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A local description of dark energy in terms of classical two-component massive spin-one uncharged fields on spacetimes with torsionful affinities

Abstract: It is assumed that the two-component spinor formalisms for curved spacetimes that are endowed with torsionful affine connexions can supply a local description of dark energy in terms of classical massive spin-one uncharged fields. The relevant wave functions are related to torsional affine potentials which bear invariance under the action of the generalized Weyl gauge group. Such potentials are thus taken to carry an observable character and emerge from contracted spin affinities whose patterns are chosen in a suitable way. New covariant calculational techniques are then developed towards deriving explicitly the wave equations that supposedly control the propagation in spacetime of the dark energy background. What immediately comes out of this derivation is a presumably natural display of interactions between the fields and both spin torsion and curvatures. The physical properties that may arise directly from the solutions to the wave equations are not brought out.

1 Introduction

Since the discovery of the cosmic dark energy [1, 2], several attempts have been made [3–9] at accomplishing a macroscopic explanation of the presently observable acceleration of the universe [10, 11], while circumventing the situations concerning some of the problems that arise in the context of the standard cosmology [4, 12]. One of the most popular approaches that were designed in this connection describes dark energy in a geometrically torsionless fash-ion as a gravitationally repulsive cosmic background modelled either by a positive cosmological constant or by a scalar field to which a physical meaning may possibly be ascribed. In this model, the dark energy density can be explicitly evaluated with the help of some auxiliary observational data, but the corresponding results nevertheless turn out to be in serious disagreement with characteristic values arising from the conventional quantum field theories. In addition, the complete physical adequacy of the scalar field taken up thereabout has not been established hitherto. Another popular approach focusses upon trivial modifications of the Lagrangian density for classical general relativity. It likewise implements alternative patterns for generally relativistic energy momentum tensors, and thereby gives rise to the need for sorting out the microscopic nature of dark energy within extended particle physics models. A somewhat interesting work carried out along these lines [13], identifies the dark energy background with a massive vector potential which is taken from the beginning to obey a non-minimal coupling to gravity. Accordingly, the Friedmann equations acquire an extra non-geometric term which is proportional to the rest mass of the dark energy particles. Moreover, the implementation of certain astronomical constraints makes it feasible to estimate the mass of the particles. The overall picture then leads to a mass value naively related to the cosmological constant, and also supplies a late-time accelerated De Sitter-like cosmic expansion.

On the basis of Einstein-Cartan's theory [14–18], a prospect has been posed by researchers for bringing forth a torsional version of the standard cosmological model (see Refs. [19, 20]). This had been partially motivated by a theoretical possibility of particularly explaining the cosmic acceleration of the universe along with its spatial flatness, its homogeneity and isotropy, without having to call for any mechanisms of cosmic inflation [3, 4]. As mentioned in Refs. [21–25], torsional gravity has also attracted a considerable deal of attention in conjunction with a prediction achieved by string theory that concerns the occurrence of couplings between torsion and spinning fields. Many insights into the understanding of both the coupling strengths of the fundamental interactions and the ratios

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between them, have thus been gained from the torsionic property of underlying spacetime geometries. Remarkably enough, the essentially unique torsionful version of the famous Infeld-van der Waerden $\gamma\varepsilon$-formalisms [26–38] had been until very recently [39] just sparsely considered in the literature [40, 41]. The main motivation for formulating this torsional extension came from the ascertainment that its geometric inner structure may allow the implementation of affine contributions which afford gauge invariant vector potentials bearing an observable character. It had then been expected that the definitive ascription of a fundamental significance to spacetime torsion would eventually become more tangible if a torsional two-component spinor description of dark energy might go hand-in-hand with the spin-torsion mechanisms that prevent the universe from being originated by a singularity [42–44].

In the present work, we take account of the torsional spinor formalisms referred to previously to bring forward a supposedly realistic description of the dynamics of dark energy in a purely local fashion. In fact, the viability for carrying out our description relies geometrically upon the possibility of choosing asymmetric spin-affine connexions that supply gauge invariant potentials for two-component massive spin-one uncharged fields on spacetimes with torsionful affinities. The paper works out the idea that the universe could have been expected beforehand to host two physical backgrounds which, as we believe, must be described in terms of affine potentials coming from the spinor structures inherently borne by generally relativistic spacetimes [45, 46]. Hence, a torsionless electromagnetic background should be locally described by the old $\gamma\varepsilon$-formalisms such as suggested in Refs. [29, 33], and a torsionful background should be describable locally in terms of geometric Proca fields within a suitably extended spinor framework. Throughout the paper, we thus adopt the attitude that identifies the former with the cosmic microwave background (CMB), and likewise think of the latter as constituting the cosmic dark energy. As was pointed out in Ref. [39] from a strictly geometric viewpoint, any torsional affine potential must be accompanied by proper torsionless contributions whence, in actuality, the implementation of this picture gives rise to one of the theoretical features of our work whereby the spacetime description of dark energy has to be united together with that of the CMB. Yet, we realize that the propagation of the CMB in regions of the universe where the values of torsional affinities are negligible may be described alone within the framework of Ref. [28].

We shall account for the well-established observational fact [8, 9] that the CMB and dark energy permeate together the whole of the universe. Because of the lo-

cality of our description, the completion of the relevant procedures will be accomplished without making it necessary to allow for any cosmological kinematics or even to call upon any ordinary cosmological presuppositions like those concerning homogeneities, isotropy, inflation and shape of physical densities. Instead, the only assumptions lying behind the implementation of our procedures are the same as the ones made before [39], according to which local spinor structures along with manifold mapping groups and the matrices that classically constitute the generalized Weyl gauge group [26–28], remain all formally unaltered when any classical spacetime consideration is shifted to the torsional framework. We stress that the defining prescriptions for any of the geometric world and spin densities tied in with the old formalisms [28, 29], may be applicable equally well herein. The information on the wave functions for both physical backgrounds is carried by adequately contracted spin curvatures which emerge as sums of typical bivector contributions from the action on arbitrary spin vectors of a characteristic torsionful second-order covariant derivative operator. It appears that the additivity property of such contracted curvatures is really passed on to the wave functions.

