

Research Article

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On the Solution of Burgers' Equation with the New Fractional Derivative

DOI 10.1515/phys-2015-0045

Received August 25, 2015; accepted November 10, 2015

Abstract: Firstly in this article, the exact solution of a time fractional Burgers' equation, where the derivative is conformable fractional derivative, with dirichlet and initial conditions is found by Hopf-Cole transform. Thereafter the approximate analytical solution of the time conformable fractional Burger's equation is determined by using a Homotopy Analysis Method (HAM). This solution involves an auxiliary parameter $\hbar$ which we also determine. The numerical solution of Burgers' equation with the analytical solution obtained by using the Hopf-Cole transform is compared.

Keywords: Hopf-Cole Transform; Time Fractional Burgers' Equation; Conformable Fractional Derivative; Time Fractional Heat Equation; Homotopy Analysis Method

MSC: 35R11, 34A08, 26A33, 34K28

PACS: 02.30.Jr, 02.60.Lj, 44.10.+i

1 Introduction

Fractional calculus (which is an important field in applied mathematics) arouses great interest for scientists [1–3]. Recently, many studies on fractional derivatives and fractional integrals, which are leading topics in fractional calculus, have been undertaken by scientists. Thus, there are many different definitions of fractional derivatives and fractionals integral such as the Riemann-Liouville definition, the Caputo definition, Grünwald-Letnikov definition and Riesz-Fischer definition. The following are some of the most common definitions:

1. Riemann-Liouville Fractional Derivative Definition: If n is a positive integer and $\alpha \in [n-1, n)$, the α derivative of a function f is given by,

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx.$$

2. Caputo Fractional Derivative Definition: If n is a positive integer and $\alpha \in [n-1, n)$, the α derivative of a function f is given by,

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx.$$

Last year R. Khalil *et al.* [4] presented a new definition of a fractional derivative and integral called the "Conformable fractional derivative and integral".

Definition Let $f: [0, \infty) \rightarrow \mathbb{R}$ be a function. The α^{th} order "conformable fractional derivative" of f is defined by,

$$T_\alpha(f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

for all $t > 0$, $\alpha \in (0, 1)$. If f is α -differentiable in some $(0, a)$, $a > 0$ and $\lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$ exists then define $f^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$ and the conformable fractional integral of a function f starting from $a \geq 0$ is defined as $I_a^\alpha(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$ where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1]$. The following theorem highlights some properties of the conformable fractional derivative [4].

Theorem 1. Let $\alpha \in (0, 1]$ and suppose f, g are α -differentiable at point $t > 0$. Then

1. $T_\alpha(cf + dg) = cT_\alpha(f) + dT_\alpha(g)$ for all $a, b \in \mathbb{R}$.
2. $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.
3. $T_\alpha(\lambda) = 0$ for all constant functions $f(t) = \lambda$.
4. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
5. $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.
6. If, in addition to f differentiable, then $T_\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}$.

This new fractional derivative definition has governed much attention in recent months. For instance T. Abdel-

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jawad [5] provide fractional versions of the chain rule, exponential functions, Gronwall's inequality, integration by parts, Taylor power series expansions and Laplace transforms. N. Benkhettoua *et al.* [6] introduced conformable time-scale fractional calculus. As physical applications of conformable fractional derivative M.A. Hammad and R. Khalil [7] give the solution for the conformable fractional heat equation and W.S. Chung [8] used the conformable fractional derivative and integral to discuss fractional Newtonian mechanics. It is clearly seen that further studies and explanations can be made regarding the physical meaning and physical applications of this new subject area.

Burgers' equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \quad (1.1)$$

was firstly presented by Bateman [9] in 1918. J.M. Burgers expounded on this in his studies between 1939-1965 [10, 11]. Since then, Burgers' equation has been used as a mathematical model in numerous areas such as number theory, gas dynamics, heat conduction, elasticity theory, turbulence theory, shock wave theory, fluid mechanics, termaviscous fluids, hydrodynamic waves and elastic waves [12–15].

Scientists conducted many studies to determine the numerical or analytical solution of the Eq. (1.1). For example, T. Ozis and A. Ozdes [16] used a direct variational method to solve Burgers' equation. E. Aksan and A. Ozdes [17] used a variational method constructed on the method of discretization in time to solve Burgers' equation. S. Kutluay *et al.* [18] obtained a the numerical solution of Burgers' equation by using finite difference methods. E. Varoğlu and L. Finn [19] made use of a weighted residue method. J. Caldwell and P. Wanless [20] used finite elements. D.J. Evans and A.R. Abdullah [21] used the group explicit method. R.C. Mittal and P. Singhal [22] used the Galerkin method to determine numerical solutions of Burgers' equation.

