Research Article Open Access

Hassan Rachid*

Peristaltic transport of a fractional Burgers' fluid with variable viscosity through an inclined tube

DOI 10.1515/phys-2015-0046

Received May 17, 2015; accepted November 19, 2015

Abstract: In the present study, we investigate the unsteady peristaltic transport of a viscoelastic fluid with fractional Burgers' model in an inclined tube. We suppose that the viscosity is variable in the radial direction. This analysis has been carried out under low Reynolds number and long-wavelength approximations. An analytical solution to the problem is obtained using a fractional calculus approach. Figures are plotted to show the effects of angle of inclination, Reynolds number, Froude number, material constants, fractional parameters, parameter of viscosity and amplitude ratio on the pressure gradient, pressure rise, friction force, axial velocity and on the mechanical efficiency.

Keywords: unsteady peristaltic transport; fractional Burgers' fluid; variable viscosity; inclined tube; fractional calculus; mechanical efficiency

PACS: 47.85.–g, 47.15.G–, 87.19.rh

1 Introduction

Peristaltic transport is a process that occurs frequently both in nature and industrial applications. In physiological processes, the transport of fluids from one place to any other place due to contraction and relaxation of the channel walls is termed peristalsis. These contractions and relaxations travel in the form of waves along the channel walls and propel the fluid. Peristalsis is involved in swallowing food through the oesophagus or transport of chyme in small intestines. The first investigation of peristalsis was by Latham [1] and Jaffrin and Shapiro [2]. Later, several studies (such as such [3–8]) analyzed the peristaltic transport phenomenon for a Newtonian fluid. Viscoelastic fluids are non- Newtonian and possess both viscous and elastic properties. These fluids play an important role in fluid mechanics. For example, Maxwell, Oldroyd, Jeffrey, Johnson-Segalman, Thein-Tanner, Micropolar, Third and Fourth grade fluids have all been the subject of work, such as [9–16] respectively. More recently in rheology, the study of viscoelastic fluids flows with fractional derivatives play an important role. The starting point of the fractional derivative of the viscoelastic model is usually a classical differential equation which is modified by replacing the time derivative of an integer order by the so-called Riemann-Liouville fractional calculus operators. This generalization allows one to precisely define non-integer order integrals or derivatives. In recent years, peristaltic transport has been studied using the fractional Maxwell model [17, 18], fractional Oldroyd-B model [19] or of fractional second-grade model [20]. These fluid models can be treated as the special cases of another kind of viscoelastic fluid, known as the fractional Burgers' fluid. The Burgers' model has been utilized to describe the motion of the Earth's maentl. This model is also the preferred model to explain the behavior of asphalt and asphalt concrete [21]. In addition, the Burgers model is sometimes used to model other geological structures, such as Olivine rocks [22]. Recently, Hayat *et al.* [23] found an exact solution for rotating flows of a generalized Burgers' fluid on an infinite insulating plate when the fluid is permeated by a transverse magnetic field. Khan *et al.* [24] have considered the accelerated flows for a fractional Burgers' model for two cases: flow induced by constantly accelerating plate, and flow induced by variable accelerated plate, and Khan *et al.* [25] analyzed the influences of Hall current on the flow of a Burgers' fluid in a pipe. Meanwhile, Siddiqui *et al.* [26] investigated the magnetohydrodynamics flow of an electrically conducting incompressible Burgers' fluid in an orthogonal rheometer. The fractional Burgers's fluid generalizes several types of fluids as mentioned bellow. It helps to study the applications of fluid engineering and biomedical engineering and it can capture the complex rheological characteristics of many real fluids better than other models. In addition, the Burgers model is applicable in transportation of industrial fluids by peristaltic pumping, as well as for peristaltic flow of physiological fluids in deformable domains such as the motion of blood, food bolus and chyme in the small intestine. The availability of the solutions for viscoelastic fluids is significant because such solutions can not only ex-

***Corresponding Author: Hassan Rachid:** Department of Physics, Faculty of Sciences Ain Chock, University Hassan II, Maarif, Casablanca 5366, Morocco, E-mail: rhassan2@yahoo.fr

plain the physics of some fundamental flows, but can also be used as a benchmark for the complicated numerical codes that have been developed for much more complex flows. For these reasons, and motivated by the application of fractional calculus in viscoelastic fluid engineering, this paper will focus on the as-yet unstudied peristaltic transport of a fractional Burgers' fluid. The aim of this paper is to analytically investigate the unsteady peristaltic flow of a viscoelastic fluid using the fractional Burgers's model, and with variable viscosity, in an inclined tube. The problem is simplified under long-wavelength and low Reynolds number approximations. The effects of different physical parameters are shown graphically. Parameters studied include, angle of inclination A, Reynolds R_e and Froude F_r numbers, material constants (λ_1, λ_2, λ_3), fractional parameters (α, β), viscosity δ and the amplitude ratio ϕ, and their effects are studied on the pressure gradient, pressure rise, fictional force, axial velocity and on the mechanical efficiency of pumping.

2 Formulation and analysis

We consider the unsteady peristaltic transport of a viscoelastic fluid through an inclined axisymmetric tube thickness with a sinusoidal wave traveling down its wall. In a cylindrical coordinate system $(\bar{R}, \bar{Z})$, the dimensional equation for the tube radius for an infinite wave train is:

$$\bar{H}(\bar{Z}, \bar{t}) = a + b \sin\left(\frac{2\pi}{\lambda}(\bar{Z} - c\bar{t})\right), \tag{1}$$

where a is the average radius of the tube, b is the amplitude of the wave, λ is the wavelength and c is the wave speed (see Figure 1).

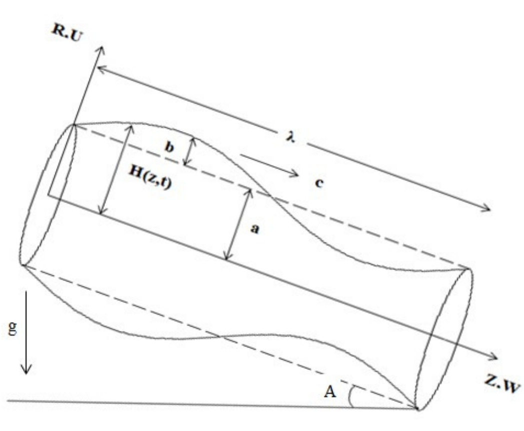

Figure 1: Geometry of the problem.

