximation on the interval (a, b), the conformal map

$$\phi(z) = \ln\left(\frac{z-a}{b-z}\right)$$

is employed. The basis functions on the interval (a, b) are derived from the composite translated sinc functions

$$S_k(z) = S(k,h)(z) \circ \phi(z) = sinc\left(\frac{\phi(z)-kh}{h}\right).$$

The inverse map of $w = \phi(z)$ is

$$z = \phi^{-1}(w) = \frac{a+be^w}{1+e^w}.$$

The sinc grid points $z_k \in (a, b)$ in D_E will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{a+be^{kh}}{1+e^{kh}}, \qquad k = 0, \pm1, \pm2, \ldots$$

Theorem 1. *Let Γ be $(0, 1)$, $F \in B(D_E)$, then for $h > 0$ sufficiently small,*

$$\int_\Gamma F(z)dz - h\sum_{j=-\infty}^{\infty}\frac{F(z_j)}{\phi'(z_j)} = \frac{i}{2}\int_{\partial D}\frac{F(z)k(\phi,h)(z)}{\sin(\pi\phi(z)/h)}dz \equiv I_F, \tag{3}$$

where

$$|k(\phi,h)|_{z\in\partial D} = \left|e^{\left[\frac{i\pi\phi(z)}{h}sgn(Im\phi(z))\right]}\right|_{z\in\partial D} = e^{\frac{-\pi d}{h}}.$$

For the sinc-Galerkin method, the infinite quadrature rule must be truncated to a finite sum. The following theorem indicates the conditions under which an exponential convergence results.

Theorem 2. *If there exist positive constants α, β and C such that*

$$\left|\frac{F(x)}{\phi'(x)}\right| \le C\begin{cases} e^{-\alpha|\phi(x)|} & x \in \psi((-\infty,\infty)) \\ e^{-\beta|\phi(x)|} & x \in \psi((0,\infty)). \end{cases} \tag{4}$$

then the error bound for the quadrature rule (3) *is*

$$\left|\int_\Gamma F(x)dx - h\sum_{j=-M}^{N}\frac{F(x_j)}{\phi'(x_j)}\right| \le C\left(\frac{e^{-\alpha Mh}}{\alpha} + \frac{e^{-\beta Nh}}{\beta}\right) + |I_F|. \tag{5}$$

The infinite sum in (3) is truncated with the use of (4) to arrive at the inequality (5). Making the selections

$$h = \sqrt{\frac{\pi d}{\alpha M}}$$

and

$$N \equiv \left[\left\lfloor \frac{\alpha M}{\beta} + 1\right\rfloor\right]$$

where $[\lfloor.\rfloor]$ is the integer part of the statement and M is the integer value which specifies the grid size, then

$$\int_\Gamma F(x)dx = h\sum_{j=-M}^{N}\frac{F(x_j)}{\phi'(x_j)} + O\left(e^{-(\pi\alpha dM)^{1/2}}\right). \tag{6}$$

These theorems are used for the integrals in the inner products that arise from the method presented here.

3 The sinc-Galerkin method

An approximate solution of $y(x)$ in (1) is represented by the formula

$$y_n(x) = \sum_{k=-M}^{N} c_k S_k(x), \qquad n = M+N+1 \tag{7}$$

where S_k is function $S(k,h)\circ\phi(x)$ for some fixed step size h. The unknown coefficients c_k in (7) are determined by orthogonalizing the residual with respect to the basis functions, i.e.

$$\begin{aligned}\langle\mu_2(x)y'', S_k\rangle &+ \langle\mu_1(x)y', S_k\rangle + \langle\mu_0(x)y, S_k\rangle \\ &+ \langle n_1(x)y^s y', S_k\rangle + \langle n_2(x)y^r(x), S_k\rangle \\ &= \langle f(x), S_k\rangle, \qquad k = -M,\ldots,N. \end{aligned} \tag{8}$$

The inner product used for the sinc-Galerkin method is defined by

$$\langle f, g\rangle = \int_a^b f(x)g(x)w(x)dx,$$

where $w(x)$ a weight function which is taken for second-order boundary value problems in the following form

$$w(x) = \frac{1}{\phi'(x)}.$$

We need the following theorems for the approximation of inner products in (8).