Therefore it can be concluded that scientists have devoted much attention to obtain the numerical and/or analytical solution for the fractional Burgers' equation. For instance A. Esen and O. Tasbozan [23] used Cubic B-spline Finite Elements to have the numerical solution of time fractional Burgers' equation. A. Esen *et al.* [24] used HAM to find the approximate analytical solution of time fractional Burgers' equation. In another study E. A.-B. Abdel-Salam *et al.* [25] used fractional Riccati expansion method to solve space-time fractional Burgers' equation. M. Inc [26] used a variational iteration method to solve the space and time fractional Burgers' equations.

Investigating and reaching exact or numerical solutions of these type of equations has a great importance in applied mathematics. The HAM, which is one of the greatest tool for finding the approximate solutions of nonlinear evolution equations (NLEEs), was first presented by Liao [27, 28]. The HAM differs from perturbation techniques in that it is not limited to any small physical parameters in the considered equation. For this reason, HAM has neither any restrictions nor limitations of perturbation techniques so that it provides us with a powerful tool to analyze strongly nonlinear problems [29–31]. Due to these advantages HAM is used as a method to obtain the approximate analytical solution of many different equations.[32–34].

In 1950 a transform

$$u = -2\nu \frac{\theta_x}{\theta} \quad (1.2)$$

was defined by Hopf [35] which helps in transforming Burgers' equation into a heat equation:

$$\frac{\partial \theta}{\partial t} = \nu \frac{\partial^2 \theta}{\partial x^2} \quad (1.3)$$

where $\theta(x, t)$ is the solution of the heat equation (1.3) and $u(x, t)$ is the solution of Eq. (1.1) [36]. In 1951 some theorems, were suggested by Cole [37] which express the relationship between Burgers' equation and the heat equation.

2 The Conformable Time Fractional Burgers' Equation

Consider the fractional Burgers' equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, \quad t > 0, \quad (2.1)$$

with the conditions,

$$u(x, 0) = \sin(\pi x) \quad (2.2)$$

and

$$u(0, t) = u(1, t) = 0, \quad (2.3)$$

where $\alpha \in (0, 1)$ and $\frac{\partial^\alpha u}{\partial t^\alpha}$ means conformable fractional derivative of function $u(x, t)$. Using the Hopf–Cole transformation (1.2), the equation evolves into a time fractional heat equation,

$$\frac{\partial^\alpha Q}{\partial t^\alpha} = \nu \frac{\partial^2 Q}{\partial x^2}, \quad (2.4)$$

where the derivative is an α -order conformable fractional derivative and $Q(x, t)$ the solution of the heat equation (2.4). The conditions are thus,

$$Q(x, 0) = e^{-\frac{1-\cos(\pi x)}{2\nu\pi}}, \quad (2.5)$$

and

$$Q_x(0, t) = Q_x(1, t) = 0. \quad (2.6)$$

Hence, using the method of separation of variables, the solution to the above linearized problem can be obtained easily as

$$Q(x, t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) e^{-\frac{n^2 \pi^2 \nu t^\alpha}{\alpha}},$$

where a_0 and a_n , $n = 1, 2, \dots$ are Fourier coefficients and can be evaluated as:

$$a_0 = \int_0^1 e^{-\frac{1-\cos(\pi x)}{2\nu\pi}} dx,$$

and

$$a_n = 2 \int_0^1 e^{-\frac{1-\cos(\pi x)}{2\nu\pi}} \cos(n\pi x) dx.$$

Then using equation (1.2), the exact solution of the equation (2.1) is

$$u(x, t) = 2\nu\pi \frac{\sum_{n=1}^{\infty} a_n e^{-\frac{n^2 \pi^2 \nu t^\alpha}{\alpha}} n \sin(n\pi x)}{a_0 + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2 \pi^2 \nu t^\alpha}{\alpha}} \cos(n\pi x)},$$

where the coefficients a_0 and a_n are:

$$a_0 = \int_0^1 e^{-\frac{1-\cos(\pi x)}{2\nu\pi}} dx,$$

and

$$a_n = 2 \int_0^1 e^{-\frac{1-\cos(\pi x)}{2\nu\pi}} \cos(n\pi x) dx, \quad n = 1, 2, 3, \dots$$