We will utilize the notation adhered to in Ref. [39]. Unless otherwise indicated in an explicit manner, the usual designation of the traditional spinor framework as $\gamma\varepsilon$-formalisms will henceforward be attributed to the torsionful two-component formalisms under consideration here. Upon writing down the world form of the pertinent field equations, we shall therefore take into account geometric electromagnetic and uncharged Proca fields for a curved spacetime $\mathfrak{M}$ that carries a world metric tensor $g_{\mu\nu}$ having the local signature $(+ - --)$ and a torsionful, metric compatible, covariant derivative operator ∇_μ. The spinor form of the field equations will be obtained by carrying out a straightforward transcription of the respective world statements. We will see that the resulting spinor field equations involve pairs of new complex conjugate current densities for each physical background, which absorb outer products carrying appropriate torsion spinors along with the wave functions themselves. In order to carry out systematically the derivation of the wave equations that control the propagation of the fields in $\mathfrak{M}$, we shall have to adapt to the torsional framework the differential calculational techniques employed for the first time in the work of Ref. [28]. What immediately comes out of this derivation is a presumably natural display of interactions between the fields and both torsion and curvatures. In either formalism, some pieces of the geometric sources originated by the field equations must thus be subject to prescribed gauge invariant subsidiary conditions which are brought

about by the inherent symmetry of the wave functions. We will not bring out at this stage any physical properties that may arise from the solutions to our wave equations, however.

Without any risk of confusion, we will use the same indexed symbol ∇_μ to write covariant derivatives in both formalisms. The symbol $\mathfrak{g}$ will sometimes be used for denoting the determinant of $g_{\mu\nu}$. For the world affine connexion associated with ∇_μ, we have the splitting

$$\Gamma_{\mu\nu\lambda} = \widetilde{\Gamma}_{\mu\nu\lambda} + T_{\mu\nu\lambda},$$

where $\widetilde{\Gamma}_{\mu\nu\lambda} = \Gamma_{(\mu\nu)\lambda}$ and $T_{\mu\nu\lambda} = \Gamma_{[\mu\nu]\lambda}$ is by definition the torsion tensor of ∇_μ. The symmetric piece $\widetilde{\Gamma}_{\mu\nu\lambda}$ may be identified with the Christoffel connexion of $g_{\mu\nu}$ in case $T_{\mu\nu\lambda}$ is rearranged adequately. We take the elements of the Weyl gauge group as non-singular complex (2×2)-matrices whose entries are defined by

$$\Lambda_A{}^B = \exp(i\theta)\delta_A{}^B,$$

where $\delta_A{}^B$ denotes the Kronecker symbol and θ is the gauge parameter of the group which shows up as an arbitrary differentiable real-valued function on $\mathfrak{M}$. The determinant $\exp(2i\theta)$ of $(\Lambda_A{}^B)$ will be denoted as Δ_Λ. A horizontal bar lying over some kernel letter will denote the operation of complex conjugation. Some minor conventions shall be explained in due course.

Our outline has been set as follows. In Section 2, we recall the contracted spin curvatures as built up in Ref. [39], and bring out the world field equations. The definition of all wave functions is shown in Section 3 together with the spinor field equations. In Section 4, the torsional calculational techniques are developed. There, we will have to consider spin curvatures somewhat further. Nonetheless, many of the curvature formulae deduced in Ref. [39] shall be taken for granted at the outset. In Section 5, the wave equations are derived. We set an outlook on future works in Section 6.

2 World field equations

The key curvature object for either formalism is a world-spin quantity $C_{\mu\nu AB}$ that occurs in the configuration

$$D_{\mu\nu}\zeta^B = C_{\mu\nu A}{}^B \zeta^A, \tag{1}$$

where ζ^A is an arbitrary spin vector and $D_{\mu\nu}$ amounts to the characteristic second-order covariant derivative operator of the torsional framework, namely,

$$D_{\mu\nu} \doteq 2(\nabla_{[\mu}\nabla_{\nu]} + T_{\mu\nu}{}^\lambda \nabla_\lambda). \tag{2}$$

In the γ-formalism, we have the tensor law

$$C'_{\mu\nu AB} = \Lambda_A{}^C \Lambda_B{}^D C_{\mu\nu CD} = \Delta_\Lambda C_{\mu\nu AB}, \tag{3}$$

whereas the object $C_{\mu\nu AB}$ for the ε-formalism is taken as an invariant spin-tensor density of weight −1, that is to say,

$$C'_{\mu\nu AB} = (\Delta_\Lambda)^{-1} \Lambda_A{}^C \Lambda_B{}^D C_{\mu\nu CD} = C_{\mu\nu AB}. \tag{4}$$

The contracted curvature $C_{\mu\nu A}{}^A$ possesses the gauge invariant additivity property[1]

$$C_{\mu\nu A}{}^A = \widetilde{C}_{\mu\nu A}{}^A + C^{(T)}_{\mu\nu A}{}^A. \tag{5}$$

In particular, $C^{(T)}_{\mu\nu A}{}^A$ accounts for the torsionfulness of ∇_μ while the whole $\widetilde{C}_{\mu\nu AB}$ is taken up by the torsionless commutator

$$2\widetilde{\nabla}_{[\mu}\widetilde{\nabla}_{\nu]}\zeta^B = \widetilde{C}_{\mu\nu A}{}^B \zeta^A, \tag{6}$$

where $\widetilde{\nabla}_\mu$ is indeed the covariant derivative operator for $\widetilde{\Gamma}_{\mu\nu\lambda}$. It turns out that we can write down the simultaneous contracted relations

$$\widetilde{C}_{\mu\nu A}{}^A = 2\partial_{[\mu}\widetilde{\vartheta}_{\nu]A}{}^A, \quad C^{(T)}_{\mu\nu A}{}^A = 2\partial_{[\mu}\vartheta^{(T)A}_{\nu]A}, \tag{7}$$

with the involved ϑ-pieces thus occurring in the skew contributions that make up in each formalism a suitably chosen asymmetric spin affinity for ∇_μ, in agreement with Equation (5). Hence, making use of the standard patterns [39]

