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The role of coherence on two-particle quantum walks

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Abstract: We investigate the dynamical properties of the two-boson quantum walk in systems with different degrees of coherence, and where the effect of the coherence on the two-boson quantum walk can be naturally introduced. A general analytical expression for the two-boson correlation function, for both pure states and mixed states, is given. We propose a possible two-photon quantum-walk scheme with a mixed initial state, and find that the two-photon correlation function and the average distance between two photons can be influenced by the initial photon distribution, the relative phase, or the degree of coherence. The propagation features of our numerical results can be explained by our analytical two-photon correlation function.

Keywords: wo-particle quantum walk, degree of coherence, two-photon correlation function, pure state, mixed state

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1 Introduction

As the quantum mechanical counterpart of the classical random walk [1], the quantum random walk has been increasingly receiving attention because of potential applications ranging from quantum information to simulation of physical phenomena. For example, the quantum walk offers an advanced tool for building a quantum algorithm [2–4] that is shown to be the basis for universal quantum computations [5–8]. We know that quantum walks include two main classes: discrete-time quantum walks, and continuous-time quantum walks [9, 10]. The continuous-

time quantum walk can evolve continuously with time through tunneling between neighbor sites and does not require quantum coin to generate superposition of states. This means the continuous-time quantum walk can be implemented via constant tunneling of quantum particles at several possible lattice sites. So far, the quantum walks of single particles have been studied in experiments by using either classical waves [11], single photons [12, 13], or single atoms [14, 15]. Additionally, quantum walks of two correlated photons trapped in waveguide lattices have also been studied in experiments [16, 17].

Note that many-particle quantum walks can exhibit more quantum features than single-particle quantum walks. The reason for this is that single-particle quantum walks can be exactly mapped to classical wave phenomena [18] but for quantum walks of more than one indistinguishable particle, the classical theory cannot provide sufficient description. In Ref. [19], the authors theoretically studied a discrete-time quantum walk of two particles and demonstrated distinctly nonclassical correlations. Meanwhile, the effect of interactions between particles on a quantum walk of two indistinguishable particles was theoretically studied in Ref. [20, 21], where the system was assumed to be completely coherent and the influence of the degree of coherence was not studied. We know that, except for the interaction between particles, other factors, *e.g.*, the initial states, the quantum-walk parameters and the degree of coherence of the system, can also affect the features of two-particle quantum walks. In particular, we know that the major challenge to experimentally realize quantum walks of correlated particles is to find a low-decoherence system that can preserve the nonclassical features of quantum walks [17]; this implies that the influence of the degree of coherence of the system on two-particle quantum walks is important. Investigating the properties of two-particle quantum walks with attention to different degrees of coherence is, therefore, worthwhile, since the decoherence effects in quantum walks have potential algorithmic applications [22].

In this paper, we propose a density matrix formalism to study a two-particle quantum walk where the degrees of coherence can be naturally introduced. With the help of the Heisenberg equation of motion, we derive a general

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analytical expression of the two-particle correlation function. As a concrete example, we propose a possible two-photon quantum-walk scheme, with a mixed initial state, to exhibit the quantum features of the two-particle quantum walk via the two-particle correlation and the average distance between the two particles. Our results indicate that the propagation of the two particles depends not only on the initial distribution of those particles, but also on the relative phase and the degree of coherence of the system. Such a propagation feature can be explained by our analytical two-particle correlation function. In the next section, we present the model and derive the analytical expression of the two-particle correlation. In Sec. 3, we propose a concrete scheme to show some dynamical features of two-particle quantum walks. Our main conclusions are summarized in Sec. 4.

