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Mixed spectral AKNS hierarchy from linear isospectral problem and its exact solutions

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Abstract: In this paper, the AKNS isospectral problem and its corresponding time evolution are generalized by embedding three coefficient functions. Starting from the generalized AKNS isospectral problem, a mixed spectral AKNS hierarchy with variable coefficients is derived. Thanks to the selectivity of these coefficient functions, the mixed spectral AKNS hierarchy contains not only isospectral equations but also nonisospectral equations. Based on a systematic analysis of the related direct and inverse scattering problems, exact solutions of the mixed spectral AKNS hierarchy are obtained through the inverse scattering transformation. In the case of reflectionless potentials, the obtained exact solutions are reduced to n-soliton solutions. This paper shows that the AKNS spectral problem being nonisospectral is not a necessary condition to construct a nonisospectral AKNS hierarchy and that the inverse scattering transformation can be used for solving some other variable-coefficient mixed hierarchies of isospectral equations and nonisospectral equations.

Keywords: mixed spectral AKNS hierarchy; spectral problem; exact solution; soliton solution; inverse scattering transformation

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1 Introduction

It is well known that solving nonlinear partial differential equations (PDEs) plays an important role in the study of nonlinear physical phenomena in many fields such as fluid dynamics, plasma physics and nonlinear optics. In the past several decades, many effective methods have been proposed for obtaining exact solutions of nonlinear PDEs, such as the inverse scattering transformation (IST) [1], Hirota's bilinear method [2], Bäcklund transformation [3, 4], Painlevé expansion [5], homogeneous balance method [6], similarity transformation method [7, 8], ansatz method [9, 10], function expansion methods, and some others [11–17]. Among these methods, the IST [1] is a systematic method. Since being put forward in 1967, the IST has achieved considerable development and received a wide range of applications [18–33]. As a famous method in mathematical physics, the IST is a milestone in the process of developing analytical methods for solving nonlinear PDEs. The IST is also known as the nonlinear Fourier transformation of nonlinear PDEs, one of the advantages of which is that it can solve the whole hierarchy of equations associated with the same spectral problem. In general, starting from the related spectral problem with a time-independent spectral parameter one could derive isospectral equations which often describe solitary waves in lossless and uniform media, while nonisospectral equations describing the solitary waves in a certain type of nonuniform media are usually resulted from the time-varying spectral problem.

When the inhomogeneities of media and nonuniformities of boundaries are taken into account, the variable-coefficient equations could describe more realistic physical phenomena than their constant-coefficient counterparts [34]. Recently, the study of nonlinear PDEs with variable coefficients has attracted much attention [28–30, 33, 34]. How to construct and solve such nonlinear PDEs is worthy of exploring. Motivated by this desire, in the present paper we shall consider a new and more general ANKS hierarchy with variable coefficients:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \left\{ \gamma(t) \left[\frac{L - 2\beta(t)E}{\alpha(t)} \right]^m + \frac{\alpha'(t)}{\alpha(t)} [L - 2\beta(t)E] x + 2x\beta'(t) \right\} \begin{pmatrix} -q \\ r \end{pmatrix}, \quad (m = 1, 2, \cdots), \tag{1}$$

where $q = q(x, t)$ and $r = r(x, t)$ are functions of the indicated variables; the derivatives of any order with respect to x of q and r vanish as x tends to infinity; $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are differentiable functions of t, $\alpha(t)$ is nonzero and bounded, $\alpha'(t) = \mathrm{d}\alpha(t)/\mathrm{d}t$, $\beta'(t) = \mathrm{d}\beta(t)/\mathrm{d}t$; E is a two or-

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der unit matrix; and the operator L is employed as

$$L = \sigma\partial + 2\begin{pmatrix} q \\ -r \end{pmatrix}\partial^{-1}(r, q), \tag{2}$$

with

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \partial = \frac{\partial}{\partial x}, \quad \partial^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{+\infty}\right)\mathrm{d}x.$$

It is obvious to see that the AKNS hierarchy (1) is a mixed hierarchy of isospectral equations and nonisospectral equations. In particular, when $\alpha(t) = 1$, $\beta(t) = 0$ and $\gamma(t) = 1$, Equation (1) becomes the constant-coefficient isospectral AKNS hierarchy [32]:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L^m \begin{pmatrix} -q \\ r \end{pmatrix}, \quad (m = 1, 2, \cdots). \tag{3}$$

