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Multivariate Padé Approximations For Solving Nonlinear Diffusion Equations

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Abstract: In this paper, multivariate Padé approximation is applied to power series solutions of nonlinear diffusion equations. As it is seen from tables, multivariate Padé approximation (MPA) gives reliable solutions and numerical results.

Keywords: Multivariate Padé approximation; nonlinear diffusion equations; variational iteration method

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1 Introduction

In recent times,univariate and multivariate Padé approximation have been successfully applied to various problems in physical and engineering sciences [1–7]. As it is indicated in [16] *"a Padé approximation can be far more accurate than a Taylor approximation. Essentially, it is a consequence of the famous 'Montessus de Ballore' theorem,which established in 1902 the uniform convergence of Padé approximants on compact subsets excluding the poles [16]"*. Although most of the power series expansions fail to converge outside the disk of convergence, a Padé approximation of a function can give better numerical solutions. The definitions and theorems of multivariate Padé approximations were constructed on univariate Padé approximations [14]. But if it is examined we realize that the applications of univariate and multivariate Padé approximation are different from each other [14], so new definitions and theorems have been constructed to overcome difficulties by the time [16]. Chisholm proposed one of the first definitions in 1973 [17]. Levin developed a general definition [18], after that Cuyt also made important contributions on homogeneous Padé approximations and uniform convergence results for the multivariate Padé approximants [19, 20]. In this paper, multivariate Padé approximations were applied to the solutions of nonlinear diffusion equations in the form [8]:

$$u_t = \left(D(u)u_x\right)_x \tag{1}$$

Subject to the initial condition:

$$u(x,0) = f(x) \tag{2}$$

The details about nonlinear diffusion equations can be seen in [8, 15].

2 Variational iteration method

The basic concepts and principles of variational iteration method can be seen in [9–13]. Sadighi and Ganji [8] constructed respectively the following correction functional and iteration formula by using the basic concepts and principles of variational iteration method:

$$u_{n+1} = u_n - \int_0^t \lambda \left\{ u_{n\tau} - \left(\tilde{D}(u_n)u_{nx}\right)_x \right\} d\tau \tag{3}$$

$$u_{n+1} = u_n - \int_0^t \left\{ u_{n\tau} - \left(\tilde{D}(u_n)u_{nx}\right)_x \right\} d\tau. \tag{4}$$

where $\delta\tilde{D}(u_n)$ is restricted variation. The Lagrange multiplier has been identified as $\lambda = -1$.

3 Multivariate Padé approximation

Consider the bivariate function $f(x,y)$ with Taylor power series development

$$f(x,y) = \sum_{i,j=0}^{\infty} c_{ij} x^i y^j \tag{5}$$

around the origin [14]. The Padé approximation problem of order for $f(x,y)$ consists in finding polynomials

$$p(x,y) = \sum_{k=0}^{m} A_k(x,y) \tag{6}$$

$$q(x,y) = \sum_{k=0}^{n} B_k(x,y) \tag{7}$$

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such that in the power series $(fq-p)(x,y)$ the coefficients of x^i and y^j by solving the following equation system;

$$\begin{cases} C_0(x,y)B_0(x,y) = A_0(x,y) \\ C_1(x,y)B_0(x,y) + C_0(x,y)B_1(x,y) = A_1(x,y) \\ \vdots \\ C_m(x,y)B_0(x,y) + \cdots + C_{m-n}(x,y)B_n(x,y) = A_m(x,y) \end{cases} \tag{8}$$

$$\begin{cases} C_{m+1}(x,y)B_0(x,y) + C_{m+1-n}(x,y)B_n(x,y) = 0 \\ \vdots \\ C_{m+n}(x,y)B_0(x,y) + \cdots + C_m(x,y)B_n(x,y) = 0 \end{cases} \tag{9}$$

where $C_k = 0$ if $k < 0$. If the equations (8) and (9) are solved then the coefficients A_k $(k = 0, \dots, m)$ and B_k $(k = 0, \dots, n)$ are obtained. So polynomials (6) and (7) are found. Polynomials $p(x,y)$ and $q(x,y)$ are called Padé equations[14]. So the multivariate Padé approximant of order (m,n) for $f(x,y)$ is defined as,

$$r_{m,n}(x,y) = \frac{p(x,y)}{q(x,y)}. \tag{10}$$

4 Applications and results

In this section multivariate Padé series solutions of nonlinear diffusion equations shall be illustrated by two examples. All the results were calculated by using the Maple software suite. The full VIM solutions of examples can be seen in Sadighi and Ganji [8].