2 A general formulation

We consider a two-particle quantum walk in a one-dimensional lattice space. The propagation of the two particles is described by the evolution of the state of a tight-binding model,

$$H = - \sum_q T_{q,q+1} (\hat{a}_q^\dagger \hat{a}_{q+1} + \text{h.c.}) + \sum_q \beta_q \hat{a}_q^\dagger \hat{a}_q, \quad (1)$$

where the operators \hat{a}_q^\dagger and \hat{a}_q respectively create and annihilate a bosonic particle at site q . Here the parameter $T_{q,q+1}$ refers to the tunneling strength of particles between the nearest neighbor sites, and

$$\beta_q = T_{q+1,q} + T_{q-1,q}. \quad (2)$$

Note that the above form of β_q was chosen to keep the probability conservation in the proposal of continuous-time quantum walk via decision tree [10]. Now, more generally, the value of β_q can be arbitrary due to the probability conservation being naturally satisfied in quantum mechanics. If the tunneling strength $T_{q,q+1}$ is a constant, β_q will become a constant for the periodic boundary condition. In this case, the value of β_q does not affect the dynamical properties of the quantum-walk system. For the open boundary condition, however, the values of β_q for the two boundary sites are different from those for the other sites. This can naturally introduce two defects to the quantum-walk system. Note that the effect of defects on single-particle quantum walks was studied in Ref. [24].

Since we consider a two-boson system, the Fock bases describing the system are

$$|1\rangle_q |1\rangle_r = \frac{1}{\sqrt{1 + \delta_{q,r}}} \hat{a}_q^\dagger \hat{a}_r^\dagger |\text{vac}\rangle, \quad (3)$$

where $\delta_{q,r}$ denotes the Kronecker delta. Equation (3) represents a two-particle state with one Particle on the q th site and the other one on the r th site. Note that $|1\rangle_q |1\rangle_r$ is regarded as identical to $|1\rangle_r |1\rangle_q$ for the indistinguishable particles considered here. The two bosonic particles can be at the same site (i.e., $q = r$) Thus dimension of the Hilbert space, expanded by the aforementioned Fock bases, is $D = L(L + 1)/2$. Here L denotes the number of sites. We know that the propagation of the two particles is determined not only by the property of the waveguide lattice but also by the two-particle input state. If the two particles are initially in a pure state, the two-particle input state can be expressed as a wavefunction, namely, a coherent superposition of the Fock bases,

$$|\psi\rangle = \sum_{q,r} c_{q,r} |1\rangle_q |1\rangle_r, \quad (4)$$

where $\sum_{q,r} |c_{q,r}|^2 = 1$. However, if the two particles are in a mixed state at the initial time, the two-particle input state needs to be described by a density matrix rather than the wavefunction. Such a density matrix is given by

$$\rho = \sum_{qr,q'r'} \rho_{qr,q'r'} (|1\rangle_q |1\rangle_r) (\langle 1|_{q'} \langle 1|_{r'}), \quad (5)$$

which is a $D \times D$ matrix. We know that $\text{Tr}\rho^2 \leq (\text{Tr}\rho)^2$ where the equals sign holds only for pure states. In the following, we will focus on a two-particle quantum walk for mixed input states.

Now we are in a position to study the propagation of the two particles with the help of the Heisenberg equation of motion for the creation operators, namely,

$$i \frac{\partial \hat{a}_q^\dagger}{\partial t} = \beta_q \hat{a}_q^\dagger + T_{q,q+1} \hat{a}_{q+1}^\dagger + T_{q,q-1} \hat{a}_{q-1}^\dagger, \quad (6)$$

where we set $\hbar = 1$ for simplicity of calculation. The creation operator \hat{a}_q^\dagger at any time can be obtained with the help of Eq. (6),

$$\hat{a}_q^\dagger(t) = \sum_r U_{q,r}(t) \hat{a}_r^\dagger(0), \quad U(t) = e^{-iHt}, \quad (7)$$

where $U_{q,r}(t)$ is the probability amplitude of a single particle transiting from the q th waveguide to the r th one. To explore the quantum behaviors of the two-particle quantum walk, let us firstly evaluate the two-particle correlation function $\Gamma_{k,l}(t) = \langle \hat{a}_k^\dagger(t) \hat{a}_l^\dagger(t) \hat{a}_l(t) \hat{a}_k(t) \rangle$ which manifests the probability that the two particles are coincident

in the k th and the l th waveguide [16, 25]. Since the two-particle input state can be described by the density matrix given in Eq. (5), the expectation value of any observable of the system can be calculated via $\langle \hat{O}(t) \rangle = \text{Tr}(\hat{O}(t)\rho)$. Then we obtain an expression for the two-particle correlation function