When $\alpha(t) = 1$ and $\beta(t) = t$, Equation (1) gives a variable-coefficient nonisospectral AKNS hierarchy:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \gamma(t)(L - 2tE)^m \begin{pmatrix} -q \\ r \end{pmatrix} + 2\begin{pmatrix} -xq \\ xr \end{pmatrix}, \quad (m = 1, 2, \cdots). \tag{4}$$

If we select $m = 2$, then Equation (1) reads

$$\begin{aligned} q_t &= \frac{\gamma(t)}{\alpha^2(t)}\left[-q_{xx} - 4\beta(t)q_x - 4\beta^2(t)q + 2q^2 r\right] \\ &+ \frac{\alpha'(t)}{\alpha(t)}\left[xq_x + q + 2x\beta(t)q\right] - 2x\beta'(t)q, \end{aligned} \tag{5}$$

$$\begin{aligned} r_t &= \frac{\gamma(t)}{\alpha^2(t)}\left[r_{xx} - 4\beta(t)r_x + 4\beta^2(t)r - 2qr^2\right] \\ &+ \frac{\alpha'(t)}{\alpha(t)}\left[xr_x + r - 2x\beta(t)r\right] + 2x\beta'(t)r, \end{aligned} \tag{6}$$

which include the following nonisospectral equations as a special case:

$$q_t = -q_{xx} + 2q^2 r + xq_x + q, \tag{7}$$

$$r_t = r_{xx} - 2qr^2 + xr_x + r. \tag{8}$$

In fact, Equations (7) and (8) can be easily obtained as long as we substitute $\alpha(t) = \mathrm{e}^t$, $\beta(t) = 0$ and $\gamma(t) = \mathrm{e}^{2t}$ into Equations (5) and (6).

The rest of the paper is organized as follows. In Section 2, we derive the mixed spectral AKNS hierarchy (1) from a generalized AKNS spectral problem. In Section 3, following the steps of IST method we present a systematic analysis on the direct and inverse scattering problems relating to the AKNS hierarchy (1). As a result, the uniform formulae of exact solutions of the AKNS hierarchy (1) are obtained. In the special case of reflectionless potentials, the obtained exact solutions are reduced to n-soliton solutions. In Section 4, we conclude this paper.

2 Derivation of the mixed spectral AKNS hierarchy

In order to construct the mixed spectral AKNS hierarchy (1), in this section we embed the coefficient functions $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ to the known AKNS spectral problem [23, 24, 32] and its time evolution [30, 32] so as to consider a generalized AKNS spectral problem with spectral parameter η independent of x and t:

$$\begin{aligned} \varphi_x = M\varphi, \quad M &= \begin{pmatrix} -\alpha(t)\eta - \beta(t) & q \\ r & \alpha(t)\eta + \beta(t) \end{pmatrix}, \\ \varphi &= \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \end{aligned} \tag{9}$$

and the time evolution:

$$\varphi_t = N\varphi, \quad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \tag{10}$$

with the boundary condition:

$$\begin{aligned} N|_{(q,r)=(0,0)} = \begin{pmatrix} -\frac{1}{2}\gamma(t)(2\eta)^m - \left[\alpha'(t)\eta + \beta'(t)\right]x & 0 \\ 0 & \frac{1}{2}\gamma(t)(2\eta)^m + \left[\alpha'(t)\eta + \beta'(t)\right]x \end{pmatrix}. \end{aligned} \tag{11}$$

Then we have the following Theorem 1.

Theorem 1. *Suppose that*

$$A = \partial^{-1}(r, q)\begin{pmatrix} -B \\ C \end{pmatrix} - x[\alpha'(t)\eta + \beta'(t)] - \frac{1}{2}\gamma(t)(2\eta)^m, \tag{12}$$

$$\begin{pmatrix} -B \\ C \end{pmatrix} = \sum_{i=1}^{m}\begin{pmatrix} -b_i \\ c_i \end{pmatrix}(2\eta)^{m-i}, \quad \begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} = \frac{2^{m-1}\gamma(t)}{\alpha(t)}\begin{pmatrix} -q \\ r \end{pmatrix}, \tag{13}$$

then the compatibility condition of Equation (9) *and* (10)*, the zero curvature equation* $M_t - N_x + [M, N] = 0$*, leads to the mixed spectral AKNS hierarchy* (1).