Example 1. Consider a slow diffusion process [15]

$$u_t = (uu_x)_x \tag{11}$$

$$u(x,0) = \frac{1}{c}x^2, \qquad t > 0 \tag{12}$$

where $c > 0$ is an arbitrary constant.

According to the iteration formulas (3), (4) Sadighi and Ganji [8] and by using decomposition method Wazwaz [15] obtained following solution,

$$u(x,t) = x^2\left(\frac{1}{c} + \frac{6t}{c^2} + \frac{36t^2}{c^3} + \frac{216t^3}{c^4} + \frac{1296t^4}{c^5} + \frac{33696t^5}{c^6} + \frac{31104t^6}{c^7} + \cdots\right) \tag{13}$$

The exact solution of (13) is given as $u(x,t) = \frac{x^2}{c-6t}$ in [15]. If the multivariate Padé approximation is applied to equation (13) for $m = 4$ and $n = 2$, according to the equation system (8) and (9) the following Padé equations are obtained;

$$p(x,t) = -\frac{36t^4\left(-1+36x^2\right)x^4}{c^7} \tag{14}$$

and

$$q(x,t) = -\frac{36t^4\left(-c+6t+36x^2c-216x^2t\right)x^2}{c^7} \tag{15}$$

So the multivariate Padé approximant of order (4, 2) for equation (13) is,

$$r_{4,2}(x,t) = \frac{\left(-1+36x^2\right)x^2}{-c+6t+36x^2c-216x^2t} \tag{16}$$

If the multivariate Padé approximation is applied to equation (13) for $m = 5$ and $n = 2$, according to the equation system (8) and (9) the following Padé equations are obtained;

$$p(x,t) = -\frac{933120t^7x^6}{c^{10}} \tag{17}$$

and

$$q(x,t) = \frac{933120t^7\,(c-6t)\,x^4}{c^{10}} \tag{18}$$

So the multivariate Padé approximant of order $(5,2)$for equation (13) is,

$$r_{5,2}(x,t) = \frac{x^2}{c-6t} \tag{19}$$

If the multivariate Padé approximation is applied to equation (13) for $m = 6$ and $n = 2$, according to the equation system (8) and (9) the following Padé equations are obtained;

$$p(x,t) = -\frac{1119744t^8(5c^4+63c^3t-600c^2t^2-3600ct^3-21600t^4)x^6}{c^{15}} \tag{20}$$

and

$$q(x,t) = -\frac{1119744t^8(5c^2+33ct-978t^2)x^4}{c^{12}} \tag{21}$$

So the multivariate Padé approximant of order $(6,2)$ for equation (13) is,

$$r_{6,2}(x,t) = \frac{(5c^4+63c^3t-600c^2t^2-3600ct^3-21600t^4)x^2}{c^3(5c^2+33ct-978t^2)}. \tag{22}$$

Example 2. Consider another slow diffusion process [**?**]

$$u_t = \left(u^2u_x\right)_x \tag{23}$$

$$u\,(x,0) = \frac{x+h}{2\sqrt{c}} \tag{24}$$

where h and c , $c > 0$, are arbitrary constants.

According to the iteration formulas (3), (4) Sadighi and Ganji [8] and by using decomposition method Wazwaz [15] obtained following solution,

$$u\,(x,t) = \frac{x+h}{2\sqrt{c}}\left(1+\frac{t}{2c}+\frac{3t^2}{8c^2}+\frac{5t^3}{16c^3}+\frac{13t^4}{64c^4}+\frac{9t^5}{80c^5}+\cdots\right) \tag{25}$$

The exact solution of (23) is given as $u\,(x,t) = \frac{1}{2}\frac{x+h}{\sqrt{c-t}}$ in [8]. If the multivariate Padé approximation is applied to equation (25) for $m = 2$ and $n = 2$, according to the equation system (8) and (9) the following Padé equations are obtained;

$$\begin{aligned} p(x,t) = {}& -t^2(64c^4hx^2+64c^4x^3+64c^3h^2tx+32c^3htx^2-32c^3tx^3 \\ & +8c^2h^3t^2-96c^2h^2t^2x-104c^2ht^2x^2-40ch^3t^3x+9h^3t^4)/4096c^{15/2} \end{aligned} \tag{26}$$

and

$$\begin{aligned} q(x,t) = {}& -t^2(16c^4x^2+16c^3htx-16c^3tx^2+2c^2h^2t^2-34c^2h^2t^2x \\ & +2c^2t^2x^2-11ch^2t^3+11cht^3+7h^2t^4)/512c^7 \end{aligned} \tag{27}$$