$$\widetilde{\vartheta}_{\mu A}{}^A = \partial_\mu \log E - 2i\Phi_\mu, \quad \vartheta^{(T)A}_{\mu A} = -2iA_\mu, \tag{8}$$

yields the purely imaginary expression

$$C_{\mu\nu A}{}^A = -2i\left(\widetilde{F}_{\mu\nu} + F^{(T)}_{\mu\nu}\right), \tag{9}$$

along with the bivectors

$$\widetilde{F}_{\mu\nu} \doteq 2\partial_{[\mu}\Phi_{\nu]}, \quad F^{(T)}_{\mu\nu} \doteq 2\partial_{[\mu}A_{\nu]}, \tag{10}$$

with Φ_μ and A_μ amounting to affine potentials subject to the gauge behaviours

$$\Phi'_\mu = \Phi_\mu - \partial_\mu\theta, \quad A'_\mu = A_\mu. \tag{11}$$

It is worthwhile to recast each of the derivatives of Equation (10) as a piece that looks formally like

$$\partial_{[\mu}\Omega_{\nu]} = \nabla_{[\mu}\Omega_{\nu]} + T_{\mu\nu}{}^\lambda \Omega_\lambda. \tag{12}$$

We mention, in passing, that the quantity E carried by the prescriptions (8) is a real positive-definite world-invariant spin-scalar density of absolute weight +1. In the

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[1] We should emphasize that the uncontracted object $C_{\mu\nu A}{}^B$ for either formalism does *not* hold the additivity property.

γ-formalism, it carries a manifestly spin-metric character, but this ceases holding for the ε-formalism. The potentials Φ_μ and A_μ are the same in both formalisms. They arise from an affine property of the covariant derivative expansions for the Hermitian connecting objects of the formalisms (for further details, see Ref. [39]).

It can be seen from Equation (11) that Φ_μ is a Maxwell potential, which we take to be physically associated to the CMB. In turn, A_μ bears gauge invariance and is likewise looked upon as a potential of mass m for the dark energy background. The world form of the first half of the overall set of field equations emerges from the usual least-action principles for Maxwell and real Proca fields in curved spacetimes [47]. It follows that, allowing for the relation

$$\frac{1}{\sqrt{-\mathfrak{g}}}\partial_\mu\left(\sqrt{-\mathfrak{g}}\mathcal{F}^{\mu\lambda}\right) = \nabla_\mu\mathcal{F}^{\mu\lambda} + 2T_\mu\mathcal{F}^{\mu\lambda} - T_{\mu\nu}{}^\lambda\mathcal{F}^{\mu\nu}, \quad (13)$$

with $T_\mu \doteq T_{\mu\tau}{}^\tau$ and the kernel letter $\mathcal{F}$ standing for either $\widetilde{F}$ or $F^{(T)}$, we get the first half of Maxwell's equations

$$\nabla^\mu\widetilde{F}_{\mu\lambda} + 2T^\mu\widetilde{F}_{\mu\lambda} - T^{\mu\nu}{}_\lambda\widetilde{F}_{\mu\nu} = 0, \quad (14)$$

along with the first half of Proca's equations

$$\nabla^\mu F^{(T)}_{\mu\lambda} + 2T^\mu F^{(T)}_{\mu\lambda} - T^{\mu\nu}{}_\lambda F^{(T)}_{\mu\nu} + m^2 A_\lambda = 0. \quad (15)$$

Obviously, in accordance with our picture, the statements (14) and (15) are the dynamical world field equations in $\mathfrak{M}$ for CMB photons and dark energy fields. Both of the second halves come about as the corresponding Bianchi identities, which may be expressed by

$$\nabla^{\mu\,\star}\mathcal{F}_{\mu\lambda} = -2{}^\star T_\lambda{}^{\mu\nu}\mathcal{F}_{\mu\nu}, \quad (16)$$

with the kernel-letter notation of (13), as well as some of the dualization schemes given in Ref. [16], having been utilized for writing Equation (16).

3 Spinor field equations

The wave functions for both backgrounds are supplied by the spinor decomposition of the bivectors carried by Equation (10). We have, in effect,

$$S^\mu_{AA'}S^\nu_{BB'}\widetilde{F}_{\mu\nu} = M_{A'B'}\phi_{AB} + M_{AB}\phi_{A'B'} \quad (17)$$

and[2]

$$S^\mu_{AA'}S^\nu_{BB'}F^{(T)}_{\mu\nu} = M_{A'B'}\psi_{AB} + M_{AB}\psi_{A'B'}, \quad (18)$$

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2 The kernel letter M will henceforth denote either γ or ε.

where the *S*-symbols are some of the connecting objects for the formalism occasionally allowed for, and the entries of the pair $(M_{AB}, M_{A'B'})$ just denote the respective covariant metric spinors. Thus, the wave functions carried by $(\phi_{AB}, \phi_{A'B'})$ and $(\psi_{AB}, \psi_{A'B'})$ come into play as massless and massive spin-one uncharged fields of opposite handednesses, with their gauge characterizations incidentally coinciding with those exhibited by Equations (3) and (4). By invoking Equation (12) together with the torsion decomposition

$$T_{AA'BB'}{}^\mu = M_{A'B'}\tau_{AB}{}^\mu + M_{AB}\tau_{A'B'}{}^\mu, \quad (19)$$

we obtain the field-potential relationships

$$\phi_{AB} = -\nabla^{C'}_{(A}\Phi_{B)C'} + 2\tau_{AB}{}^\mu\Phi_\mu \quad (20)$$

and

$$\psi_{AB} = -\nabla^{C'}_{(A}A_{B)C'} + 2\tau_{AB}{}^\mu A_\mu. \quad (21)$$

The contravariant form of (20) and (21) is written in both formalisms as

$$\phi^{AB} = \nabla^{(A}_{C'}\Phi^{B)C'} + 2\tau^{AB\mu}\Phi_\mu \quad (22)$$

and

$$\psi^{AB} = \nabla^{(A}_{C'}A^{B)C'} + 2\tau^{AB\mu}A_\mu, \quad (23)$$

where we have implemented the eigenvalue equations

$$\nabla_\mu\gamma_{AB} = i\alpha_\mu\gamma_{AB}, \quad \nabla_\mu\gamma^{AB} = -i\alpha_\mu\gamma^{AB}, \quad (24)$$

together with their conjugates and the definition

$$\alpha_\mu \doteq \partial_\mu\Phi + 2(\Phi_\mu + A_\mu), \quad (25)$$

with the quantity Φ being nothing else but the polar argument of the independent component of γ_{AB} (see Equation (40) below).