Proof. We substitute Equation (12) into Equation (10), then $M_t - N_x + [M, N] = 0$ can be reduced to:

$$\begin{aligned} \begin{pmatrix} q \\ r \end{pmatrix}_t &= L\begin{pmatrix} -B \\ C \end{pmatrix} - 2[\alpha(t)\eta + \beta(t)]\begin{pmatrix} -B \\ C \end{pmatrix} \\ &+\gamma(t)(2\eta)^m\begin{pmatrix} -q \\ r \end{pmatrix} + 2x[\alpha'(t)\eta + \beta'(t)]\begin{pmatrix} -q \\ r \end{pmatrix}. \end{aligned} \tag{14}$$

Substituting Equation (13) into Equation (14) and comparing the coefficients of the same order of η yields

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = [L - 2\beta(t)E] \begin{pmatrix} -b_m \\ c_m \end{pmatrix} + 2x\beta'(t) \begin{pmatrix} -q \\ r \end{pmatrix}, \quad (15)$$

$$\begin{pmatrix} -b_m \\ c_m \end{pmatrix} = \frac{L - 2\beta(t)E}{2\alpha(t)} \begin{pmatrix} -b_{m-1} \\ c_{m-1} \end{pmatrix} + x\frac{\alpha'(t)}{\alpha(t)} \begin{pmatrix} -q \\ r \end{pmatrix}, \quad (16)$$

$$\begin{pmatrix} -b_i \\ c_i \end{pmatrix} = \frac{L - 2\beta(t)E}{2\alpha(t)} \begin{pmatrix} -b_{i-1} \\ c_{i-1} \end{pmatrix}, \quad (i = 2, \cdots, m-1), \quad (17)$$

$$\begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} = \frac{2^{m-1}\gamma(t)}{\alpha(t)} \begin{pmatrix} -q \\ r \end{pmatrix}. \quad (18)$$

From Equations (15)–(18) we have

$$\begin{pmatrix} -b_m \\ c_m \end{pmatrix} = \left\{ \gamma(t)\frac{[L - 2\beta(t)E]^{m-1}}{\alpha^m(t)} + x\frac{\alpha'(t)}{\alpha(t)} \right\} \begin{pmatrix} -q \\ r \end{pmatrix}, \quad (19)$$

and finally reach the mixed spectral AKNS hierarchy (1) by substituting Equation (19) into Equation (15). Therefore, the proof is finished. □

Remark 1. We introduced in this section three functions $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ so that the AKNS hierarchy (1) has much generality. For example, when $\alpha(t) = 1$, $\beta(t) = 0$ and $\gamma(t) = 1$, the matrixes M and N in Equations (9) and (10) become the ones [32] which can be utilized to derive the known AKNS hierarchy (3) by similar manipulations.

3 Exact solutions of the mixed spectral AKNS hierarchy

In this section, the IST method will be extended to solve the mixed spectral AKNS hierarchy (1). Firstly, we give a systematic analysis of the direct and inverse scattering problems relating to Equation (1). Secondly, we obtain the uniform formulae of exact solutions of Equation (1). Finally, we reduce the obtained exact solutions to generate n-soliton solutions.

3.1 The direct scattering problem

For convenience, we replace η with ik, here and hereafter i always stands for the imaginary unit in similar circumstances. The following Lemma 1 can be used.

Lemma 1. *If the real potentials* $(q(x), r(x))^T$ *satisfy*

$$\int_{-\infty}^{+\infty} \left| x^j q(x) \right| \mathrm{d}x < +\infty, \quad \int_{-\infty}^{+\infty} \left| x^j r(x) \right| \mathrm{d}x < +\infty, \quad (j = 0, 1), \quad (20)$$

then the spectral problem (9) *has a set of Jost solutions* $\varphi(x, k)$, $\bar{\varphi}(x, k)$, $\psi(x, k)$ *and* $\bar{\psi}(x, k)$, *which are bounded for all values of* x *and also have the following asymptotic behaviors when* $|x| \to +\infty$*:*

$$\varphi(x, k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{[ik\alpha(t)+\beta(t)]x}, \quad (x \to +\infty), \quad (21)$$

$$\bar{\varphi}(x, k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-[ik\alpha(t)+\beta(t)]x}, \quad (x \to +\infty), \quad (22)$$

$$\psi(x, k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-[ik\alpha(t)+\beta(t)]x}, \quad (x \to -\infty), \quad (23)$$

$$\bar{\psi}(x, k) \sim \begin{pmatrix} 0 \\ -1 \end{pmatrix} \mathrm{e}^{[ik\alpha(t)+\beta(t)]x}, \quad (x \to -\infty), \quad (24)$$

$$\varphi_x(x, k) \sim [ik\alpha(t) + \beta(t)] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{e}^{[ik\alpha(t)+\beta(t)]x}, \quad (x \to +\infty), \quad (25)$$