$$\begin{aligned} \Gamma_{k,l}(t) = & \sum_{q \neq r, q' \neq r'} \rho_{qr, q' r'} \left(U_{kq'} U_{lr'} U_{lq}^* U_{kr}^* + U_{kq'} U_{lr'} U_{lr}^* U_{kq}^* + U_{kr'} U_{lq'} U_{lq}^* U_{kr}^* + U_{kr'} U_{lq'} U_{lr}^* U_{kq}^* \right) \\ & + \sum_{q, q' \neq r'} \sqrt{2} \rho_{qq, q' r'} \left(U_{kq'} U_{lr'} U_{lq}^* U_{kq}^* + U_{kr'} U_{lq'} U_{lq}^* U_{kq}^* \right) + \sum_{q \neq r, q'} \sqrt{2} \rho_{qr, q' q'} \left(U_{kq'} U_{lq'} U_{lq}^* U_{kr}^* + U_{kq'} U_{lq'} U_{lr}^* U_{kq}^* \right) \\ & + \sum_{q, q'} 2 \rho_{qq, q' q'} U_{kq'} U_{lq'} U_{lq}^* U_{kq}^*, \end{aligned} \quad (8)$$

which presents a general form for either pure initial input states or mixed ones. One can obtain the two-particle correlation at any time, as long as the density matrix corresponding to the input state is given. With the help of such an expression for the two-particle correlation function, many dynamical features of two-particle quantum walk can be explained.

3 Two-photon quantum walk for a concrete input state

In order to expose the quantum properties of two-particle quantum walks more clearly, we turn to a concrete example where the particles are assumed to be photons. We know that each beam can become two coherent beams after propagating through a grating [28] and the pure two-photon input states can be experimentally realized via injecting two coherent beams into a waveguide lattice [16]. Then we suppose that there are two incoherent light beams and the relation of their intensities is $\cos^2 \delta : \sin^2 \delta$. The two incoherent beams propagate through separate gratings, and are then simultaneously injected into the waveguide arrays. The two incoherent beams will create two pure two-photon states, ψ_1 and ψ_2 respectively [16, 28]. Because the initial two beams are not coherent, the initial state of the system needs to be described by the density matrix

$$\rho = \cos^2 \delta |\psi_1\rangle\langle\psi_1| + \sin^2 \delta |\psi_2\rangle\langle\psi_2|. \quad (9)$$

As an example, we take $\psi_1 = \cos \frac{\theta}{2} |2\rangle_1 + \sin \frac{\theta}{2} e^{i\phi} |2\rangle_0$ and $\psi_2 = |2\rangle_1$, then the initial density matrix is

$$\rho = \rho_{00,11} |2\rangle_0\langle 2|_1 + \rho_{11,00} |2\rangle_1\langle 2|_0 + \rho_{11,11} |2\rangle_1\langle 2|_1 + \rho_{00,00} |2\rangle_0\langle 2|_0 \quad (10)$$

with $\rho_{00,11} = \cos^2 \delta \cos \frac{\theta}{2} \sin \frac{\theta}{2} e^{i\phi}$, $\rho_{11,00} = \rho_{00,11}^*$, $\rho_{11,11} = \cos^2 \delta \cos^2 \frac{\theta}{2} + \sin^2 \delta$, and $\rho_{00,00} = \cos^2 \delta \sin^2 \frac{\theta}{2}$. Here $|2\rangle_q$ stands for $|1\rangle_q |1\rangle_q$ whose definition was given in Eq. (3). We find that the other matrix elements of ρ are zeros, except for the above four elements. Since these elements can be changed via δ , θ and ϕ , without losing the generality, we redefine them as $\rho_{00,00} = \alpha$, $\rho_{11,11} = 1 - \alpha$ and $\rho_{00,11} = \rho_{11,00}^* = e^{i\phi} \sqrt{\eta - 1 - 4\alpha^2 + 4\alpha}/2$ with $0 \leq \alpha \leq 1$. Here the parameter $\eta = 2\text{Tr}(\rho^2) - 1$ ($0 \leq \eta \leq 1$) is introduced to characterize the degree of coherence [29]. We have $\eta = 1$ when the system is in a pure state, otherwise $\eta < 1$. With the help of Eq. (8), the two-photon correlation of the system for the mixed state shown in Eq. (10) yields