The viscoelastic fluid is modeled as a fractional Burgers' model given by:

$$\left(1 + \bar{\lambda_1}^{\alpha}\tilde{D}_{\bar{t}}^{\alpha} + \bar{\lambda_2}^{\alpha}\tilde{D}_{\bar{t}}^{2\alpha}\right)\bar{\mathbf{S}} = \bar{\mu}(\bar{r})\left(1 + \bar{\lambda_3}^{\beta}\tilde{D}_{\bar{t}}^{\beta}\right)\dot{\gamma}, \tag{2}$$

where $\bar{t}$, $\bar{\mathbf{S}}$, $\dot{\gamma}$ and $\bar{\mu}(\bar{r})$ are the time, shear stress, rate of shear strain and the viscosity function, respectively. $\bar{\lambda_1}$ and $\bar{\lambda_3}$ ($< \bar{\lambda_1}$) are the relaxation and the retardation times, respectively, $\bar{\lambda_2}$ is the new material parameter of the Burgers' fluid having the dimension of t^2, α and β are the fractional time derivative parameters such that $0 \le \alpha \le \beta \le 1$. $\tilde{D}_t^{\alpha}$ is the upper convected fractional derivative defined by:

$$\tilde{D}_{\bar{t}}^{\alpha}(\bar{\mathbf{S}}) = D_{\bar{t}}^{\alpha}(\bar{\mathbf{S}}) + (\bar{\mathbf{V}}.\nabla)(\bar{\mathbf{S}}) - \bar{\mathbf{L}}(\bar{\mathbf{S}}) - (\bar{\mathbf{S}})\bar{\mathbf{L}}^T \tag{3}$$

in which:

$$\dot{\gamma} = (\nabla\bar{\mathbf{V}}) + (\nabla\bar{\mathbf{V}})^T, \tag{4}$$

where $\bar{\mathbf{L}}$ is the velocity gradient and $\bar{\mathbf{V}}$ is the velocity vector, and $D_{\bar{t}}^{\alpha} = \partial_{\bar{t}}^{\alpha}$ is the fractional differentiation operator of order α with respect to t and may be defined as fractional complex transform [27]

$$D_t^{\alpha}C(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t_0}^{t}(s-t)^{n-\alpha-1}\left[C_0(s)-C(s)\right]\,ds, \tag{5}$$

or it can convert a fractional differential equation to a partial differential equation as given in [28].

Here $\Gamma(.)$ denotes the Gamma function and

$$\tilde{D}_t^{2\alpha}(\mathbf{S}) = \tilde{D}_t^{\alpha}(\tilde{D}_t^{\alpha}(\mathbf{S})). \tag{6}$$

We choose a cylindrical coordinate system $(\bar{r}, \bar{z})$ where the $\bar{z}$-axis is the longitudinal direction and the $\bar{r}$-axis is transverse to it.

The equations of motion for the flow of an incompressible fluid are given by:

$$\rho\left[\frac{\partial\bar{u}}{\partial\bar{t}} + \bar{u}\frac{\partial\bar{u}}{\partial\bar{r}} + \bar{w}\frac{\partial\bar{u}}{\partial\bar{z}}\right] = -\frac{\partial\bar{p}}{\partial\bar{r}} + \frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}(\bar{r}\bar{S}_{\bar{r}\bar{r}}) + \frac{\partial\bar{S}_{\bar{r}\bar{z}}}{\partial\bar{z}} - \rho\, g\, \cos(A) \tag{7}$$

$$\rho\left[\frac{\partial\bar{w}}{\partial\bar{t}} + \bar{u}\frac{\partial\bar{w}}{\partial\bar{r}} + \bar{w}\frac{\partial\bar{w}}{\partial\bar{z}}\right] = -\frac{\partial\bar{p}}{\partial\bar{z}} + \frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}(\bar{r}\bar{S}_{\bar{r}\bar{z}}) + \frac{\partial\bar{S}_{\bar{z}\bar{z}}}{\partial\bar{z}} + \rho\, g\, \sin(A) \tag{8}$$

$$\frac{\partial\bar{u}}{\partial\bar{r}} + \frac{\bar{u}}{\bar{r}} + \frac{\partial\bar{w}}{\partial\bar{z}} = 0, \tag{9}$$

where ρ is the fluid density, $\bar{u}$ and $\bar{w}$ are the velocity components in the wave frame and $\bar{p}$ is the pressure.

We assume that the extra stress $\bar{\mathbf{S}}$ depends on r and t only. After using the initial condition $\bar{S}(t = 0)$, Equations (2-6) yield $\bar{S}_{\bar{r}\bar{r}} = \bar{S}_{\bar{z}\bar{z}} = \bar{S}_{\bar{r}\bar{\theta}} = \bar{S}_{\bar{\theta}\bar{z}} = 0$ and

$$\left(1 + \bar{\lambda}_1{}^{\alpha}\frac{\partial^{\alpha}}{\partial \bar{t}^{\alpha}} + \bar{\lambda}_2{}^{\alpha}\frac{\partial^{2\alpha}}{\partial \bar{t}^{2\alpha}}\right)\bar{S}_{\bar{r}\bar{z}} = \bar{\mu}(\bar{r}) \left(1 + \bar{\lambda}_3{}^{\beta}\frac{\partial^{\beta}}{\partial \bar{t}^{\beta}}\right)\frac{\partial \bar{w}}{\partial \bar{r}}, \tag{10}$$

where $\bar{S}_{\bar{r}\bar{z}}$ is the tangential stress.

For further analysis, we introduce the following dimensionless parameters:

$$z = \frac{\bar{z}}{\lambda}; Z = \frac{\bar{Z}}{\lambda}; r = \frac{\bar{r}}{a}; R = \frac{\bar{R}}{a}; H = \frac{\bar{H}}{a}; t = \frac{c\bar{t}}{\lambda};$$

$$u = \frac{\lambda\bar{u}}{ac}; U = \frac{\lambda\bar{U}}{ac}; w = \frac{\bar{w}}{c}; W = \frac{\bar{W}}{c}; \lambda_1 = \frac{c\bar{\lambda}_1}{\lambda};$$

$$\lambda_2 = \frac{c\bar{\lambda}_2}{\lambda}; \lambda_3 = \frac{c\bar{\lambda}_3}{\lambda}; p = \frac{a^2\bar{p}}{\mu\lambda c}; \mu(r) = \frac{\bar{\mu}(\bar{r})}{\mu_0};$$

$$Q = \frac{\bar{Q}}{\pi c a^2}; \varepsilon = \frac{a}{\lambda}; Re = \frac{\rho\, c\, a}{\mu_0}; Fr = \frac{c^2}{ga}; \phi = \frac{b}{a}, \tag{11}$$

where $\bar{U}$ and $\bar{W}$ are the velocity components in the fixed frame, μ_0 is the viscosity on the $\bar{z}$-axis ($\bar{r} = 0$), ε is the dimensionless wave number, Re is the Reynolds number, Fr is the Froude number and ϕ is the amplitude ratio with $0 < \phi < 1$.