So the multivariate Padé approximant of order $(2,2)$for equation (25) is,

$$\begin{aligned} r_{2,2}(x,t) = {}& (64c^4hx^2+64c^4x^3+64c^3h^2tx+32c^3htx^2-32c^3tx^3+8c^2h^3t^2 \\ & -96c^2h^2t^2x-104c^2ht^2x^2-40ch^3t^3x+9h^3t^4)/(8\sqrt{c}(16c^4x^2+16c^3htx-16c^3tx^2+2c^2h^2t^2 \\ & -34c^2h^2t^2x+2c^2t^2x^2-11ch^2t^3+11cht^3+7h^2t^4) \end{aligned} \tag{28}$$

If the multivariate Padé approximation is applied to equation (25) for $m = 3$ and $n = 2$, according to the equation system (8) and (9) the following Padé equations are obtained;

$$\begin{aligned} p(x,t) = {}& -t^4(480tx^3c^4-329h^3t^5-1372h^3t^4c+1120h^3c^3t^2 \\ & +320x^3c^5-160t^2x^3c^3+2296h^3t^3c^2+1760h^2c^4tx+320hc^5x^2 \\ & +4408h^2t^2c^3x+2240htc^4x^2+1124h^2t^3c^2x+3128ht^2c^3x^2 \\ & -1437h^2t^4cx-1172ht^3x^2c^2)/(163840c^{(21/2)}) \end{aligned} \tag{29}$$

and

$$q(x,t) = -t^4(280h^2t^2c^2 + 440htc^3x + 434h^2t^3c + 602ht^2c^2x \\ +80x^2c^4 + 80x^2c^3t - 759t^3xch - 110t^2x^2c^2 - 665h^2t^4)/(20480c^9) \tag{30}$$

So the multivariate Padé approximant of order $(3,2)$for equation (25) is,

$$r_{3,2}(x,t) = (480tx^3c^4 - 329h^3t^5 - 1372h^3t^4c + 1120h^3c^3t^2 \\ +320x^3c^5 - 160t^2x^3c^3 + 2296h^3t^3c^2 + 1760h^2c^4tx + 320hc^5x^2 \\ +4408h^2t^2c^3x + 2240htc^4x^2 + 1124h^2t^3c^2x + 3128ht^2c^3x^2 \\ -1437h^2t^4cx - 1172ht^3x^2c^2)/(8c^{(3/2)}(280h^2t^2c^2 + 440htc^3x + 434h^2t^3c \\ +602ht^2c^2x + 80x^2c^4 + 80x^2c^3t - 759t^3xch - 110t^2x^2c^2 - 665h^2t^4)) \tag{31}$$

If the multivariate Padé approximation is applied to equation (25) for $m = 4$ and $n = 2$, according to the equation system (8) and (9) the following Padé equations are obtained;

$$p(x,t) = -t(320c^5ht^3x^2 + 320c^5t^3x^3 + 1760c^4h^2t^4x + 2240c^4ht^4x^2 + 480c^4t^4x^3 \\ +1120c^3h^3t^5 + 5560c^3h^2t^5x + 4280c^3ht^5x^2 - 160c^3t^5x^3 + 3160c^2h^3t^5 \\ +1700c^2h^2t^6x - 1460c^2ht^6x^2 - 1660ch^3t^7 - 1725ch^2t^7x - 365h^3t^8 \\ -128c^3h^2x - 128c^3hx^2 - 96ch^3t - 64c^2h^2tx + 32c^2htx^2 + 32ch^3t^2 \\ +32ch^2t^2x + 4h^3t^3)/\left(163840c^{(21/2)}\right) \tag{32}$$

and

$$q(x,t) = -(80c^4t^3x^2 + 440c^3ht^4x + 80c^3t^4x^2 + 280c^2h^2t^5 + 890c^2ht^5x - 110c^2t^5x^2 \\ +650ch^2t^6 - 975cht^6x - 845h^2t^7 - 32c^2hx - 24ch^2t + 24chtx + 20h^2t^2)t/\left(20480c^9\right) \tag{33}$$