We next carry out the spinor translation of the individual pieces of Equations (14)-(16), with the purpose of paving the way for deriving the field equations at issue. Evidently, it will suffice to carry through the apposite procedures for either of the $\mathcal{F}$-bivectors of Equation (13). For the ∇-term of (15), say, we have

$$\nabla^{AA'}F^{(T)}_{AA'BB'} = \nabla^{AA'}(M_{A'B'}\psi_{AB}) + \text{c.c.}, \quad (26)$$

with the symbol "c.c." denoting an overall complex conjugate piece. In the γ-formalism, the right-hand side of Equation (26) reads

$$\nabla^{AA'}(\gamma_{A'B'}\psi_{AB}) + \text{c.c.} = (\nabla^A_{B'}\psi_{AB} - i\alpha^A_{B'}\psi_{AB}) + \text{c.c.}. \quad (27)$$

As $\nabla_\mu\varepsilon_{AB} = 0$ in both formalisms, the ε-formalism counterpart of (27) may be obtained by dropping the α-term

from it. By combining (18) and (19), we readily find the patterns

$$T^{AA'} F^{(T)}_{AA'BB'} = \left(\tau^{AM}{}_{MB'} - \tau_{B'M}{}^{AM'} \right) \psi_{AB} + \text{c.c.} \qquad (28)$$

and

$$T^{AA'MM'}{}_{BB'} F^{(T)}_{AA'MM'} = 2\tau^{AM}{}_{BB'} \psi_{AM} + \text{c.c.,} \qquad (29)$$

which just represent $T^{\mu} F^{(T)}_{\mu\lambda}$ and $T^{\mu\nu}{}_{\lambda} F^{(T)}_{\mu\nu}$ in either formalism. The γ-formalism version of the left-hand side of Equation (16) is given by

$$\nabla^{AA'*} F^{(T)}_{AA'BB'} = i \left[\left(\nabla^{A'}_B \psi_{A'B'} + i\alpha^{A'}_B \psi_{A'B'} \right) - \text{c.c.} \right], \qquad (30)$$

whereas the piece $^{*}T_{\lambda}{}^{\mu\nu} F^{(T)}_{\mu\nu}$ gets in each formalism translated into

$$^{*}T_{BB'}{}^{AA'MM'} F^{(T)}_{AA'MM'}$$
$$= i \left[\left(\tau_B{}^{AM}{}_{B'} \psi_{AM} - \text{c.c.} \right) + \left(\tau_{B'M}{}^{AM'} \psi_{AB} - \text{c.c.} \right) \right]. \qquad (31)$$

Towards completing our derivation procedures, it is convenient to require the unprimed and primed wave functions for either background to bear algebraic independence throughout $\mathfrak{M}$. This requirement enables us to manipulate the configurations involved in the spinor transcription we have carried out above in such a way that the left-right handedness characters of the fields become transparently separable. Therefore, by taking into account the equality

$$\tau^{AM}{}_{MB'} - \tau_{B'M}{}^{AM'} = T^A_{B'}, \qquad (32)$$

we obtain the field equation

$$\nabla^{AA'} (M_{A'B'} \psi_{AB}) + \frac{1}{2} m^2 A_{BB'} = s_{BB'}, \qquad (33)$$

with the complex dark energy source

$$s_{BB'} = 2 \left(\tau^{AM}{}_{BB'} \psi_{AM} - T^A_{B'} \psi_{AB} \right). \qquad (34)$$

It should be remarked that the term $\tau_B{}^{AM}{}_{B'} \psi_{AM}$, which is borne by Equation (31), cancels out at an intermediate step of the manipulations that yield the statement (33), and thence also so does its complex conjugate. In the γ-formalism, we then have

$$\nabla^A_{B'} \psi_{AB} - i\alpha^A_{B'} \psi_{AB} + \frac{1}{2} m^2 A_{BB'} = s_{BB'}, \qquad (35)$$

with the corresponding ε-formalism statement being spelt out as

$$\nabla^A_{B'} \psi_{AB} + \frac{1}{2} m^2 A_{BB'} = s_{BB'}. \qquad (36)$$

For the CMB, we get the γ-formalism massless field equation

$$\nabla^A_{B'} \phi_{AB} - i\alpha^A_{B'} \phi_{AB} = \mathfrak{s}_{BB'}, \qquad (37)$$

along with its ε-formalism counterpart

$$\nabla^A_{B'} \phi_{AB} = \mathfrak{s}_{BB'} \qquad (38)$$

and the geometric source

$$\mathfrak{s}_{BB'} = 2 \left(\tau^{AM}{}_{BB'} \phi_{AM} - T^A_{B'} \phi_{AB} \right). \qquad (39)$$

It was demonstrated in Ref. [28] that the wave-function pattern $\phi_A{}^B$ for the torsionless framework bears a commonness feature in that it is the same in both the classical formalisms. Inasmuch as the traditional algebraic definitions for metric spinors and connecting objects are formally appropriate for the torsionful framework as well, we can right away write the $\gamma\varepsilon$-relationships

$$C^{(\gamma)\ B}_{\mu\nu A} = C^{(\varepsilon)\ B}_{\mu\nu A} \Leftrightarrow C^{(\gamma)}_{\mu\nu AB} = \gamma C^{(\varepsilon)}_{\mu\nu AB}, \qquad (40)$$

where γ is the independent component of γ_{AB}. Consequently,[3] we can say that each of the pairs $(\phi_A^B, \phi_{A'}^{B'})$ and $(\psi_A^B, \psi_{A'}^{B'})$ possesses a commonness property which is seemingly similar to the classical one, in addition to holding in both formalisms a gauge invariant spin-tensor character. In each formalism, we thus have the field equations

$$\nabla^{AB'} \psi_A^B + \frac{1}{2} m^2 A^{BB'} = s^{BB'} \qquad (41)$$

and

$$\nabla^{AB'} \phi_A^B = \mathfrak{s}^{BB'}, \qquad (42)$$

where the ϕ-field relation, for instance,

$$\gamma_{CB} \nabla^{AB'} \phi_A^C = \nabla^{AB'} \phi_{AB} - i\alpha^{AB'} \phi_{AB}, \qquad (43)$$

has been used in the γ-formalism case.