Using the above non-dimensional quantities and under the assumptions of the long-wavelength approximation (i.e., $\varepsilon \ll 1$ or $\lambda \gg a$) and the low Reynolds number (i.e., $Re \to 0$), the continuity equation is satisfied and the equations of motion (7-9) can be reduced to:

$$\left(1 + \lambda_1^{\alpha}D_t^{\alpha} + \lambda_2^{\alpha}D_t^{2\alpha}\right)\left(\frac{\partial p}{\partial z} - \frac{Re}{Fr}\sin(A)\right) = \left(1 + \lambda_3^{\beta}D_t^{\beta}\right)\left(\frac{1}{r}\left(\mu(r) r\frac{\partial w}{\partial r}\right)\right). \tag{12}$$

The boundary conditions are:

$$w = -1 \qquad \text{at} \qquad r = H \tag{13}$$

$$\frac{\partial w}{\partial r} = 0 \qquad \text{at} \qquad r = 0, \tag{14}$$

where $H = 1 + \phi\sin(2\pi z)$ is the dimensionless equation of the tube radius in the wave frame.

Integrating (12), and using the boundary conditions (13-14), we obtain:

$$\left(1 + \lambda_1^{\alpha}D_t^{\alpha} + \lambda_2^{\alpha}D_t^{2\alpha}\right)\left(\frac{\partial p}{\partial z} - \frac{Re}{Fr}\, sin(A)\right) \left(\int_H^r \frac{r}{\mu(r)}\, dr\right) = 2\left(1 + \lambda_3^{\beta}D_t^{\beta}\right)(w + 1). \tag{15}$$

The volume rate of flow in the fixed coordinate system (R, Z) is given as:

$$Q(Z, t) = 2\int_0^H W\, R\; dR. \tag{16}$$

Using the transformations between the laboratory and the wave frames, in the dimensionless form, given by:

$$z = Z - t\ ;\ r = R\ ;\ u = U\ ;\ w = W - 1 \tag{17}$$

the flow rate (16) becomes:

$$Q(Z, t) = q + H^2, \tag{18}$$

where q is the volume flow rate in the moving coordinate system, and is given by:

$$q = 2\int_0^H w\, r\; dr. \tag{19}$$

Equation (15) becomes:

$$\left(1 + \lambda_1^{\alpha}D_t^{\alpha} + \lambda_2^{\alpha}D_t^{2\alpha}\right)\left(\frac{\partial p}{\partial z} - \frac{Re}{Fr}\, sin(A)\right) = \left(1 + \lambda_3^{\beta}D_t^{\beta}\right)\frac{Q(Z, t)}{\left[\int_0^H \left(\int_H^r \frac{r}{\mu(r)}\, dr\right) r\; dr\right]}. \tag{20}$$

The time-averaged flow rate is defined by:

$$Q = \int_0^1 Q(Z, t)\; dt = q + 1 + \frac{\phi^2}{2}. \tag{21}$$

Using (21), (18) becomes:

$$Q(Z, t) = Q + H^2 - 1 - \frac{\phi^2}{2}. \tag{22}$$

On substituting (22) into (20), we then obtain:

$$\left(1 + \lambda_1^{\alpha}D_t^{\alpha} + \lambda_2^{\alpha}D_t^{2\alpha}\right)\left(\frac{\partial p}{\partial z} - \frac{Re}{Fr}\, sin(A)\right) = \left(1 + \lambda_3^{\beta}D_t^{\beta}\right)\left(\frac{Q + H^2 - 1 - \frac{\phi^2}{2}}{\left[\int_0^H \left(\int_H^r \frac{r}{\mu(r)}\, dr\right) r\; .dr\right]}\right) \tag{23}$$