Theorem 3. *The following relations hold:*

$$\langle\mu_2(x)y'', S_k\rangle \approx h\sum_{j=-M}^{N}\sum_{i=0}^{2}\frac{y(x_j)}{\phi'(x_j)h^i}\delta_{kj}^{(i)}g_{2,i}(x_j) \tag{9}$$

$$\langle\mu_1(x)y', S_k\rangle \approx -h\sum_{j=-M}^{N}\sum_{i=0}^{1}\frac{y(x_j)}{\phi'(x_j)h^i}\delta_{kj}^{(i)}g_{1,i}(x_j) \tag{10}$$

and for $G(x) = n_2(x)y^r(x)$, $G(x) = \mu_0(x)y(x)$ and $G(x) = f(x)$

$$\langle G, S_k\rangle \approx h\frac{G(x_k)w(x_k)}{\phi'(x_k)}. \tag{11}$$

The proof of this theorem and values of $g_{k,i}(x)$ can be found in [21].

Theorem 4. *The following relation holds:*

$$\langle n_1(x)y^s y', S_k\rangle \approx -\frac{h}{s+1}\sum_{j=-M}^{N}\frac{y^{s+1}(x_j)}{\phi'(x_j)} \times\left[\frac{1}{h}\delta_{kj}^{(1)}(\phi' n_1 w)(x_j)+\delta_{kj}^{(0)}(n_1 w)'(x_j)\right]. \quad (12)$$

Proof. For $n_1(x)y^s y'$, the inner product with sinc basis elements is given by

$$\langle n_1 y^s y', S_k\rangle = \int_a^b y^s y'(S_k n_1 w)dx.$$

Integrating by parts to remove the first derivative from the dependent variable y leads to the equality

$$\langle n_1 y^s y', S_k\rangle = B_1 - \frac{1}{s+1}\int_a^b y^{s+1}(S_k n_1 w)' dx, \quad (13)$$

where the boundary term is

$$B_1 = \left[\frac{1}{s+1}(y^{s+1}S_k n_1 w)\right]_{x=a}^{b} = 0$$

and expanding the derivatives under the integral in (13) yields

$$\langle n_1 y^s y', S_k\rangle = -\frac{1}{s+1}\int_a^b y^{s+1}\left[S_k^{(1)}\phi'(n_1 w)+S_k^{(0)}(n_1 w)'\right]dx. \quad (14)$$

Applying the sinc quadrature rule given by (6) to the right-hand side of (3.8) and deleting the error term yields (12). □

Replacing each term of (8) with the approximations defined in (9)-(12), respectively, and replacing $y(x_j)$ by c_j, and dividing by h, we obtain the following theorem:

Theorem 5. *If the assumed approximate solution of the boundary-value problem* (1),(2) *is* (7)*, then the discrete sinc-Galerkin system for the determination of the unknown coefficients* $\{c_j\}_{j=-M}^{N}$ *is given, for* $k = -M, \ldots, N$, *by*

$$\sum_{j=-M}^{N}\left\{\sum_{i=0}^{2}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{2,i}(x_j)}{\phi'(x_j)}c_j-\sum_{i=0}^{1}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{1,i}(x_j)}{\phi'(x_j)}c_j -\frac{1}{s+1}\left[\frac{1}{h}\delta_{kj}^{(1)}(n_1 w)(x_j)c_j^{s+1}+\frac{(n_1 w)'(x_k)}{\phi'(x_k)}c_k^{s+1}\right]\right\} +\frac{\mu_0(x_k)w(x_k)}{\phi'(x_k)}c_k+\frac{n_2(x_k)w(x_k)}{\phi'(x_k)}c_k^r=\frac{f(x_k)w(x_k)}{\phi'(x_k)}. \quad (15)$$