$$\bar{\varphi}_x(x, k) \sim -[ik\alpha(t) + \beta(t)] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{-[ik\alpha(t)+\beta(t)]x}, \quad (x \to +\infty), \quad (26)$$

$$\psi_x(x, k) \sim -[ik\alpha(t) + \beta(t)] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{-[ik\alpha(t)+\beta(t)]x}, \quad (x \to -\infty), \quad (27)$$

$$\bar{\psi}_x(x, k) \sim [ik\alpha(t) + \beta(t)] \begin{pmatrix} 0 \\ -1 \end{pmatrix} \mathrm{e}^{[ik\alpha(t)+\beta(t)]x}, \quad (x \to -\infty), \quad (28)$$

$$\varphi_k(x, k) \sim ix\alpha(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{[ik\alpha(t)+\beta(t)]x}, \quad (x \to +\infty), \quad (29)$$

$$\bar{\varphi}_k(x, k) \sim -ix\alpha(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-[ik\alpha(t)+\beta(t)]x}, \quad (x \to +\infty), \quad (30)$$

$$\psi_k(x, k) \sim -ix\alpha(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-[ik\alpha(t)+\beta(t)]x}, \quad (x \to -\infty), \quad (31)$$

$$\bar{\psi}_k(x, k) \sim ix\alpha(t) \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{[ik\alpha(t)+\beta(t)]x}, \quad (x \to -\infty), \quad (32)$$

and these Jost solutions can be expressed as:

$$\varphi(x,k)=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=\begin{pmatrix}-\int_x^{\infty}\mathrm{e}^{-[ik\alpha(t)+\beta(t)](x-y)}q(y)\varphi_2(y,k)\mathrm{d}y\\ \mathrm{e}^{[ik\alpha(t)+\beta(t)]x}-\int_x^{\infty}\mathrm{e}^{[ik\alpha(t)+\beta(t)](x-y)}r(y)\varphi_1(y,k)\mathrm{d}y\end{pmatrix}, \tag{33}$$

$$\bar{\varphi}(x,k)=\begin{pmatrix}\bar{\varphi}_1\\ \bar{\varphi}_2\end{pmatrix}=\begin{pmatrix}\mathrm{e}^{-[ik\alpha(t)+\beta(t)]x}-\int_x^{\infty}\mathrm{e}^{-[ik\alpha(t)+\beta(t)](x-y)}q(y)\bar{\varphi}_2(y,k)\mathrm{d}y\\ -\int_x^{\infty}\mathrm{e}^{[ik\alpha(t)+\beta(t)](x-y)}r(y)\bar{\varphi}_1(y,k)\mathrm{d}y\end{pmatrix}, \tag{34}$$

$$\psi(x,k)=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}\mathrm{e}^{-[ik\alpha(t)+\beta(t)]x}+\int_{-\infty}^{x}\mathrm{e}^{-[ik\alpha(t)+\beta(t)](x-y)}q(y)\psi_2(y,k)\mathrm{d}y\\ \int_{-\infty}^{x}\mathrm{e}^{[ik\alpha(t)+\beta(t)](x-y)}r(y)\psi_1(y,k)\mathrm{d}y\end{pmatrix}, \tag{35}$$

$$\bar{\psi}(x,k)=\begin{pmatrix}\bar{\psi}_1\\ \bar{\psi}_2\end{pmatrix}=\begin{pmatrix}\int_{-\infty}^{x}\mathrm{e}^{-[ik\alpha(t)+\beta(t)](x-y)}q(y)\bar{\psi}_2(y)\mathrm{d}y\\ -\mathrm{e}^{-[ik\alpha(t)+\beta(t)]x}+\int_{-\infty}^{x}\mathrm{e}^{[ik\alpha(t)+\beta(t)](x-y)}r(y)\bar{\psi}_1(y)\mathrm{d}y\end{pmatrix}. \tag{36}$$

Since the Jost solutions $\varphi(x,k)$ and $\bar{\varphi}(x,k)$ are independent, we suppose that

$$\psi(x,k)=a(k)\bar{\varphi}(x,k)+b(k)\varphi(x,k), \tag{37}$$

$$\bar{\psi}(x,k)=-\bar{a}(k)\varphi(x,k)+\bar{b}(k)\bar{\varphi}(x,k). \tag{38}$$

Using the Wronskians [30] of $\varphi(x,k)=(\varphi_1(x,k),\varphi_2(x,k))^T$ and $\psi(x,k)=(\psi_1(x,k),\psi_2(x,k))^T$, from Equations (37) and (38) we have

$$a(k)=W(\psi(x,k),\varphi(x,k)),\quad \bar{a}(k)=W(\bar{\psi}(x,k),\bar{\varphi}(x,k)), \tag{39}$$

$$b(k)=W(\bar{\varphi}(x,k),\psi(x,k)),\quad \bar{b}(k)=W(\bar{\psi}(x,k),\varphi(x,k)). \tag{40}$$