4 Calculational techniques

By this point, we shall build up the techniques that yield in both formalisms the wave equations for the fields being considered. In fact, these techniques constitute a torsional version of the differential ones which had been developed originally within the classical $\gamma\varepsilon$-framework [28]. Let us begin with the operator decomposition

$$S^{\mu}_{AA'} S^{\nu}_{BB'} D_{\mu\nu} = M_{A'B'} \check{D}_{AB} + M_{AB} \check{D}_{A'B'}. \qquad (44)$$

Whence, implementing Equations (2) and (19), gives

$$\check{D}_{AB} = \Delta_{AB} + 2\tau_{AB}{}^{\mu} \nabla_{\mu}, \quad \Delta_{AB} \doteq -\nabla^C_{(A} \nabla_{B)C'}, \qquad (45)$$

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together with the complex conjugate of (45). The operators $\check{D}_{AB}$ and $\varDelta_{AB}$ both are linear and possess the Leibniz rule property.

It may be useful to utilize Equation (24) for reexpressing the γ-formalism operator $\varDelta_{AB}$ as

$$\varDelta_{AB} = \nabla_{C'(A}\nabla_{B)}^{C'} - i\alpha_{C'(A}\nabla_{B)}^{C'}. \tag{46}$$

In the ε-formalism, one has

$$\varDelta_{AB} = -\nabla_{(A}^{C'}\nabla_{B)C'} = \nabla_{C'(A}\nabla_{B)}^{C'}. \tag{47}$$

It is worth noticing that the γ-formalism contravariant form of $\varDelta_{AB}$ appears as

$$\varDelta^{AB} = -(\nabla^{C'(A}\nabla_{C'}^{B)} + i\alpha^{C'(A}\nabla_{C'}^{B)}), \tag{48}$$

or, equivalently, as

$$\varDelta^{AB} = \nabla_{C'}^{(A}\nabla^{B)C'}. \tag{49}$$

Because α_μ bears gauge invariance [39], the conjugate $\check{D}$-operators for the γ-formalism behave under gauge transformations as covariant spin tensors. In the ε-formalism, they correspondingly behave as invariant spin-tensor densities of weight -1 and antiweight -1.

Equations (1) and (44) suggest that some of the elementary $\check{D}$-derivatives should be prescribed in either formalism by

$$\check{D}_{AB}\zeta^C = \varpi_{ABM}{}^C\zeta^M, \quad \check{D}_{A'B'}\zeta^C = \varpi_{A'B'M}{}^C\zeta^M, \tag{50}$$

with the spin-curvature expansion

$$C_{AA'BB'CD} = M_{A'B'}\varpi_{ABCD} + M_{AB}\varpi_{A'B'CD}, \tag{51}$$

and the relationships

$$\varpi_{ABCD}^{(\gamma)} = \gamma^2\varpi_{ABCD}^{(\varepsilon)}, \quad \varpi_{A'B'CD}^{(\gamma)} = |\,\gamma\,|^2\,\varpi_{A'B'CD}^{(\varepsilon)}, \tag{52}$$

which clearly agree with (40). We can show [39] that the spinor pair

$$\mathbf{G} = \left(\varpi_{AB(CD)}, \varpi_{A'B'(CD)}\right) \tag{53}$$

constitutes the irreducible decomposition of the Riemann tensor for ∇_μ. Its unprimed entry is expandable as[4]

$$X_{ABCD} = \Psi_{ABCD} - M_{(A|(C}\xi_{D)|B)} - \frac{1}{3}\varkappa M_{A(C}M_{D)B}, \tag{54}$$

with

$$\Psi_{ABCD} = X_{(ABCD)}, \quad \xi_{AB} = X^M{}_{(AB)M}, \quad \varkappa = X_{LM}{}^{LM}, \tag{55}$$

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4 From now on, we will for simplicity employ the definitions $X_{ABCD} \doteq \varpi_{AB(CD)}$ and $\Xi_{A'B'CD} \doteq \varpi_{A'B'(CD)}$.

and the Ψ-spinor defining a typical wave function for gravitons in $\mathfrak{M}$. Likewise, the contracted pieces $(\varpi_{ABM}{}^M, \varpi_{A'B'M}{}^M)$ fulfill the additivity relations (5) and (9), and are thereby proportional to the wave functions of (17) and (18) according to the schemes

$$\widetilde{\varpi}_{ABM}{}^M = -2i\phi_{AB}, \quad \widetilde{\varpi}_{A'B'M}{}^M = -2i\phi_{A'B'} \tag{56}$$

and

$$\varpi_{ABM}^{(T)}{}^M = -2i\psi_{AB}, \quad \varpi_{A'B'M}^{(T)}{}^M = -2i\psi_{A'B'}. \tag{57}$$

Hence, we can cast the prescriptions (50) into the form

$$\check{D}_{AB}\zeta^C = X_{ABM}{}^C\zeta^M - i\,(\phi_{AB} + \psi_{AB})\,\zeta^C \tag{58}$$

and

$$\check{D}_{A'B'}\zeta^C = \Xi_{A'B'M}{}^C\zeta^M - i\,(\phi_{A'B'} + \psi_{A'B'})\,\zeta^C. \tag{59}$$