Now we define some notation to represent the system (15) in matrix-vector form. Let $D(y)$ denote a diagonal matrix whose diagonal elements are $y(x_{-M}), y(x_{-M+1}), , y(x_N)$ and non-diagonal elements are zero; also for $0 \le i \le 2$, let $I^{(i)}$ denote the matrices

$$I^{(i)} = [\delta_{jk}^{(i)}], \qquad j, k = -M, \ldots, N,$$

where D and I are square matrices of dimension $n \times n$. In order to calculate the unknown coefficients c_k in the nonlinear system (15), we rewrite this system using the above notations in matrix-vector form as

$$AC + BC^s + EC^r = F, \quad (16)$$

where

$$A = \sum_{j=0}^{2}\frac{1}{h^j}I^{(j)}D\left(\frac{g_{2,j}}{\phi'}\right)-\sum_{j=0}^{1}\frac{1}{h^j}I^{(j)}D\left(\frac{g_{1,j}}{\phi'}\right)+I^{(0)}D\left(\frac{g_{0,0}}{\phi'}\right)$$

$$B = -\frac{1}{s+1}\left[\frac{1}{h}I^{(1)}D\left(n_1 w\right)+I^{(0)}D\left(\frac{(n_1 w)'}{\phi'}\right)\right]$$

$$E = D\left(\frac{n_2 w}{\phi'}\right)$$

$$F = D\left(\frac{wf}{\phi'}\right)\mathbf{1}$$

$$C^j = \left(c_{-M}^j, c_{-M+1}^j, \ldots, c_{N-1}^j, c_N^j\right)^T, \quad j = 1, s, r.$$

Now we have a nonlinear system of n equations in n unknown coefficients given by (16). Solving by *Newton's method*, we can obtain the unknown coefficients c_k that are necessary for approximating the solution in (7).

4 Computational examples

In this section, some numerical examples are presented to show the accuracy of the method introduced here using `MATHEMATICA 10`. In all examples, $d = \pi/2$, $\alpha = \beta = 1/2$, and $N = M$. E_N shows the maximum absolute error between the exact solution and numerical solution at collocation points by this method. Also, R_N in the examples indicates the experimental rate of convergence that calculated as [28]

$$R_N = \frac{\log[E_{N/2}/E_N]}{\log 2}.$$

Example 1 Consider the following nonlinear boundary value problem

$$y''(x) + xy^2(x)y'(x) + x^2y^3(x) = f(x)$$

Table 1: Maximum error and convergence rate for *Example 4*.

N	Max. Error E_N	Rate of convergence R_N
2	4.49×10^{-2}	
4	9.94×10^{-3}	2.17
8	8.23×10^{-4}	3.59
16	3.22×10^{-5}	4.67
32	4.64×10^{-7}	6.11
64	8.00×10^{-10}	9.18

Table 2: Numerical results for *Example 4*.

x	Exact sol.	Error for $N = 2$	Error for $N = 64$
0	0	0	0
0.1	0.0729	1.48×10^{-2}	7.80×10^{-10}
0.2	0.1024	3.85×10^{-2}	8.00×10^{-10}
0.3	0.1029	4.49×10^{-2}	1.94×10^{-10}
0.4	0.0864	3.78×10^{-2}	1.24×10^{-10}
0.5	0.0625	2.45×10^{-2}	2.07×10^{-10}
0.6	0.0384	1.12×10^{-2}	2.26×10^{-10}
0.7	0.0189	2.15×10^{-3}	2.42×10^{-10}
0.8	0.0064	1.32×10^{-3}	3.94×10^{-10}
0.9	0.0009	2.09×10^{-3}	5.87×10^{-10}
1	0	0	0

subject to the homogeneous boundary conditions

$$y(0) = 0, \qquad y(1) = 0,$$

where

$$f(x) = (1-x)\big(6(-1+x)+6x+(-1+x)^8x^5+(-1+x)^7x^3(-1+4x)\big).$$

The exact solution of this problem is given by $y(x) = x(1-x)^3$. The numerical solutions which are obtained by using the presented method for this problem are given in Table 1 and Table 2. Additionally, graphs of the exact and approximate solutions for different values of collocation points N are provided in Figure 1.