Theorem 2. *The function $a(k)(\bar{a}(k))$ is analytical in the upper half k-plane (the lower half k-plane).*

Proof. We suppose that

$$\begin{aligned}\phi(x,k) &= (\phi_1(x,k),\phi_2(x,k))^T\\ &= \psi(x,k)-a(k)\bar{\varphi}(x,k)=b(k)\varphi(x,k),\end{aligned} \tag{41}$$

then using Equatios (21)–(23) yields:

$$\begin{aligned}\phi_1(x,k)&=\mathrm{e}^{-[ik\alpha(t)+\beta(t)]x}-a(k)\mathrm{e}^{-[ik\alpha(t)+\beta(t)]x}\\ &\quad+\int_{-\infty}^{\infty}\mathrm{e}^{-[ik\alpha(t)+\beta(t)](x-y)}q(y)\psi_2(y,k)\mathrm{d}y\\ &\quad-\int_x^{\infty}\mathrm{e}^{-[ik\alpha(t)+\beta(t)](x-y)}q(y)\phi_2(y,k)\mathrm{d}y,\end{aligned} \tag{42}$$

$$\begin{aligned}\phi_2(x,k)&=\int_{-\infty}^{\infty}\mathrm{e}^{-[ik\alpha(t)+\beta(t)](x-y)}r(y)\psi_1(y,k)\mathrm{d}y\\ &\quad-\int_x^{\infty}\mathrm{e}^{-[ik\alpha(t)+\beta(t)](x-y)}r(y)\phi_1(y,k)\mathrm{d}y.\end{aligned} \tag{43}$$

Thus, $\phi_1(x,k)$ and $\phi_2(x,k)$ can be rewritten as:

$$\phi_1(x,k)=-\int_x^{\infty}\mathrm{e}^{-[ik\alpha(t)+\beta(t)](x-y)}q(y)\phi_2(y,k)\mathrm{d}y, \tag{44}$$

$$\begin{aligned}\phi_2(x,k) &= b(k)\mathrm{e}^{[ik\alpha(t)+\beta(t)]x}\\ &\quad-\int_x^{\infty}\mathrm{e}^{-[ik\alpha(t)+\beta(t)](x-y)}r(y)\phi_1(y,k)\mathrm{d}y,\end{aligned} \tag{45}$$

then it follows from Equations (39), (40), (44) and (45) that

$$a(k)=1+\int_{-\infty}^{\infty}\mathrm{e}^{[ik\alpha(t)+\beta(t)]y}q(y)\psi_2(y,k)\mathrm{d}y, \tag{46}$$

$$b(k) = \int_{-\infty}^{\infty} \mathrm{e}^{-[ik\alpha(t)+\beta(t)]y} r(y)\psi_1(y,k)\mathrm{d}y. \tag{47}$$

We therefore complete the proof. □

It should be noted that $a(k)(\bar{a}(k))$ has only a finite number of zeros because that $a(k) \to 1(\bar{a}(k) \to 1)$ when $k \to \infty$. Assuming $a(k)(\bar{a}(k))$ has a zero $\kappa_j(\bar{\kappa}_j)$, we can see from Equations (39) and (40) that $\psi(x,\kappa_j)$ and $\varphi(x,\kappa_j)(\bar{\psi}(x,\bar{\kappa}_j)$ and $\bar{\varphi}(x,\bar{\kappa}_j))$ are linearly dependent, then there must exist constants b_j and $\bar{b}_j$ such that

$$\psi(x,\kappa_j) = b_j\varphi(x,\kappa_j), \tag{48}$$

$$\bar{\psi}(x,\bar{\kappa}_j) = \bar{b}_j\bar{\varphi}(x,\bar{\kappa}_j). \tag{49}$$

From Equation (9), we have

$$\begin{aligned}\varphi_{kx}(x,k) &= \varphi_{xk}(x,k) = [M\varphi(x,k)]_k \\ &= M_k\varphi(x,k) + M\varphi_k(x,k),\end{aligned} \tag{50}$$

then using Equation (50) and Equation (9) yields

$$\begin{aligned}W_x(\psi(x,k),\varphi_k(x,k)) &= i\alpha(t)\big[\varphi_1(x,k)\psi_2(x,k) \\ &+ \varphi_2(x,k)\psi_1(x,k)\big].\end{aligned} \tag{51}$$