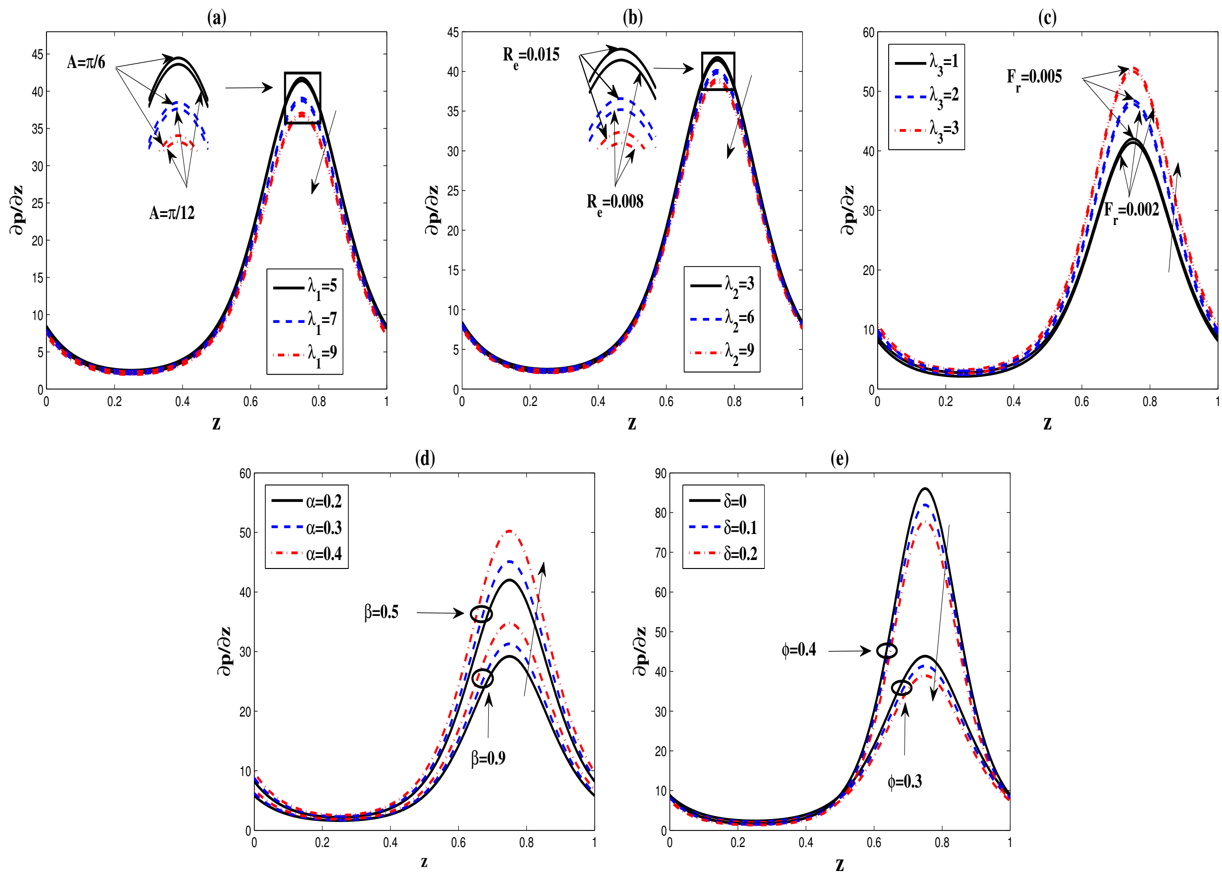

Figure 2: Pressure gradient $\frac{\partial p}{\partial z}$ versus axial distance z when $Q = -2$ and $t = 1$ corresponding to (a) different values of A and λ_1 with $Re = 0.008$, $Fr = 0.005$, $\lambda_2 = 1$, $\lambda_3 = 3$, $\alpha = 0.4$, $\beta = 0.7$, $\phi = 0.3$ and $\delta = 0.1$ (b) different values of Re and λ_2 with $A = \frac{\pi}{12}$, $Fr = 0.005$, $\lambda_1 = 5$, $\lambda_3 = 3$, $\alpha = 0.4$, $\beta = 0.7$, $\phi = 0.3$ and $\delta = 0.1$$(c)$ different values of Fr and λ_3 with $A = \frac{\pi}{12}$, $Re = 0.008$, $\lambda_1 = 5$, $\lambda_2 = 1$, $\alpha = 0.4$, $\beta = 0.7$, $\phi = 0.3$ and $\delta = 0.1$ (d) different values of α and β with $A = \frac{\pi}{12}$, $Re = 0.008$, $Fr = 0.005$, $\lambda_1 = 5$, $\lambda_2 = 1$, $\lambda_3 = 3$, $\phi = 0.3$ and $\delta = 0.1$ (e) different values of ϕ and δ with $A = \frac{\pi}{12}$, $Re = 0.008$, $Fr = 0.005$, $\lambda_1 = 5$, $\lambda_2 = 1$, $\lambda_3 = 3$ and $\alpha = 0.4$, $\beta = 0.7$.

3 Method of solution

In order to obtain a solution for Equation (23), using the definition of the fractional differential operator (5), we obtain the pressure gradient as:

$$\frac{\partial p(z,t)}{\partial z} = \frac{Re}{Fr} sin(A) + \left(\frac{1 + \lambda_3^{\beta} \frac{t^{-\beta}}{\Gamma(1-\beta)}}{1 + \lambda_1^{\alpha} \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \lambda_2^{\alpha} \frac{t^{-2\alpha}}{\Gamma(1-2\alpha)}} \right) \left(\frac{Q + H^2 - 1 - \frac{\phi^2}{2}}{\left[\int_0^H \left(\int_H^r \frac{r}{\mu(r)}\, dr \right) r\, dr \right]} \right). \tag{24}$$

From this solution it is clear that for $A = 0$, $\delta = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\alpha = \beta = 1$ we obtain the classical solution of peristaltic transport of a Newtonian fluid [29]. For $\lambda_2 = 0$, $\lambda_2 = \lambda_3 = 0$ and $\lambda_1 = \lambda_2 = 0$ (with $\alpha = \beta = 1$) we find the solutions of the cases of fractional Oldroyd-B, fractional Maxwell and fractional second-grade models, respectively. In addition, the first term $\frac{Re}{Fr}\sin(A)$ interprets the effect of gravitation because of the inclination of the tube.

4 The pumping characteristics

The pressure rise Δp and the frictional force F_λ at the walls of the tube, in the non-dimensional form, are given by:

$$\Delta p = \int_0^1 \frac{\partial p(z,t)}{\partial z}\, dz \tag{25}$$

$$F_\lambda = \int_0^1 H^2 \left(-\frac{\partial p(z,t)}{\partial z} \right) dz. \tag{26}$$