3 Fundamentals of the HAM

In this work, HAM is applied to the treated problem. In order to explain the fundamentals of the method we consider the following differential equation,

$$\mathcal{N}[u(x, t)] = 0,$$

where $\mathcal{N}$ is a nonlinear operator, x and t show independent variables, $u(x, t)$ is an unknown function. By using the generalization of HAM, Liao [27, 28] has constructed a zero-order deformation equation

$$(1-p)\mathcal{L}[\phi(x, t; p) - u_0(x, t)] = p\hbar\mathcal{N}[\phi(x, t; p)], \quad (3.1)$$

where $p \in [0, 1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, $\mathcal{L}$ is an auxiliary linear operator, $u_0(x, t)$ is an initial guess of $u(x, t)$, $\phi(x, t; p)$ is an unknown function, respectively. Thus, it would be sensible to choose auxiliary parameters and operators in HAM. When we choose $p = 0$ and $p = 1$ then we obtain

$$\phi(x, t; 0) = u_0(x, t), \quad \phi(x, t; 1) = u(x, t)$$

respectively. So, as the embedding parameter p increases from 0 to 1, the solutions $\phi(x, t; p)$ differ from the initial value $u_0(x, t)$ to the solution $u(x, t)$. If $\phi(x, t; p)$ is expanded in Taylor series with respect to the embedding parameter p , we obtain:

$$\phi(x, t; p) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) p^m$$

where

$$u_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m \phi(x, t; p)}{\partial p^m} \right|_{p=0}. \quad (3.2)$$

When the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ are chosen in a suitable manner, the series (which are denoted above) converges at $p = 1$, and

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t).$$

This must be one of the solutions of the original nonlinear equation, as shown by Liao [28, 31]. According to (3.2), the governing equation can be reduced from the zero-order deformation equation (3.1). Define the vector:

$$\mathbf{u}_n = \{u_0(x, t), u_1(x, t), \dots, u_n(x, t)\}.$$

If we differentiate Eq. (3.1) m times with respect to the embedding parameter p and then setting $p = 0$ and dividing by $m!$, we obtain the m th-order deformation equation:

$$\mathcal{L}[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar R_m(\mathbf{u}_{m-1}) \quad (3.3)$$

where

$$R_m(\mathbf{u}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}[\phi(x, t; p)]}{\partial p^{m-1}} \right|_{p=0}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

We emphasize that $u_m(x, t)$ for $m \geq 1$ is governed by Eq. (3.3) with the boundary condition that comes from the problem. It can then be easily solved by using symbolic computation software such as Mathematica.

4 Application of HAM

We handle the time fractional Burgers equation as

$$D_t^\alpha u + uu_x - \nu u_{xx} = 0, \quad (4.1)$$

with initial conditions

$$u(x, 0) = \sin(\pi x), \quad (4.2)$$

where $\alpha \in (0, 1)$ and the derivative is a conformable fractional derivative. When $\nu > 0$, it is often referred to as the viscous Burgers equation, and when $\nu = 0$, it is often referred to as the inviscid Burgers equation. For convenience and to shorten the article, ν is taken as 1 for all calculations in this paper. One can use any other value of ν for calculations. For investigating the series solution of Eq. (4.1) with initial condition (4.2), we choose the linear operator,

$$\mathcal{L}[\phi(x, t; p)] = D_t^\alpha \phi(x, t; p),$$

with the property,

$$\mathcal{L}[c] = 0,$$

where c is constant. From Eq. (4.1), we now define the nonlinear operator as,

$$\begin{aligned} \mathcal{N}[\phi(x, t; p)] &= \frac{\partial^\alpha \phi(x, t; p)}{\partial t^\alpha} + \phi(x, t; p) \frac{\partial \phi(x, t; p)}{\partial x} \\ &\quad - \frac{\partial^2 \phi(x, t; p)}{\partial x^2}. \end{aligned}$$

From Theorem 1 the nonlinear operator can be written as follows,

$$\begin{aligned} \mathcal{N}[\phi(x, t; p)] &= t^{1-\alpha} \frac{\partial \phi(x, t; p)}{\partial t} + \phi(x, t; p) \frac{\partial \phi(x, t; p)}{\partial x} \\ &\quad - \frac{\partial^2 \phi(x, t; p)}{\partial x^2}. \end{aligned}$$

Therefore the zero-order deformation equation is established as:

$$(1-p)\mathcal{L}[\phi(x, t; p) - u_0(x, t)] = p\hbar \mathcal{N}[\phi(x, t; p)]. \quad (4.3)$$