Example 2 Consider the nonlinear singular boundary value problem

$$y''(x) - \frac{1}{x}y'(x) + \frac{1}{x(x-1)}y^3(x)y'(x) - \frac{1}{x-1}y^2(x) = f(x)$$

subject to the homogeneous boundary conditions

$$y(0) = 0, \qquad y(1) = 0,$$

Table 3: Maximum error and convergence rate for *Example 4*.

N	Max. Error E_N	Rate of convergence R_N
2	1.35×10^{-1}	
4	4.65×10^{-2}	1.53
8	7.92×10^{-3}	2.55
16	7.01×10^{-4}	3.49
32	3.34×10^{-5}	4.39
64	3.27×10^{-7}	6.67

Table 4: Numerical results for *Example 4*.

x	Exact sol.	Error for $N = 2$	Error for $N = 64$
0	0	0	0
0.1	0.03090	5.15×10^{-2}	2.40×10^{-7}
0.2	0.11755	1.23×10^{-1}	2.81×10^{-7}
0.3	0.24270	1.35×10^{-1}	1.38×10^{-7}
0.4	0.38042	1.02×10^{-1}	1.03×10^{-7}
0.5	0.50000	5.71×10^{-2}	1.58×10^{-7}
0.6	0.57063	2.64×10^{-2}	2.54×10^{-8}
0.7	0.56631	2.70×10^{-2}	1.24×10^{-7}
0.8	0.47022	5.11×10^{-2}	3.27×10^{-7}
0.9	0.27811	3.36×10^{-2}	2.78×10^{-7}
1	0	0	0

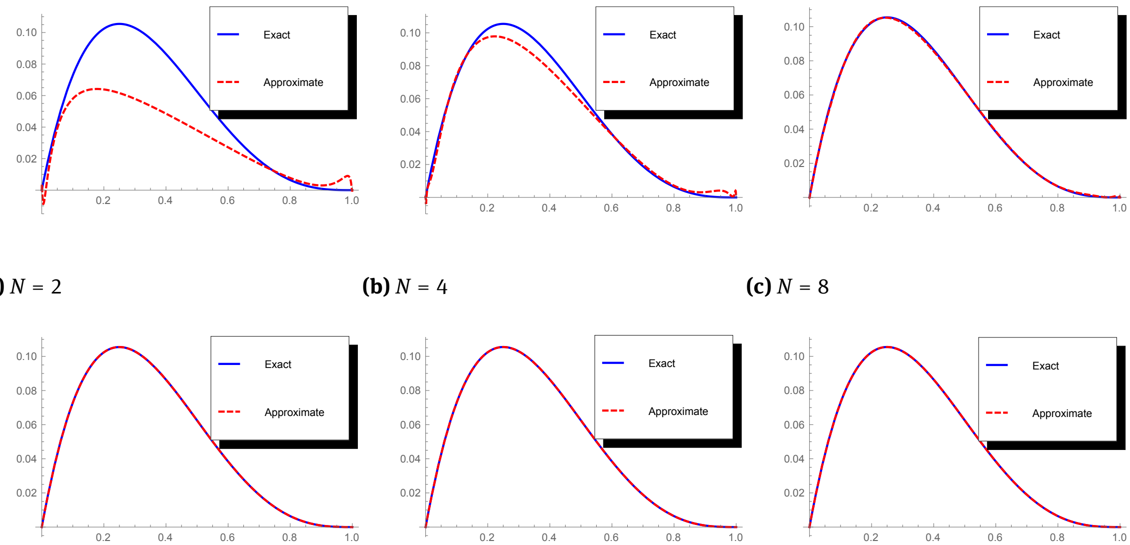
where

$$\begin{aligned} f(x) &= \pi\cos\pi x - \frac{\sin\pi x}{x} - \pi^2 x\sin\pi x - \frac{x^2(\sin\pi x)^2}{x-1} \\ &+ \frac{\pi x^3\cos\pi x(\sin\pi x)^3}{x-1} + \frac{x^2(\sin\pi x)^4}{x-1}. \end{aligned}$$