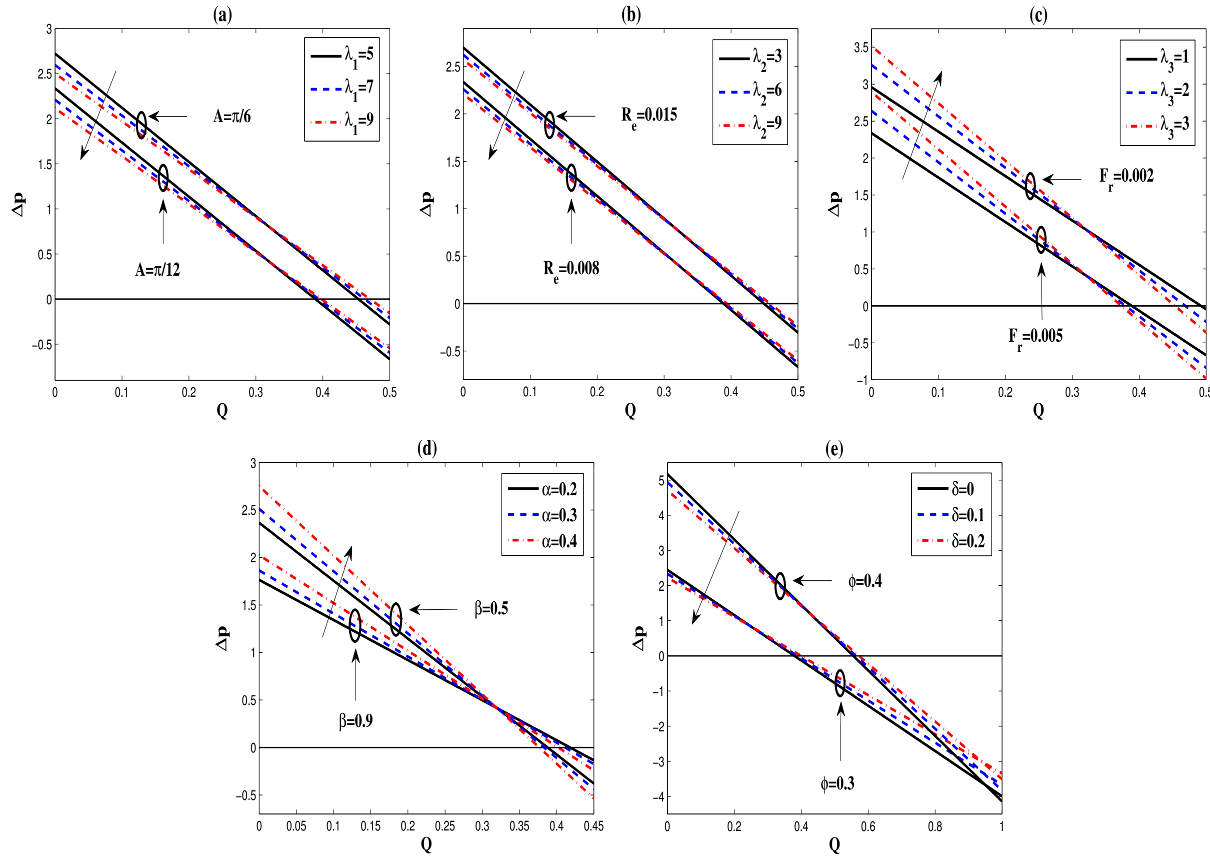

Figure 3: Pressure rise Δp versus time-averaged flow rate Q when $t = 1$ corresponding to (a) different values of A and λ_1 with $Re = 0.008$, $Fr = 0.005$, $\lambda_2 = 1$, $\lambda_3 = 3$, $\alpha = 0.4$, $\beta = 0.7$, $\phi = 0.3$ and $\delta = 0.1$ (b) different values of Re and λ_1 with $A = \frac{\pi}{12}$, $Fr = 0.005$, $\lambda_1 = 5$, $\lambda_3 = 3$, $\alpha = 0.4$, $\beta = 0.7$, $\phi = 0.3$ and $\delta = 0.1$(c) different values of Fr and λ_3 with $A = \frac{\pi}{12}$, $Re = 0.008$, $\lambda_1 = 5$, $\lambda_2 = 1$, $\alpha = 0.4$, $\beta = 0.7$, $\phi = 0.3$ and $\delta = 0.1$ (d) different values of α and β with $A = \frac{\pi}{12}$, $Re = 0.008$, $Fr = 0.005$, $\lambda_1 = 5$, $\lambda_2 = 1$, $\lambda_3 = 3$, $\phi = 0.3$ and $\delta = 0.1$ (e) different values of ϕ and δ with $A = \frac{\pi}{12}$, $Re = 0.008$, $Fr = 0.005$, $\lambda_1 = 5$, $\lambda_2 = 1$, $\lambda_3 = 3$ and $\alpha = 0.4$, $\beta = 0.7$.

5 Mechanical efficiency of pumping

The mechanical efficiency is defined as the ratio between the average rate per wavelength at which work is done by the moving fluid against a pressure head and the average rate at which the walls do work on the fluid [9, 29].
We find the expression of the mechanical efficiency E as follows :

$$E = \frac{Q\Delta p}{H(z=0)^2\,\Delta p + F_\lambda}. \qquad (27)$$

In order to analyze the effect of viscosity variation $\mu(r)$ on peristaltic transport, we assume that $\mu(r)$ in a dimensionless form is given by [30]:

$$\begin{aligned} \mu(r) &= e^{-\delta r} \\ &\ \ or \qquad\qquad\qquad\qquad (28) \\ \mu(r) &= 1-\delta r \quad for \quad \delta << 1, \end{aligned}$$

where δ is the viscosity parameter.

6 Results and discussions

The analytical expressions for the pressure gradient $\frac{\partial p}{\partial z}$, the pressure rise Δp, the frictional force F_λ and the mechanical efficiency E are derived in the previous sections. In order to compute these physical quantities with respect to parameters of interest for the problem, we observe that the integrals in Equations (25-26) are not integrable in the closed form. They are evaluated numerically using mathematics software.

6.1 Behavior of pressure gradient

Plots of the pressure gradient $\frac{\partial p}{\partial z}$ versus axial difference z for various parameters are shown in Figures 2,a-e. The