Precisely, if we choose $p = 0$ and $p = 1$ then we obtain

$$\phi(x, t; 0) = u_0(x, t) = u(x, 0), \quad \phi(x, t; 1) = u(x, t).$$

Thus, since the embedding parameter p increases from 0 to 1, the solution $\phi(x, t; p)$ varies from the initial value $u_0(x, t)$ to the solution $u(x, t)$. By expanding $\phi(x, t; p)$ in Taylor series with respect to the embedding parameter p :

$$\phi(x, t; p) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) p^m, \quad (4.4)$$

where

$$u_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m \phi(x, t; p)}{\partial p^m} \right|_{p=0}. \quad (4.5)$$

If the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ are properly chosen, the above series converges at $p = 1$, and,

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t),$$

which must be one of the solutions of the original nonlinear equations, as proved by Liao [28]. By differentiating Eq. (4.3) m times with respect to the embedding parameter p , we obtain the m th-order deformation equation:

$$\mathcal{L}[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar R_m(\mathbf{u}_{m-1}), \quad (4.6)$$

where,

$$\begin{aligned} R_m(\mathbf{u}_{m-1}) &= t^{1-\alpha} \frac{\partial u_{m-1}(x, t)}{\partial t} \\ &\quad + \sum_{n=0}^{m-1} u_n(x, t) \frac{\partial u_{m-1-n}(x, t)}{\partial x} - \frac{\partial^2 u_{m-1}(x, t)}{\partial x^2}, \end{aligned}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}.$$

The solutions of the m th-order deformation Eq. (4.6) for $m \geq 1$ leads to

$$u_m(x, t) = \chi_m u_{m-1}(x, t) + \hbar \mathcal{L}^{-1}[R_m(\mathbf{u}_{m-1})]. \quad (4.7)$$

By using Eq.(4.7) with initial condition given by (4.2) we successively obtain

$$\begin{aligned} u_0(x, t) &= \sin(\pi x), \\ u_1(x, t) &= (\pi \hbar t^\alpha (\pi + \cos(\pi x)) \sin(\pi x)) / \alpha, \\ &\vdots \end{aligned}$$

Therefore, the series solutions expressed by HAM can be written in the form

$$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \dots \quad (4.8)$$

To demonstrate the efficiency of the method, we compare the HAM solutions of the time conformable fractional Burgers equation given by Eq. (4.8) with its exact solutions

$$u(x, t) = \frac{2\pi\nu \sum_{n=1}^{\infty} d_n e^{-\frac{(n\pi\sqrt{\nu})^2}{\alpha} t^\alpha} n \sin(n\pi x)}{d_0 + \sum_{n=1}^{\infty} d_n e^{-\frac{(n\pi\sqrt{\nu})^2}{\alpha} t^\alpha} \cos(n\pi x)}$$

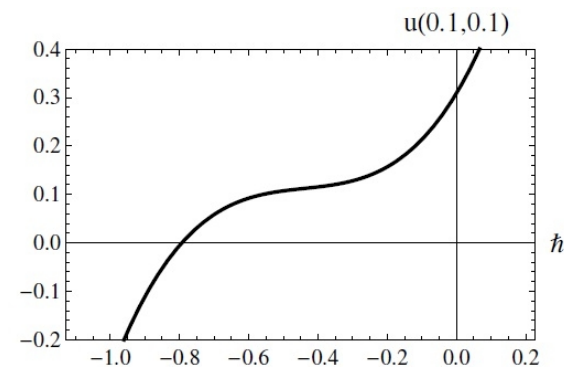
where

$$d_0 = \int_0^1 e^{-\frac{1-\cos(\pi x)}{2\pi\nu}} dx \quad (4.9)$$

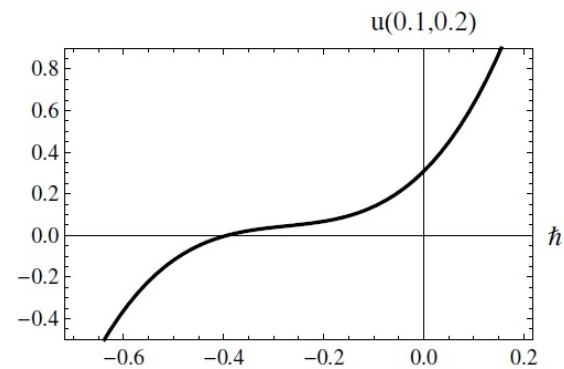
and

$$d_n = 2 \int_0^1 e^{-\frac{1-\cos(\pi x)}{2\pi v}} \cos(n\pi x) dx, \quad n = 1, 2, 3, \dots \quad (4.10)$$

The auxiliary parameter $\hbar$, which is in our HAM solution series, provides us with a simple way to adjust and control the convergence of the solution series. To obtain an appropriate range for $\hbar$, we consider the so-called $\hbar$ -curve to choose a proper value of $\hbar$ which provide that the solution series is convergent, as pointed by Liao [28], by discovering the valid region of $\hbar$ which corresponds to the line segments nearly parallel to the horizontal axis.