Similarly, we have

$$\begin{aligned}W_x(\psi_k(x,k),\varphi(x,k)) &= -i\alpha(t)\big[\varphi_1(x,k)\psi_2(x,k) \\ &+ \varphi_2(x,k)\psi_1(x,k)\big].\end{aligned} \tag{52}$$

Integrating Equation (51) from x to l and Equation (52) from $-l$ to x, respectively, and then subtracting them, we obtain when $k = \kappa_j$

$$W_k(\psi(x,\kappa_j),\varphi(x,\kappa_j)) = -2i\alpha(t)b_j\int_{-\infty}^{\infty}\varphi_1(x,\kappa_j)\varphi_2(x,\kappa_j)dx. \tag{53}$$

From Equation (39), we have

$$W_k(\psi(x,\kappa_j),\varphi(x,\kappa_j)) = a_k(\kappa_j), \tag{54}$$

then using Equations (53) and (54) yields

$$2\int_{-\infty}^{\infty}\varphi_1(x,\kappa_j)\varphi_2(x,\kappa_j)\mathrm{d}x = \frac{-a_k(\kappa_j)}{ib_j\alpha(t)}. \tag{55}$$

By similar manipulations we have

$$2\int_{-\infty}^{\infty}\bar{\varphi}_1(x,\bar{\kappa}_j)\bar{\varphi}_2(x,\bar{\kappa}_j)\mathrm{d}x = \frac{-\bar{a}_k(\bar{\kappa}_j)}{i\bar{b}_j\alpha(t)}. \tag{56}$$

If κ_j and $\bar{\kappa}_j$ are the single roots of $a(k)$ and $\bar{a}(k)$, respectively, there must exist constants c_j and $\bar{c}_j$ such that

$$2\int_{-\infty}^{\infty}c_j^2\varphi_1(x,\kappa_j)\varphi_2(x,\kappa_j)\mathrm{d}x = 1, \tag{57}$$

$$2\int_{-\infty}^{\infty}\bar{c}_j^2\bar{\varphi}_1(x,\bar{\kappa}_j)\bar{\varphi}_2(x,\bar{\kappa}_j)\mathrm{d}x = 1, \tag{58}$$

then from Equations (55)–(58) we obtain

$$c_j^2 = -\frac{ib_j\alpha(t)}{a_k(\kappa_j)}, \tag{59}$$

$$\bar{c}_j^2 = -\frac{i\bar{b}_j\alpha(t)}{\bar{a}_k(\kappa_j)}. \tag{60}$$

Definition 1. We name c_j and $\bar{c}_j$ satisfying Equations (57) and (58) to be the normalization constants for the eigenfunctions $\varphi(x,\kappa_j)$ and $\bar{\varphi}(x,\bar{\kappa}_j)$, respectively; $c_j\varphi(x,\kappa_j)$ and $\bar{c}_j\bar{\varphi}(x,\bar{\kappa}_j)$ are named the normalization eigenfunctions.

Beside the discrete spectra κ_j and $\bar{\kappa}_j$, the spectral problem (9) also includes continuous spectral k which cannot be normalized but contain the whole real axis of the k-plane. Note that the Jost functions $\varphi(x,k)$, $\bar{\varphi}(x,k)$, $\psi(x,k)$ and $\bar{\psi}(x,k)$ satisfying Equations (9), (21)–(32) are bounded. Therefore, for any real number k the linear expressions (37) and (38) still hold in the real axis. Thus, we have

$$T(k)\psi(x,k) = \bar{\varphi}(x,k) + R(k)\varphi(x,k), \tag{61}$$

$$\bar{T}(k)\bar{\psi}(x,k) = -\varphi(x,k) + \bar{R}(k)\bar{\varphi}(x,k), \tag{62}$$

where $T(k) = \frac{1}{a(k)}$, $\bar{T}(k) = \frac{1}{\bar{a}(k)}$ and $R(k) = \frac{b(k)}{a(k)}$, $\bar{R}(k) = \frac{\bar{b}(k)}{\bar{a}(k)}$ are the transmission coefficients and the reflection coefficients, respectively.

Definition 2. The sets

$$\left\{k(\mathrm{Im}k=0), R(k), \kappa_j(\mathrm{Im}\kappa_j>0), c_j, j=1,2,\cdots,n\right\}, \tag{63}$$

$$\left\{k(\mathrm{Im}k=0), \bar{R}(k), \bar{\kappa}_j(\mathrm{Im}\bar{\kappa}_j<0), \bar{c}_j, j=1,2,\cdots,\bar{n}\right\} \tag{64}$$

are named the scattering data of the spectral problem (9).