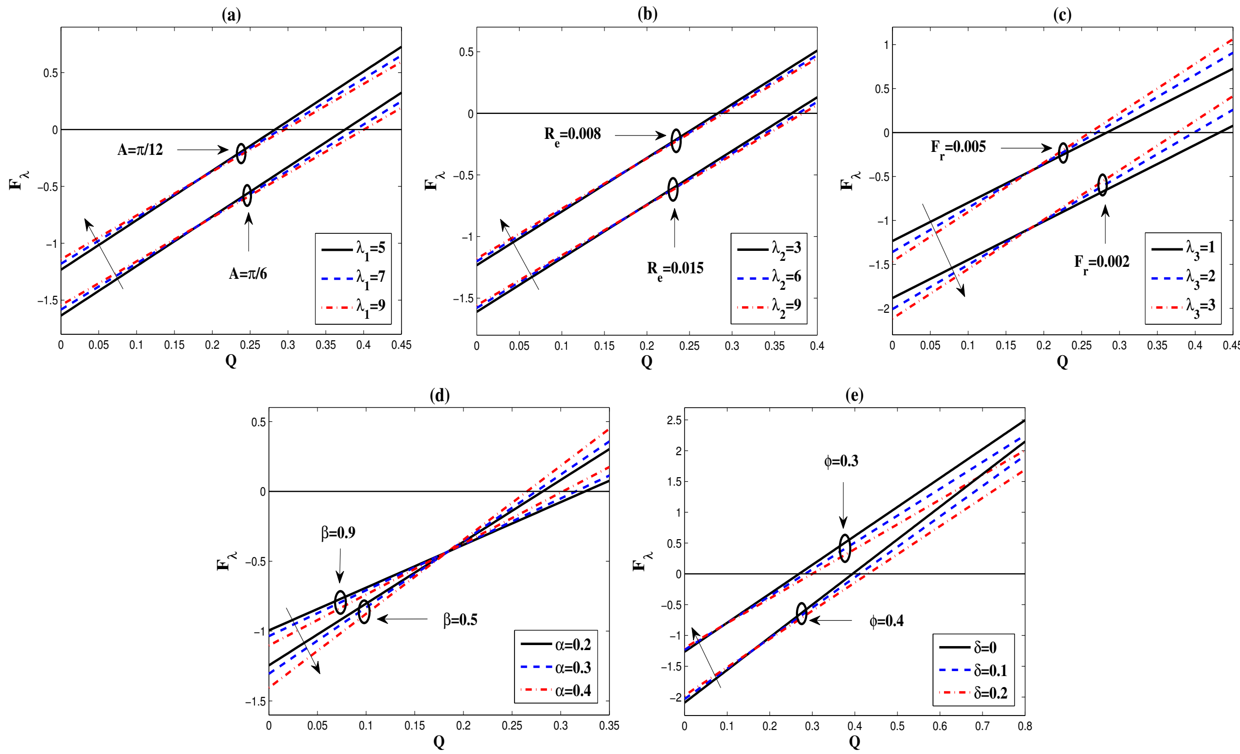

Figure 4: Frictional force F_λ versus time-averaged flow rate Q when $t = 1$ corresponding to (*a*) different values of A and λ_1 with $Re = 0.008$, $Fr = 0.005$, $\lambda_2 = 1$, $\lambda_3 = 3$, $\alpha = 0.4$, $\beta = 0.7$, $\phi = 0.3$ and $\delta = 0.1$ (*b*) different values of Re and λ_2 with $A = \frac{\pi}{12}$, $Fr = 0.005$, $\lambda_1 = 5$, $\lambda_3 = 3$, $\alpha = 0.4$, $\beta = 0.7$, $\phi = 0.3$ and $\delta = 0.1$(*c*) different values of Fr and λ_3 with $A = \frac{\pi}{12}$, $Re = 0.008$, $\lambda_1 = 5$, $\lambda_2 = 1$, $\alpha = 0.4$, $\beta = 0.7$, $\phi = 0.3$ and $\delta = 0.1$ (*d*) different values of α and β with $A = \frac{\pi}{12}$, $Re = 0.008$, $Fr = 0.005$, $\lambda_1 = 5$, $\lambda_2 = 1$, $\lambda_3 = 3$, $\phi = 0.3$ and $\delta = 0.1$ (*e*) different values of ϕ and δ with $A = \frac{\pi}{12}$, $Re = 0.008$, $Fr = 0.005$, $\lambda_1 = 5$, $\lambda_2 = 1$, $\lambda_3 = 3$ and $\alpha = 0.4$, $\beta = 0.7$.

axial differences are all shown at one wavelength for different values of angle of inclination A, Reynolds number R_e, Froude number F_r, material constants $(\lambda_1, \lambda_2, \lambda_3)$, fractional parameters (α, β), parameter of viscosity δ and amplitude ratio ϕ. These figures show that in the wider part of the tube, $\frac{\partial p}{\partial z}$ is relatively small, where the flow can easily pass without giving any large pressure gradient. However, in the narrow part of the tube a much larger pressure gradient is required to maintain the same flux, especially for the narrowest position near $z = 0.75$. This is in good agreement with the physical situation. Moreover the maximum of the pressure gradient increases with increasing A, Re, λ_3, ϕ or α while it decreases when the increase in Fr, λ_1, λ_2, δ or β is increased.

6.2 Behavior of pressure rise

The effects of various parameters on the pressure rise Δp with a time-averaged flow rate Q are shown in Figures 3,a-e. We have the following peristaltic regions; pumping region ($\Delta p > 0$), free-pumping region ($\Delta p = 0$) and co-pumping region ($\Delta p < 0$). These figures show a linear relationship between Δp and Q in all three of these regions, i.e, the pressure rise decreases with increasing time-averaged flow rate.

From Figures 3,a-c we observe that the time-averaged flow rate Q increases with increasing A or Re, while it decreases in all regions as Fr is increased. But for λ_1, λ_2 or λ_3, these Figures show that the curves intersect in the pumping region at $Q = 0.32$. It can be seen that the flow rate Q decreases with increasing λ_1 or λ_2 in the pumping region, while it increases in both the free-pumping and the co-pumping regions. From Figure 3, c it is found that the flow rate Q increases with increasing λ_3 in the pumping region while it decreases in both the free-pumping and the co-pumping regions. Figures 3, d display the pressure rise Δp against the time-averaged flow rate Q for different values of α and β. This Figure indicates that all the curves are also intersecting in the pumping region at $Q = 0.32$. In addition, Q increases with increasing α in the pumping region but it decreases in both the free-pumping and the co-pumping regions while we observe an opposite behavior of Q versus β compared to α. From Figure 3, e it can be seen