(a) $x = 0.1, t = 0.1$



(b) $x = 0.1, t = 0.2$

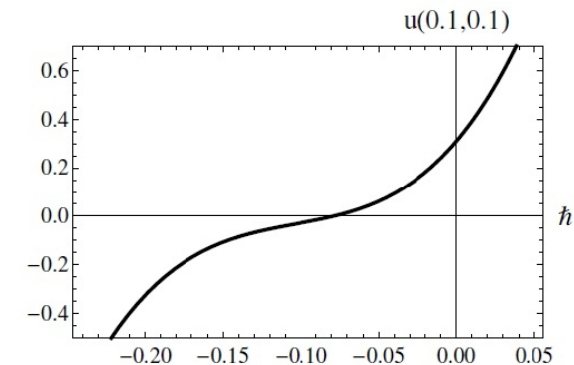
Figure 1: The $\hbar$ -curves of 4th-order approximate solution obtained by the HAM for $\alpha = 1$.

Table 1: Exact and numerical solutions for $\alpha = 1$ and $\hbar = -0.5$.

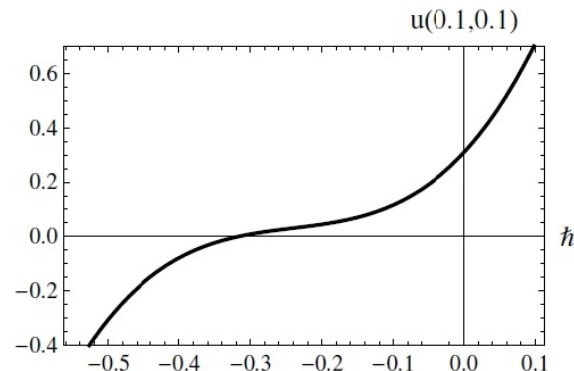
t	x	Numerical	Exact	Absolute Error
0.1	0.1	0.107347	0.109538	0.002191
	0.2	0.209011	0.209792	0.000781
	0.3	0.294529	0.291896	0.002633

Table 2: Exact and numerical solutions for $\alpha = 1$ and $\hbar = -0.3$.

t	x	Numerical	Exact	Absolute Error
0.2	0.1	0.039200	0.041929	0.002729
	0.2	0.077707	0.079994	0.002287
	0.3	0.107657	0.110622	0.002965



(a) $\alpha = 0.5$



(b) $\alpha = 0.75$

Figure 2: The $\hbar$ -curves of 4th-order approximate solution obtained by the HAM for $x = 0.1$ and $t = 0.1$.

Table 3: Exact and numerical solutions for $\alpha = 0.5$ and $\hbar = -0.075$.

t	x	Numerical	Exact	Absolute Error
0.1	0.1	0.006068	0.000599	0.005469
	0.2	0.009353	0.001114	0.008239
	0.3	0.007259	0.001569	0.005690

Table 4: Exact and numerical solutions for $\alpha = 0.75$ and $h = -0.25$.

t	x	Numerical	Exact	Absolute Error
0.1	0.1	0.028019	0.029248	0.001229
	0.2	0.054543	0.055752	0.001209
	0.3	0.0723617	0.076991	0.004629

5 Conclusion

In this paper, we argue new exact and numerical solutions of time fractional Burgers' equation, which is known as one dimensional nonlinear time fractional partial differential equation. It is known that Burgers' equation is one of the rare nonlinear equations whose solution can be obtained analytically. Burgers' equation has a great importance in applied sciences. The Hopf-Cole transform and conformable fractional derivative definition (which is a new fractional derivative definition) are used for the exact solution of the time fractional Burgers' equation. For numerical solutions, the homotopy analysis method, which is a powerful and efficient technique in finding the numerical solution of the conformable time fractional Burgers' equation is applied successfully. This conformable fractional derivative definition is a convenient definition in the exact solution procedure of fractional differential equations. Conformable fractional derivatives are easier to use when compared to the other fractional derivatives, as its derivative definition does not include any integral term. It has been also shown that the HAM solution of the problem, converges very rapidly to the exact one by choosing a convenient auxiliary parameter h . In conclusion, some tables which compare the numerical and analytical solutions are provided to show that the HAM is a powerful and efficient technique in finding the numerical solution of the conformable time fractional Burgers' equation.

Acknowledgement: The authors are grateful to the anonymous referees for their careful checking of the details and for their helpful comments that contributed to the improvement of this paper.

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