The exact solution of this problem is $y(x) = x\sin\pi x$. The numerical solutions which are obtained by using the presented method for this problem are given in Table 3 and Table 4. Additionally, graphs of the exact and approximate solutions for different values of N are provided in Figure 2.

5 Conclusion

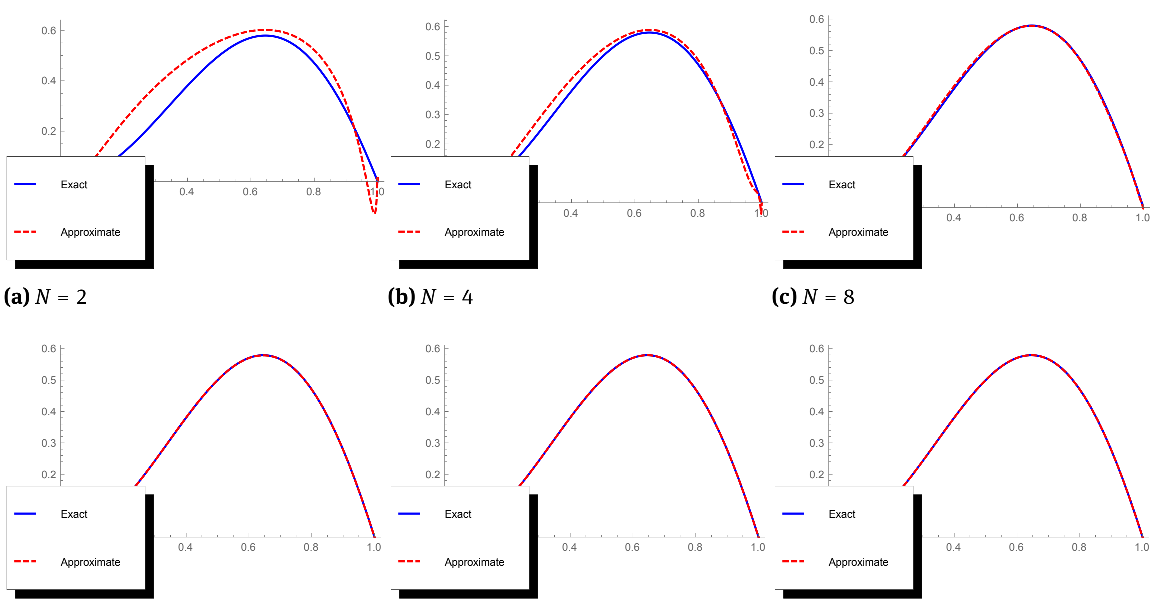
In this paper, the sinc-Galerkin method is introduced to obtain the approximate solutions of a class of nonlinear differential equations. In order to illustrate the accuracy of the presented method, the numerical results are compared with results from exact solutions. From these comparisons, it is concluded that the sinc-Galerkin method



(a) $N = 2$ **(b)** $N = 4$ **(c)** $N = 8$

(d) $N = 16$ **(e)** $N = 32$ **(f)** $N = 64$

Figure 1: Graphs of exact and approximate solutions for Example 4 when $N = 2, 4, 8, 16, 32, 64$.



(a) $N = 2$ **(b)** $N = 4$ **(c)** $N = 8$

(d) $N = 16$ **(e)** $N = 32$ **(f)** $N = 64$

Figure 2: Graphs of exact and approximate solutions for Example 4 when $N = 2, 4, 8, 16, 32, 64$.

provides a good approximate solution and shows promise for solving other types of nonlinear differential equations.

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