$$\Gamma_{q,r} = 2\gamma \text{Re}(e^{i\phi} U_{q0} U_{r0} U_{r1}^* U_{q1}^*) + 2\alpha |U_{r0} U_{q0}|^2 + 2(1 - \alpha) |U_{r1} U_{q1}|^2, \quad (11)$$

where $\gamma = \sqrt{\eta - 1 + 4\alpha(1 - \alpha)}$. This implies that the two-photon correlation depends not only on the initial probability distribution of the two photons, but also on the degree of coherence and the relative initial phase of the system. The first term in Eq. (11) is a coherent one that reveals well the quantum nature of two-photon quantum walk. Taking a pure initial state (*i.e.*, $\eta = 1$) as an example, the two-photon correlation function (11) becomes $\Gamma = |\sqrt{2\alpha} e^{i\phi} U_{r0} U_{q0} + \sqrt{2(1 - \alpha)} U_{r1} U_{q1}|^2$, which implies that the two-photon correlation can take place when the two photons from the 0th site propagate to the q th and the r th sites, respectively, or when the two photons from the 1st site to the q th and the r th sites, respectively. Due to the two photons being indistinguishable, the two paths can interfere, which is essentially Hanbury Brown Twiss (HBT) interference [30].

Now we investigate the quantum features of a two-photon quantum walk by considering a waveguide array consisting of $(2l+1)$ identical waveguides. In this case, the tunneling strengths between nearest-neighbor arrays are all the same, *i.e.*, $T_{q,r} = C$ with C being a constant, and β_q becomes a constant $2C$ for the periodical boundary condition considered. Then $U_{q,r}(t)$ becomes $e^{i2Ct} i^{q-r} J_{q-r}(2Ct)$ where J_q is the q th order Bessel function [26, 27]. With the help of Eq. (11), we can write out the two-photon correlation function in terms of Bessel functions.

$$\Gamma_{q,r}(\tau) = -2\gamma \cos \phi J_q(\tau) J_r(\tau) J_{r-1}(\tau) J_{q-1}(\tau) + 2\alpha [J_q(\tau) J_r(\tau)]^2 + 2(1-\alpha) [J_{r-1}(\tau) J_{q-1}(\tau)]^2. \quad (12)$$

where $\tau = 2Ct$.