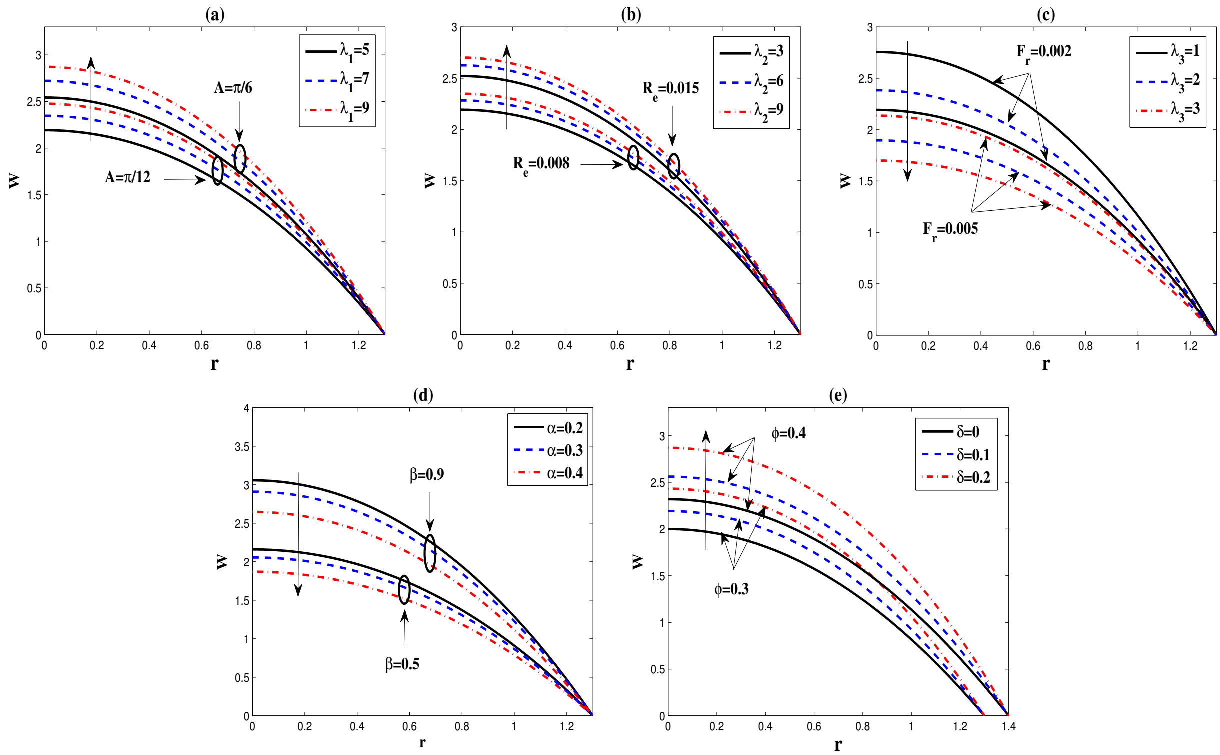

Figure 5: Axial velocity W versus radial direction r when $t = 1$ corresponding to (a) different values of A and λ_1 with $Re = 0.008$, $Fr = 0.005$, $\lambda_2 = 1$, $\lambda_3 = 3$, $\alpha = 0.4$, $\beta = 0.7$, $\phi = 0.3$ and $\delta = 0.1$ (b) different values of Re and λ_2 with $A = \frac{\pi}{12}$, $Fr = 0.005$, $\lambda_1 = 5$, $\lambda_3 = 3$, $\alpha = 0.4$, $\beta = 0.7$, $\phi = 0.3$ and $\delta = 0.1$(c) different values of Fr and λ_3 with $A = \frac{\pi}{12}$, $Re = 0.008$, $\lambda_1 = 5$, $\lambda_2 = 1$, $\alpha = 0.4$, $\beta = 0.7$, $\phi = 0.3$ and $\delta = 0.1$ (d) different values of α and β with $A = \frac{\pi}{12}$, $Re = 0.008$, $Fr = 0.005$, $\lambda_1 = 5$, $\lambda_2 = 1$, $\lambda_3 = 3$, $\phi = 0.3$ and $\delta = 0.1$ (e) different values of ϕ and δ with $A = \frac{\pi}{12}$, $Re = 0.008$, $Fr = 0.005$, $\lambda_1 = 5$, $\lambda_2 = 1$, $\lambda_3 = 3$ and $\alpha = 0.4$, $\beta = 0.7$.

that the time-averaged flow rate Q decreases with increasing δ in the pumping region while it increases in both the free-pumping and the co-pumping regions. For ϕ we observe, from the same figure, that the curves intersect in the co-pumping region where Q increases with the increase in ϕ.

6.3 Behavior of frictional force

We calculated the fictional force F_λ from Equation (26). The effects of A, R_e, F_r, λ_1, λ_2, λ_3, α, β, δ and ϕ on F_λ are depicted in Figures 4,a-e. It can be observed from these figures that, when plotted versus physical parameters, the frictional force F_λ behavior is the opposite of that shown by the pressure rise Δp.

6.4 Axial velocity

In Figures 5,a-e we represent the axial velocity w for different values of the same physical parameters. These figures indicate that the velocity w increases with increasing A, λ_1, λ_2, β, δ or ϕ, while it decreases with an increase in R_e, F_r, λ_3 or α.

6.5 Behavior of mechanical efficiency

Figures 6,a-e plot the mechanical efficiency E versus the ratio of the time-averaged flow rate Q and maximum flow rate Q_0 (i.e $\frac{Q}{Q_0}$). These figures show the effects of interest parameters on E. It is found that the mechanical efficiency increases, attains a maximum value, and decreases to zero. It is also observed that the mechanical efficiency E increases with increasing A, R_e, λ_1, λ_2, β, δ or ϕ, and it decreases if F_r, λ_3 or α are increased.

7 Conclusions

In this work we have analytically studied the peristaltic flow of a fractional Burgers' fluid with variable viscosity in an inclined tube. The problem was simplified under the assumptions of long-wavelength approximation and