The prescriptions for computing $\check{D}$-derivatives of a covariant spin vector η_A can be obtained out of (50) by assuming that

$$\check{D}_{AB}\left(\zeta^C\eta_C\right) = 0, \quad \check{D}_{A'B'}\left(\zeta^C\eta_C\right) = 0, \tag{60}$$

and carrying out Leibniz expansions thereof. We thus have

$$\check{D}_{AB}\eta_C = -\left[X_{ABC}{}^M\eta_M - i\,(\phi_{AB} + \psi_{AB})\,\eta_C\right] \tag{61}$$

and

$$\check{D}_{A'B'}\eta_C = -\left[\Xi_{A'B'C}{}^M\eta_M - i\,(\phi_{A'B'} + \psi_{A'B'})\,\eta_C\right], \tag{62}$$

along with the complex conjugates of Equations (58)-(62). The $\check{D}$-derivatives of a differentiable complex spin-scalar density α of weight $\mathfrak{w}$ on $\mathfrak{M}$ are written out explicitly as

$$\check{D}_{AB}\alpha = 2i\mathfrak{w}\alpha\,(\phi_{AB} + \psi_{AB}), \quad \check{D}_{A'B'}\alpha = 2i\mathfrak{w}\alpha\,(\phi_{A'B'} + \psi_{A'B'}). \tag{63}$$

These configurations may in both formalisms be regarded as immediate consequences of the integrability condition

$$D_{\mu\nu}\alpha = 2i\mathfrak{w}\alpha\left(\widetilde{F}_{\mu\nu} + F_{\mu\nu}^{(T)}\right). \tag{64}$$

When acting on a world-spin scalar h, the $\check{D}$-operators then recover the defining relation $D_{\mu\nu}h = 0$ as

$$\check{D}_{AB}h = 0, \quad \check{D}_{A'B'}h = 0, \tag{65}$$

whence

$$\varDelta_{AB}h = -2\tau_{AB}{}^\mu\nabla_\mu h. \tag{66}$$

Of course, the patterns for $\check{D}$-derivatives of some spin-tensor density can be specified from Leibniz expansions like

$$\check{D}_{AB}\left(\alpha B_{C\ldots D}\right) = \left(\check{D}_{AB}\alpha\right)B_{C\ldots D} + \alpha\check{D}_{AB}B_{C\ldots D}, \tag{67}$$

with $B_{C...D}$ being a spin tensor.

As for the old $\gamma\varepsilon$-framework, whenever $\check{D}$-derivatives of Hermitian quantities are actually computed in either formalism, the wave function contributions carried by the expansions (58)-(62) get cancelled. Such a cancellation likewise happens when we let the $\check{D}$-operators act freely upon spin tensors having the same numbers of covariant and contravariant indices of the same kind. For $\mathfrak{w} < 0$, it still occurs in the expansion (67) when $B_{C...D}$ is taken to carry $-2\mathfrak{w}$ indices and $\mathrm{Im}\alpha \neq 0$ everywhere. A similar property also holds for situations that involve outer products between contravariant spin tensors and complex spin-scalar densities having suitable positive weights. The gauge behaviours specified in the foregoing Section tell us that such weight-valence properties neatly fit in with the case of the ε-formalism wave functions.

In carrying out the procedures for deriving our wave equations, it may become necessary to take account of the algebraic rules

$$2\nabla^{A'}_{[B}\nabla_{A]A'} = M_{AB}\Box = \nabla^{A'}_C \left(M_{BA}\nabla^C_{A'}\right) \qquad (68)$$

and

$$2\nabla^{[A}_{A'}\nabla^{B]A'} = M^{AB}\Box = \nabla^C_{A'}\left(M^{BA}\nabla^{A'}_C\right), \qquad (69)$$

along with the operator splittings

$$\nabla^C_A\nabla_{BC'} = \frac{1}{2}M_{BA}\Box - \Delta_{AB}, \ \nabla^A_{A'}\nabla^{BA'} = \Delta^{AB} + \frac{1}{2}M^{AB}\Box \quad (70)$$

and the gauge invariant definition

$$\Box \doteq \nabla_{AA'}\nabla^{AA'}. \qquad (71)$$

Owing to the applicability in both formalisms of the metric compatibility condition

$$\nabla_\mu(M_{AB}M_{A'B'}) = 0, \qquad (72)$$

we can reset (71) as

$$\Box = \nabla^{AA'}\nabla_{AA'}. \qquad (73)$$

In addition, from the equations

$$\Box\gamma_{AB} = \Theta\gamma_{AB}, \ \Box\gamma^{AB} = \overline{\Theta}\gamma^{AB}, \qquad (74)$$

whose derivation involves using the eigenvalue carried by (24) together with

$$\Theta \doteq -\alpha^\mu\alpha_\mu + i\nabla_\mu\alpha^\mu, \qquad (75)$$

we also get the symbolic γ-formalism devices

$$(\Box\iota_A{}^C)\gamma_{CB} = \left(\Box - 2i\alpha^\mu\nabla_\mu + \overline{\Theta}\right)\iota_{AB} \qquad (76)$$

and

$$\gamma^{AC}(\Box\iota_C{}^B) = \left(\Box + 2i\alpha^\mu\nabla_\mu + \Theta\right)\iota^{AB}, \qquad (77)$$

which obey the valence-interchange rule[5]

$$i\alpha^\mu\nabla_\mu \leftrightarrow -i\alpha^\mu\nabla_\mu, \ \Theta \leftrightarrow \overline{\Theta}. \qquad (78)$$

In the γ-formalism, the $\Box$-correlations for ι_{AB} and ι^{AB} can then be achieved from

$$\gamma_{AC}\gamma_{BD}\Box\iota^{CD} = \left(\Box - 4i\alpha^\mu\nabla_\mu - Y\right)\iota_{AB} \qquad (79)$$

and

$$\gamma^{AC}\gamma^{BD}\Box\iota_{CD} = \left(\Box + 4i\alpha^\mu\nabla_\mu - \overline{Y}\right)\iota^{AB}, \qquad (80)$$

which conform to Equation (78) with $Y = 2(\alpha^\mu\alpha_\mu - \overline{\Theta})$.