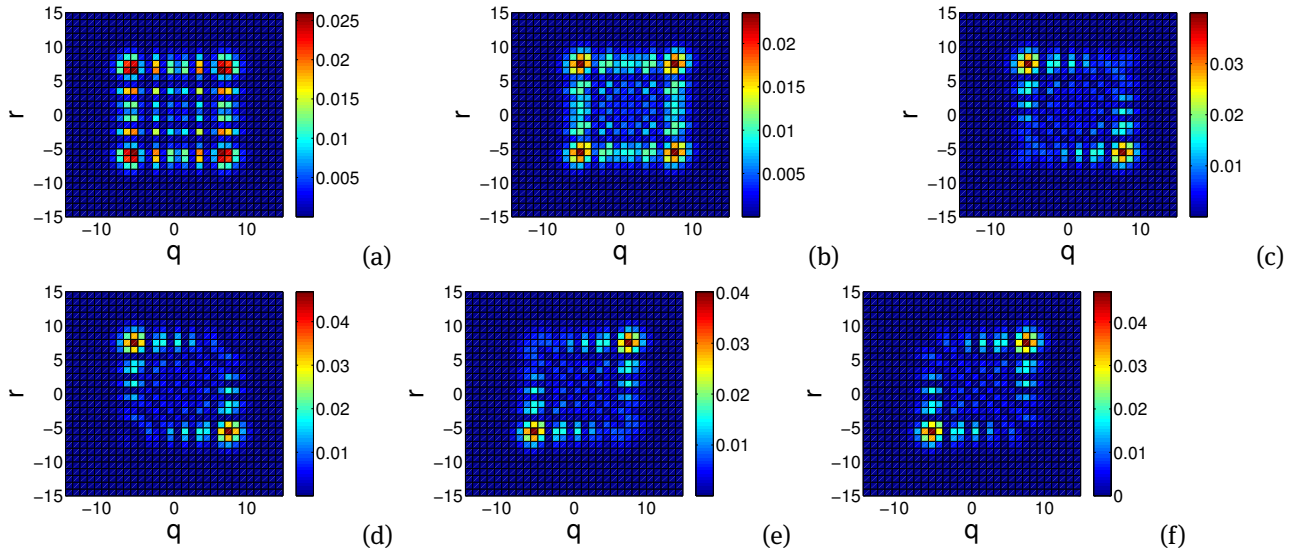


Figure 1: (Color online) Two-photon correlation at time $t = 4(1/C)$ for different initial conditions. The initial condition is (a) $\alpha = 1, \eta = 1, \phi = 0$ (b) $\alpha = 0.5, \eta = 0, \phi = 0$ (c) $\alpha = 0.5, \eta = 0.5, \phi = 0$ (d) $\alpha = 0.5, \eta = 1, \phi = 0$ (e) $\alpha = 0.5, \eta = 0.5, \phi = \pi$, and (f) $\alpha = 0.5, \eta = 1, \phi = \pi$.

In Fig. 1, we plotted the two-photon correlation matrix at time $t = 4(1/C)$ for different initial conditions, where $1/C$ is the unit of time (*i.e.*, the time t is in the unit of the inverse of tunneling strength). We can see that each particle can be found on either side of the origin after propagation, which is reflected in the four symmetric peaks in Fig. 1 (a). For this case (*i.e.*, $\alpha = 1$ and $\eta = 1$), the system is in a pure state and Eq. (11) becomes $\Gamma_{q,r} = 2|U_{r0}U_{q0}|^2$ which is the same as the result in Ref. [16]. Such a correlation function is just a product of the two classical probability distributions, so there is no interference and the photons propagate in the ballistic direction. From Fig. 1 (b), we find that, just like Fig. 1 (a), the two photons also favor localizing at the four corners of the correlation map. The reason for this is that there is no interference because the system is completely incoherent, which is confirmed by the fact that the coherent term in Eq. (11) vanishes in the case of $\eta = 0$. Whereas, with the increase of the degree of coherence, the coherent term emerges in the two-photon correlation function $\Gamma_{q,r}$, so the $\Gamma_{q,r}$ exhibits the properties of interference. Due to the Hanbury Brown-Twiss (HBT) interference, two local maxima emerge in the off-diagonal regions of the correlation matrix (see Fig. 1 (c) and (d)). That implies that the two photons favor being far from each other, which is in contrast to the case of Fig. 1 (e) and (f) where, except the initial relative phase ϕ , the other parameters are the same as those in Fig. 1 (c) and (d), respectively. This is reasonable because the coherent term in Eq. (12) is in proportion to $\cos \phi$ for the periodical waveguide lattice we considered. Additionally, comparing the values of the maxima in Fig. 1 (c) and (d), it is easy to find that the larger the degree of coherence, the more distinct the interference effect of the system will be.

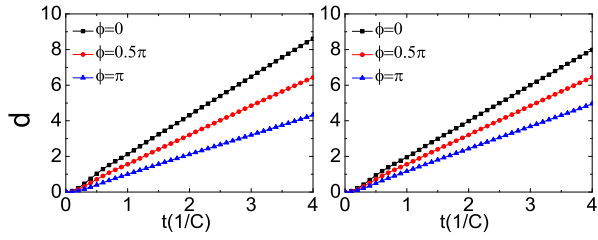


Figure 2: (Color online) The time evolution of the distance between two photons for different initial conditions. The parameters are $\alpha = 0.5$, and $\eta = 1$ (left panel), $\eta = 0.5$ (right panel).

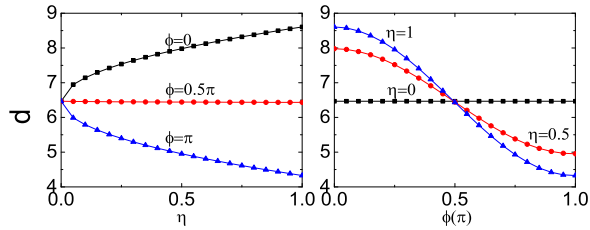


Figure 3: (Color online) Dependence of the distance between two photons at time $t = 4(1/C)$ on the degree of coherence (left panel) and the initial relative phase (right panel). The parameter is $\alpha = 0.5$.