So the multivariate Padé approximant of order $(4,2)$for equation (25) is,

$$r_{4,2}(x,t) = (320c^5ht^3x^2 + 320c^5t^3x^3 + 1760c^4h^2t^4x + 2240c^4ht^4x^2 + 480c^4t^4x^3 \\ +1120c^3h^3t^5 + 5560c^3h^2t^5x + 4280c^3ht^5x^2 - 160c^3t^5x^3 + 3160c^2h^3t^5 \\ +1700c^2h^2t^6x - 1460c^2ht^6x^2 - 1660ch^3t^7 - 1725ch^2t^7x - 365h^3t^8 \\ -128c^3h^2x - 128c^3hx^2 - 96ch^3t - 64c^2h^2tx + 32c^2htx^2 + 32ch^3t^2 \\ +32ch^2t^2x + 4h^3t^3)/(8c^{(3/2)}(80c^4t^3x^2 + 440c^3ht^4x + 80c^3t^4x^2 + 280c^2h^2t^5 + 890c^2ht^5x \\ -110c^2t^5x^2 + 650ch^2t^6 - 975cht^6x - 845h^2t^7 - 32c^2hx - 24ch^2t + 24chtx + 20h^2t^2)) \tag{34}$$

As it is presented above in Example 1, If the numerical results are compared at $c = 1$ following table obtained (Table 1).

Table 1: Comparison of Exact solution of equation (11) and MPA solutions of equation (13) for example 1.

x	t	Exact solution	$r_{4,2}(x,t)$	$r_{5,2}(x,t)$	$r_{6,2}(x,t)$
0.001	0.001	$0.1006036217 \times 10^{-5}$	$0.1006036217 \times 10^{-5}$	$0.1006036217 \times 10^{-5}$	$0.1006036217 \times 10^{-5}$
0.002	0.002	$0.4048582996 \times 10^{-5}$	$0.4048582996 \times 10^{-5}$	$0.4048582996 \times 10^{-5}$	$0.4048583000 \times 10^{-5}$
0.003	0.003	$0.9164969450 \times 10^{-5}$	$0.9164969450 \times 10^{-5}$	$0.9164969450 \times 10^{-5}$	$0.9164969506 \times 10^{-5}$
0.004	0.004	0.00001639344262	0.00001639344262	0.00001639344262	0.00001639344305
0.005	0.005	0.00002577319588	0.00002577319588	0.00002577319588	0.00002577319791
0.006	0.006	0.00003734439834	0.00003734439834	0.00003734439834	0.00003734440562
0.007	0.007	0.00005114822547	0.00005114822547	0.00005114822547	0.00005114824697
0.008	0.008	0.00006722689076	0.00006722689076	0.00006722689076	0.00006722694565
0.009	0.009	0.00008562367865	0.00008562367865	0.00008562367865	0.00008562380421
0.01	0.01	0.0001063829787	0.0001063829787	0.0001063829787	0.0001063832422

As it is presented above in Example 2, According to the numerical results at $c = 1$ and $h = 1$ following table obtained (Table 2).

Table 2: Comparison of Exact solution of equation (23) and MPA solutions of equation (25) for example 2.

x	t	Exact solution	$r_{2,2}(x,t)$	$r_{3,2}(x,t)$	$r_{4,2}(x,t)$
0.001	0.001	0.5007504375	0.5007504379	0.5007504380	0.5007504379
0.002	0.002	0.5015017525	0.5015017528	0.5015017524	0.5015017526
0.003	0.003	0.5022539465	0.5022539468	0.5022539469	0.5022539468
0.004	0.004	0.5030070220	0.5030070221	0.5030070220	0.5030070220
0.005	0.005	0.5037609805	0.5037609808	0.5037609810	0.5037609806
0.006	0.006	0.5040143180	0.5045158248	0.5045158248	0.5045158248
0.007	0.007	0.5047697970	0.5052715562	0.5052715560	0.5052715560
0.008	0.008	0.5060281770	0.5060281775	0.5060281776	0.5060281770
0.009	0.009	0.5067856900	0.5067856904	0.5067856905	0.5067856899
0.01	0.01	0.5075440965	0.5075440974	0.5075440974	0.5075440965

5 Conclusion

In this paper, rational power series solution of various kinds of nonlinear diffusion equations were constructed by multivariate Padé approximations.The approximation is effective, easy to use, and reliable.The main benefit of the approximation is to offer rational approximation in a rapid convergent rational series form.

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