3.2 The inverse scattering problem

Lemma 2. *Supposing the improper integrals* (20) *consisting of the real potentials* $q(x)$ *and* $r(x)$ *are convergent, then*

there must exist the following unique differentiable vectors:

$$\begin{aligned} K(x,y) &= (K_1(x,y), K_2(x,y))^T, \\ \bar{K}(x,y) &= (\bar{K}_1(x,y), \bar{K}_2(x,y))^T, \quad (x>y), \end{aligned} \tag{65}$$

$$\begin{aligned} J(x,y) &= (J_1(x,y), J_2(x,y))^T, \\ \bar{J}(x,y) &= (\bar{J}_1(x,y), \bar{J}_2(x,y))^T, \quad (x>y), \end{aligned} \tag{66}$$

such that

$$\varphi(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{[ik\alpha(t)+\beta(t)]x} + \int_x^\infty K(x,y) e^{[ik\alpha(t)+\beta(t)]y} dy, \tag{67}$$

$$\bar{\varphi}(x,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-[ik\alpha(t)+\beta(t)]x} + \int_x^\infty \bar{K}(x,y) e^{-[ik\alpha(t)+\beta(t)]y} dy, \tag{68}$$

$$\psi(x,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-[ik\alpha(t)+\beta(t)]x} + \int_{-\infty}^x J(x,y) e^{-[ik\alpha(t)+\beta(t)]y} dy, \tag{69}$$

$$\bar{\psi}(x,k) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{[ik\alpha(t)+\beta(t)]x} + \int_{-\infty}^x \bar{J}(x,y) e^{[ik\alpha(t)+\beta(t)]y} dy, \tag{70}$$

where $K(x,y)$ satisfies the integral equations

$$K_1(x,y) = -\frac{1}{2} q\left(\frac{x+y}{2}\right) - \int_x^{\frac{x+y}{2}} q(s) K_2(s, x+y-s) ds, \tag{71}$$

$$K_2(x,y) = -\int_x^\infty r(s) K_1(s, -x+y+s) ds, \tag{72}$$

particularly, when $y = s$

$$K_1(x,x) = -\frac{1}{2} q(x), \tag{73}$$

$$K_2(x,x) = -\frac{1}{2} \int_x^\infty q(s) r(s) ds. \tag{74}$$

Theorem 3. *If set*

$$\begin{aligned} F_c(x) &= \frac{1}{2\pi} \int_{-\infty}^\infty R(k) e^{[ik\alpha(t)+\beta(t)]x} dk, \\ F_d(x) &= \sum_{j=1}^n c_j^2 e^{[i\kappa_j \alpha(t)+\beta(t)]x}, \end{aligned} \tag{75}$$

$$\begin{aligned} \bar{F}_c(x) &= \frac{1}{2\pi} \int_{-\infty}^\infty \bar{R}(k) e^{-[ik\alpha(t)+\beta(t)]x} dk, \\ \bar{F}_d(x) &= -\sum_{j=1}^{\bar{n}} \bar{c}_j^2 e^{-[i\bar{\kappa}_j \alpha(t)+\beta(t)]x}, \end{aligned} \tag{76}$$

$$F(x) = F_c(x) + F_d(x), \quad \bar{F}(x) = \bar{F}_c(x) + \bar{F}_d(x), \tag{77}$$

then the vector functions $K(x,y)$ and $\bar{K}(x,y)$ satisfy

$$\bar{K}(x+y) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F(x+y) + \int_x^\infty K(x,z) F(z+y) dz = 0, \tag{78}$$

$$K(x+y) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{F}(x+y) + \int_x^\infty \bar{K}(x,z) \bar{F}(z+y) dz = 0. \tag{79}$$

Proof. Letting both sides of Equation(61) be substracted by $\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-[ik\alpha(t)+\beta(t)]x}$ simultaneously, then taking the Fourier transformation to k, we have

$$\begin{aligned} &\int_{-\infty}^\infty \left(T(k)\psi(x,k) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-[ik\alpha(t)+\beta(t)]x} \right) e^{[ik\alpha(t)+\beta(t)]y} dk \\ &= \int_{-\infty}^\infty \left(\bar{\varphi}(x,k) + R(k)\varphi(x,k) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-[ik\alpha(t)+\beta(t)]x} \right) \\ &\times \ e^{[ik\alpha(t)+\beta(t)]y} dk, \end{aligned} \tag{80}$$

the left side of which equals to

$$\begin{aligned} &\int_{-\infty}^\infty \left(T(k)\psi(x,k) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-[ik\alpha(t)+\beta(t)]x} \right) e^{[ik\alpha(t)+\beta(t)]y} dk \\ &= 2\pi\alpha(t) \sum_{j=1}^n \frac{\psi(x,\kappa_j)}{a_k(\kappa_j)} e^{[i\kappa_j \alpha(t)+\beta(t)]y}. \end{aligned} \tag{81}$$