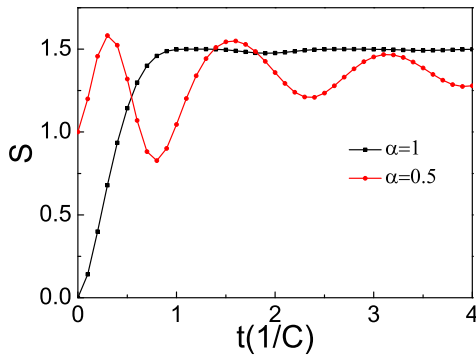


Figure 4: (Color online) Time evolution of the von Neumann entropy of the set of sites in the left half of the system. The parameters are $\eta = 1$, $\phi = 0$, and $L = 15$.

To exhibit the propagation properties of the two photons, we also calculate the average distance between the two photons,

$$d = -2\gamma \cos \phi \sum_{q>r} (q-r) J_q(\tau) J_r(\tau) J_{r-1}(\tau) J_{q-1}(\tau) + \sum_{q>r} (q-r) \left(2\alpha [J_q(\tau) J_r(\tau)]^2 + 2(1-\alpha) [J_{r-1}(\tau) J_{q-1}(\tau)]^2 \right). \quad (13)$$

Here the first term is for coherence, and is affected by the degree of coherence of the system due to $\gamma = \sqrt{\eta - 1 + 4\alpha(1 - \alpha)}$. We plot the time evolution of the distance between the two photons in Fig. 2. We can see that the distance between the two photons is affected not only by the relative phase, but also by the degree of coherence. In Fig. 3, we plot the dependence of the distance d at time $t = 4(1/C)$ on the degree of coherence and the initial relative phase. From the left panel of this figure, we can see that the distance between the two photons becomes larger with increase in the degree of coherence when $\phi = 0$, which is in contrast to the case of $\phi = \pi$. The reason for this phenomenon is that the HBT interference causes the two photons to be far from each when $0 \leq \phi < \pi/2$, which can be confirmed by Fig. 1 (c) and (d) where there are two maxima in the off-diagonal regions. Therefore, the distance between the two photons becomes larger with the increase of the degree of coherence, due to the fact that the increase of the degree of coherence makes the interference effect more significant. The case of $\pi/2 \leq \phi < \pi$ is in contrast to that of $0 \leq \phi < \pi/2$ because the two photons favor staying together when $\pi/2 \leq \phi < \pi$, which can be confirmed by Fig. 1 (e) and (f). Note that when $\phi = \pi/2$, the interference term in the two-photon correlation function becomes zero, so the degree of coherence does not affect the distance between two particles (see the line marked with circles in the left panel of Fig. 3). The right panel of Fig. 3 shows that the relative phase of the system at the initial time can affect the distance between two photons in the case of $\eta > 0$, but such an effect vanishes in the case of $\eta = 0$. This is reasonable because the distance between two photons is in proportion to $\cos \phi \sqrt{\eta - 1 - 4\alpha^2 + 4\alpha}$, which can be found in Eq. (13). Additionally, we can calculate the von Neumann entropy to show the evolution of the entanglement of the system. We split the system into two halves, L and R , and build a reduced density matrix ρ_L of the subsystem L at any time [31]. Then we can calculate the von Neumann entropy of ρ_L as

$$S = - \sum_i \lambda_i \log_2 \lambda_i, \quad (14)$$

where λ_i are the non-zero eigenvalues of the matrix ρ_L . In Fig. 4, we plot the time evolution of the von Neumann entropy of the left half of the system for different initial conditions.

4 Conclusion

We proposed a density matrix formalism to study the properties of two-particle quantum walks where the effect of

coherence was introduced naturally. We gave the general analytical expression of the two-particle correlation function, which is correct for systems in both mixed states and pure states. We suggested a possible two-photon scheme to explore quantum features of two-particle random walks with mixed initial states. For such a concrete scheme, we calculated the two-photon correlation and the average distance between the two photons. The corresponding results showed that the propagation of the two photons depends not only on the initial distribution of those photons, but also on the relative phase and degree of coherence of the system. Such propagation features of the two photons were explained using an analytical expression of the two-particle correlation function we obtained.

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