5 Wave equations

To obtain the entire set of wave equations that govern the propagation of both physical backgrounds in $\mathfrak{M}$, we initially follow up the simpler procedure which consists in implementing the calculational techniques exhibited anteriorly to work out the field equation of either formalism for the common dark energy wave function ψ^B_A. It will become manifest that a gauge invariant condition for each entry of the pairs $(\psi^B_A, \psi^{B'}_{A'})$ and $(\phi^B_A, \phi^{B'}_{A'})$ can be established as a geometric consequence of the symmetry of the underlying fields. Rather than elaborating upon Equation (35), which could unnecessarily produce some complicated manipulations, we will deduce the γ-formalism wave equations for the unprimed pair (ψ_{AB}, ψ^{AB}) by appealing to the differential devices (76) and (77). We may certainly get the wave equations for any primed ψ-fields by taking complex conjugates. The wave equations for all ϕ-fields shall then arise in a trivial way, provided that the field equations for both backgrounds carry formally the same couplings between the wave functions and torsion spinors.

We start by operating with $\nabla^C_{B'}$ on the configuration of Equation (41). Thus, recalling the contravariant splitting of (70) leads us to the statement

$$\Delta^{AC}\psi^B_A - \frac{1}{2}M^{AC}\Box\psi^B_A + \frac{1}{2}m^2\nabla^C_{B'}A^{BB'} = \nabla^C_{B'}s^{BB'}. \qquad (81)$$

It is obvious that both first-order derivative kernels of (81) are of the type

$$\nabla^C_{B'}u^{BB'} = \nabla^{(B}_{B'}u^{C)B'} - \frac{1}{2}M^{BC}\nabla_\mu u^\mu, \qquad (82)$$

---

[5] The rule (78) had also arisen in Ref. [28] in connection with the derivation of the wave equations for the CMB and gravitons in torsionless environments.

with the symmetric piece for the potential being given by

$$\nabla_{B'}^{(B} A^{C)B'} = \psi^{BC} - 2\tau^{BC\mu} A_\mu, \tag{83}$$

in accordance with (23). By virtue of the relation (45), the Δ-piece of (81) may be rewritten in either formalism as

$$\Delta^{AC}\psi_A^B = \check{D}^{AC}\psi_A^B - 2\tau^{AC\mu}\nabla_\mu\psi_A^B. \tag{84}$$

Furthermore, calling for (58) and (61) along with the expansion (54), after some computations, we get the contributions

$$\check{D}^{A(B}\psi_A^{C)} = \Psi^{ABC}{}_M\psi_A^M + \frac{2}{3}\varkappa\psi^{BC} - \psi_M^{(B}\xi^{C)M} \tag{85}$$

and

$$\check{D}^{A[B}\psi_A^{C]} = M^{BC}\psi_{AM}\xi^{AM}. \tag{86}$$

We can see that the symmetry property of the wave functions entails imparting symmetry in the indices B and C to the $\square$-block of (81), which means that

$$M^{A[B}\square\psi_A^{C]} = \frac{1}{2}M^{BC}M^A{}_D\square\psi_A^D \equiv 0. \tag{87}$$

In both formalisms, Equation (87) thus implies that

$$2\Delta^{A[C}\psi_A^{B]} = M^{BC}\left(\frac{1}{2}m^2\nabla_\mu A^\mu - \nabla_\mu s^\mu\right), \tag{88}$$

while the relations (84) and (86) yield the expression

$$\Delta^{A[C}\psi_A^{B]} = M^{BC}\left(\tau_M^{A\mu}\nabla_\mu\psi_A^M - \psi_{AM}\xi^{AM}\right). \tag{89}$$

So, utilizing Equation (34) and working out the $\tau\nabla\psi$-term of (89) to the extent that

$$\tau_M^{A\mu}\nabla_\mu\psi_A^M = -\left[\frac{1}{2}\nabla_\mu s^\mu + \nabla_{CB'}\left(T^{AB'}\psi_A^C\right) + \psi_A^M\nabla_\mu\tau_M^{A\mu}\right], \tag{90}$$

we arrive at the condition[6]

$$\frac{1}{4}m^2\nabla_\mu A^\mu + \nabla_{CB'}\left(T^{AB'}\psi_A^C\right) + \psi_A^M\nabla_\mu\tau_M^{A\mu} - \psi_A^M\xi_M^A = 0. \tag{91}$$

For ϕ_A^B, we similarly obtain the massless condition

$$\nabla_{CB'}\left(T^{AB'}\phi_A^C\right) + \phi_A^M\nabla_\mu\tau_M^{A\mu} - \phi_A^M\xi_M^A = 0, \tag{92}$$

along with the complex conjugates of (91) and (92).

The property (87) stipulates in either formalism that the only contributions to the wave equation for ψ_A^B are those produced by the symmetric pieces in B and C of the corresponding configuration (81). Hence, carrying out a symmetrization over the indices B and C of (81), likewise

fitting together the pieces of Equations (83)-(85) and rearranging indices adequately thereafter, we end up with the dark energy equation

$$\left(\square + \frac{4}{3}\varkappa + m^2\right)\psi_A^B + 2\Psi^{LB}{}_{MA}\psi_L^M = 2\beta_A^B, \tag{93}$$

with

$$\beta^{AB} = \nabla_{B'}^{(A} s^{B)B'} + \psi_M^{(A}\xi^{B)M} + 2(\nabla_\mu\psi_M^{(A})\tau^{B)M\mu} + m^2\tau^{AB\mu}A_\mu. \tag{94}$$

We should emphasize that the statements (91)-(93) are formally the same in both formalisms, and additionally bear gauge invariance because of the behaviour of A_μ as specified by Equation (11). Indeed, it is the masslessness of the CMB fields that ensures the absence from (92) of a term proportional to $\nabla_\mu\Phi^\mu$.