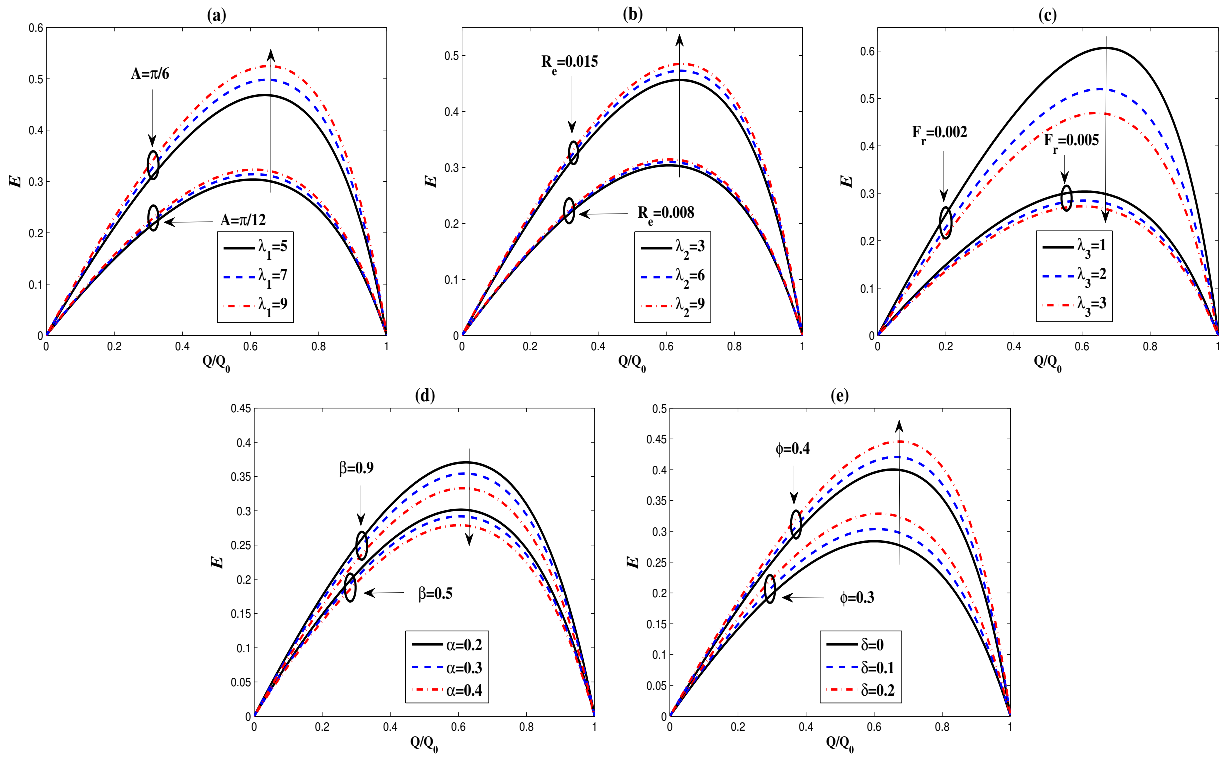

Figure 6: Mechanical efficiency E versus $\frac{Q}{Q_0}$ when $t = 1$ corresponding to (*a*) different values of A and λ_1 with $Re = 0.008$, $Fr = 0.005$, $\lambda_2 = 1$, $\lambda_3 = 3$, $\alpha = 0.4$, $\beta = 0.7$, $\phi = 0.3$ and $\delta = 0.1$ (*b*) different values of Re and λ_2 with $A = \frac{\pi}{12}$, $Fr = 0.005$, $\lambda_1 = 5$, $\lambda_3 = 3$, $\alpha = 0.4$, $\beta = 0.7$, $\phi = 0.3$ and $\delta = 0.1$(*c*) different values of Fr and λ_3 with $A = \frac{\pi}{12}$, $Re = 0.008$, $\lambda_1 = 5$, $\lambda_2 = 1$, $\alpha = 0.4$, $\beta = 0.7$, $\phi = 0.3$ and $\delta = 0.1$ (*d*) different values of α and β with $A = \frac{\pi}{12}$, $Re = 0.008$, $Fr = 0.005$, $\lambda_1 = 5$, $\lambda_2 = 1$, $\lambda_3 = 3$, $\phi = 0.3$ and $\delta = 0.1$ (*e*) different values of ϕ and δ with $A = \frac{\pi}{12}$, $Re = 0.008$, $Fr = 0.005$, $\lambda_1 = 5$, $\lambda_2 = 1$, $\lambda_3 = 3$ and $\alpha = 0.4$, $\beta = 0.7$.

low Reynolds number. An analytical solution to the problem was obtained using a fractional calculus approach. Interaction of various emerging parameters with peristaltic flow was studied with the help of illustrations. The computations have shown that the pressure rise Δp decreases with increasing time-averaged flow rate Q for all interest parameters, while the frictional force F_λ vs Q shows the opposite behavior. In addition, an increase of angle of inclination A, material constant λ_3, fractional parameter α, amplitude ratio ϕ or the Reynolds number causes an increase of the time-averaged flow rate in the pumping region, and for a given value of Δp, while opposing behavior is seen for material constants λ_1, λ_2, fractional parameter β, Froude number or the parameter of viscosity δ. The axial velocity w increases with increasing A, λ_1, λ_2, β, δ or ϕ, while it decreases with any increase in R_e, F_r, λ_3 or α. The mechanical efficiency E increases with an increase in A, λ_1, λ_2, β, R_e, δ or ϕ, while it decreases with the increase in F_r, λ_3 or α.

Acknowledgement: The author is grateful to the referees and editors for their constructive suggestions.

References

[1] T.W. Latham, MS Thesis, Fluid motion in a peristaltic pump, (Massachusetts Institute of Technology, Cambridge, 1966)
[2] M.Y. Jaffrin, A.H. Shapiro, Ann. Rev. Fluid Mech. 3, 13 (1971)
[3] J.C. Burns, T. Parkes, J. Fluid Mech. 29, 731 (1967)
[4] T. Hayat, N. Ali, Appl. Math. Mod. 32, 761 (2008)
[5] M.J. Manton, J. Fluid Mech. 68, 467 (1975)
[6] M. Mishra, A. Ramachandra Rao, Z. A. M. P. 54, 532 (2003)
[7] H. Rachid, M.T. Ouazzani, Eur. Phys. J. Plus 130, 122 (2015)
[8] A. Ramachandra Rao, M. Mishra, Acta Mechanica 168, 35 (2004)
[9] H. Rachid, M.T. Ouazzani, Int. J. Appl. Mech. 6, 1450061 (2014)
[10] M.H. Haroun, J. Applied Mech. Tech. Phys. 46, 842 (2005)
[11] M. Kothandapani, S. Srinivas, Int. J. Non-Linear Mech. 43, 915 (2008)
[12] M. Elshahed, M.H. Haroun, Math. Prob. Eng. 2005, 663 (2005)
[13] Abd El Naby, Abd El Hakeem, J. Applied Mech. 76, 064504 (2009)
[14] N. Ali, T. Hayat, Comput. Math. Appl. 55, 589 (2008)
[15] T. Hayat, Y. Wang, A.M. Siddiqui, K. Hutter, S. Asghar, J. Mech. Med. Biol. 12, 1691 (2002)
[16] O. U. Mehmood, N. Mustapha, S. Shafie, Applied Math. Mech. 33, 1313 (2012)
[17] D. Tripathi, S.K. Pandey, S. Das, Appl. Math. Computer 215, 3645 (2010)

[18] H. Rachid, Abstr. Appl. Anal. 2015, 360918 (2015)
[19] D. Tripathi, Math. BioSci. 233, 90 (2011)
[20] V.K. Narla, K.M. Prasad, J.V. Ramanamurthy, Chin. J. Eng. 2013, 582390 (2013)
[21] A.R. Lee, A.H.D. Markwick, J. Soc. Chem. Ind. 56, 146 (1937)
[22] B.H. Tan, I. Jackson, J.D.F. Gerald, Phys. Chem. Miner. 28, 641 (2001)
[23] T. Hayat, S.B. Khan, M. Khan, Appl. Math. Model., 32, 749 (2008)
[24] M. Khan, S. Hyder Ali, Haitao Qi, Nonlinear Aanal-Real., 10, 2286 (2009)
[25] M. Khan, S. Asghar, T. Hayat, Nonlinear Aanal-Real., 10, 974 (2009)
[26] A. M. Siddiqui, M.A. Rana, Naseer Ahmed, Appl. Math. Model., 34, 2881 (2010)
[27] J.-H. He, Z.-B. Li, J. King Saud Univ. Sci. , DOI:10.1016/j.jksus.2015.07.001, (2015) in press
[28] F.-J. Liu, Z.-B. Li, S. Zhang, H.-Y. Liu, Therm. Sci., 19, 1155 (2015)
[29] A.H. Shapiro, M.Y. Jaffrin, S.L. Weinberg, J. Fluid Mech. 37, 799 (1969)
[30] L.M. Srivastava, V.P. Srivastava, Biorheology 20, 153 (1983)