It now becomes clear that the application to Equation (93) of the correlations supplied by (76) and (77), allows us to attain quite easily the γ-formalism version of the wave equations for ψ_{AB} and ψ^{AB}. In effect, we have

$$\left(\square - 2i\alpha^\mu\nabla_\mu + \overline{\Theta} + \frac{4}{3}\varkappa + m^2\right)\psi_{AB} - 2\Psi_{AB}{}^{LM}\psi_{LM} = 2\beta_{AB} \tag{95}$$

and

$$\left(\square + 2i\alpha^\mu\nabla_\mu + \Theta + \frac{4}{3}\varkappa + m^2\right)\psi^{AB} - 2\Psi^{AB}{}_{LM}\psi^{LM} = 2\beta^{AB}, \tag{96}$$

which satisfy the rule (78). For the ε-formalism, we obtain

$$\left(\square + \frac{4}{3}\varkappa + m^2\right)\psi_{AB} - 2\Psi_{AB}{}^{LM}\psi_{LM} = 2\beta_{AB} \tag{97}$$

and

$$\left(\square + \frac{4}{3}\varkappa + m^2\right)\psi^{AB} - 2\Psi^{AB}{}_{LM}\psi^{LM} = 2\beta^{AB}. \tag{98}$$

We notice that the ε-formalism lower-index version of β^{AB} is expressed simply as

$$\beta_{AB} = \nabla_{B'(A} s_{B)}^{B'} - \psi_{(A}^M\xi_{B)M} - 2(\nabla_\mu\psi_{(A}^M)\tau_{B)M}{}^\mu + m^2\tau_{AB}{}^\mu A_\mu. \tag{99}$$

Due to the occurrence of the same formal geometric patterns on the right-hand sides of the field equations of Section 3, we can promptly obtain the CMB wave equations from the statements (93)-(98) by first setting $m = 0$ and then replacing wave functions appropriately. In either formalism, we thus have

$$\left(\square + \frac{4}{3}\varkappa\right)\phi_A^B + 2\Psi^{LB}{}_{MA}\phi_L^M = 2\eta_A^B, \tag{100}$$

with

$$\eta^{AB} = \nabla_{B'}^{(A} s^{B)B'} + \phi_M^{(A}\xi^{B)M} + 2(\nabla_\mu\phi_M^{(A})\tau^{B)M\mu} \tag{101}$$

---

[6] When Equations (88)-(90) are combined together, the terms that involve $\nabla_\mu s^\mu$ explicitly get cancelled.

and $\mathfrak{s}_\mu$ being given by (39). The γ-formalism equations for (ϕ_{AB}, ϕ^{AB}) accordingly appear as

$$\left(\Box - 2i\alpha^\mu \nabla_\mu + \overline{\Theta} + \frac{4}{3}\varkappa\right)\phi_{AB} - 2\Psi_{AB}{}^{LM}\phi_{LM} = 2\eta_{AB}$$
(102)

and

$$\left(\Box + 2i\alpha^\mu \nabla_\mu + \Theta + \frac{4}{3}\varkappa\right)\phi^{AB} - 2\Psi^{AB}{}_{LM}\phi^{LM} = 2\eta^{AB},$$
(103)

whereas the ε-formalism counterparts of Equations (102) and (103) are stated as

$$\left(\Box + \frac{4}{3}\varkappa\right)\phi_{AB} - 2\Psi_{AB}{}^{LM}\phi_{LM} = 2\eta_{AB}$$
(104)

and

$$\left(\Box + \frac{4}{3}\varkappa\right)\phi^{AB} - 2\Psi^{AB}{}_{LM}\phi^{LM} = 2\eta^{AB}.$$
(105)

6 Concluding remarks and outlook

The description we have just proposed here has been based upon the belief that the spinor structures of generally relativistic spacetimes should support locally a geometric description of the microwave and dark energy backgrounds of the universe. Because of the fact that any torsional affine potentials must always enter geometric prescriptions together with adequate torsionless companions, we could definitely establish that any torsional two-component spinor description of the dark energy background must be accompanied by a description of the CMB. We saw that all wave functions couple to the pieces of the spinor decomposition for the torsion tensor of $\mathfrak{M}$. They also interact with curvatures via couplings like, say, the $\Psi\psi$ and $\Psi\phi$ ones carried by Equations (97) and (104). However, they do not interact with one another whence we can say that one background propagates in $\mathfrak{M}$ as if the other were absent. This result appears to be in full agreement with the suggestion made earlier in Ref. [33] by which the CMB may propagate alone in spacetimes equipped with torsionless affinities as Infeld-van der Waerden photons.

One of the striking aspects of the procedures implemented in Section 5, is related to the gauge invariance of the condition (92), which takes place because the masslessness of the CMB fields annihilates either $\gamma\varepsilon$-contribution that carries the non-invariant piece $\nabla_\mu\Phi^\mu$. It should be stressed that the occurrence of the massive condition (91) rests upon the torsionfulness intrinsically borne by Equation (89). In the limiting case of the torsionless framework, the derivative $\Delta^{A[C}\phi_A^{B]}$ becomes an identically

vanishing contribution in both formalisms, and Equations (93)-(98) all "evaporate" together with the source $\mathfrak{s}^\mu$ and the curvature spinor ξ_{AB}. Under this circumstance, the world-spin scalar $\varkappa$ bears reality and satisfies the equality

$$4\varkappa = R,$$

with R being the Ricci scalar of ∇_μ. Hence, the electromagnetic wave equations of Ref. [32] may be recovered, with the physical significance described in Ref. [33] being of course effectively ascribed to them.

We expect that the subsidiary conditions involved in the derivation of the wave equations for the dark energy background could perhaps shed some light on the physical meaning of the right-hand side of Einstein-Cartan's field equations. We also believe that a distributional treatment of our wave equations could be of considerable importance insofar as it may provide us with local theoretical evaluations of the dynamical properties of dark energy, including the feasibility of performing explicit calculations towards making direct comparisons with data coming from the observed anisotropy of the CMB. One of our hopes is that the role played by spacetime torsion could be actually further strengthened. It is considerably interesting to point out that the calculational techniques developed in Section 4 can supply geometric tools for describing the propagation of gravitons and Dirac particles in torsional cosmic environments, in combination with the mechanisms that may avert gravitational singularities as particularly exhibited in Ref. [43].

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