In view of Equations (48), (59) and (67), we can rewrite Equation (81) as

$$\begin{aligned} &2\pi\alpha(t) \sum_{j=1}^n \frac{\psi(x,i\kappa_j)}{a_k} e^{[i\kappa_j \alpha(t)+\beta(t)]y} \\ &= -2\pi \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sum_{j=1}^n c_j^2 e^{[i\kappa_j \alpha(t)+\beta(t)](x+y)} \\ &-2\pi \int_x^\infty K(x,y) \sum_{j=1}^n c_j^2 e^{[i\kappa_j \alpha(t)+\beta(t)](y+z)} dz. \end{aligned} \tag{82}$$

Substituting Equation (60) into the first term of the right side of Equation (80) and using the inverse Fourier transformation, we have

$$\int_{-\infty}^{\infty}\left(\bar{\varphi}(x,k)-\begin{pmatrix}1\\0\end{pmatrix}\mathrm{e}^{-[ik\alpha(t)+\beta(t)]x}\right)\mathrm{e}^{[ik\alpha(t)+\beta(t)]y}\mathrm{d}k = 2\pi\bar{K}(x,y). \tag{83}$$

Taking advantage of Equation (59), we obtain from the second term of the right side of Equation (80)

$$\begin{aligned}&\int_{-\infty}^{\infty}R(k)\varphi(x,k)\mathrm{e}^{[ik\alpha(t)+\beta(t)]y}\mathrm{d}k\\ =\;&\begin{pmatrix}0\\1\end{pmatrix}\int_{-\infty}^{\infty}R(k)\mathrm{e}^{[ik\alpha(t)+\beta(t)](x+y)}\mathrm{d}k\\ +\;&\int_{x}^{\infty}K(x,z)\int_{-\infty}^{\infty}R(k)\mathrm{e}^{[ik\alpha(t)+\beta(t)](y+z)}\mathrm{d}k\mathrm{d}z.\end{aligned} \tag{84}$$

Finally, using Equations (75), (82)–(84), we can write (79) as

$$-\begin{pmatrix}0\\1\end{pmatrix}F_d(x+y)-\int_{x}^{\infty}K(x,z)F_d(z+y)\mathrm{d}z \tag{85}$$

$$=\bar{K}(x+y)+\begin{pmatrix}0\\1\end{pmatrix}F_c(x+y)+\int_{x}^{\infty}K(x,z)F_c(z+y)\mathrm{d}z=0, \tag{86}$$

which is namely Equation (78). In the same way, we can prove Equation (79). The proof is ended. □

3.3 The time dependence of the scattering data

Lemma 3. *Suppose that*

$$\bar{L}^{*}=-\sigma\partial+2\begin{pmatrix}-r\\q\end{pmatrix}\partial^{-1}(q,r),\quad \bar{L}=\sigma L\sigma,$$

then $\bar{L}^{*}$ is the conjugation operator of $\bar{L}$.

Theorem 4. *The scattering data in Equations* (63) *and* (64) *for the spectral problem* (9) *possess the following time dependence:*

$$\kappa_j(t)=\kappa_j,\quad \bar{\kappa}_j(t)=\bar{\kappa}_j, \tag{87}$$

$$c_j^2(t)=c_j^2(0)\mathrm{e}^{(2i\kappa_j)^m\int_0^t\gamma(t)\mathrm{d}s+2\ln|a(t)|}, \tag{88}$$

$$\bar{c}_j^2(t)=\bar{c}_j^2(0)\mathrm{e}^{-(2i\bar{\kappa}_j)^m\int_0^t\gamma(t)\mathrm{d}s+2\ln|\bar{a}(t)|}, \tag{89}$$

$$a(t,k)=a(0,k),\quad \bar{a}(t,k)=\bar{a}(0,k), \tag{90}$$

$$\begin{aligned}b(t,k)&=b(0,k)\mathrm{e}^{(2ik)^m\int_0^t\gamma(s)\mathrm{d}s},\\ \bar{b}(t,k)&=\bar{b}(0,k)\mathrm{e}^{-(2ik)^m\int_0^t\gamma(s)\mathrm{d}s},\end{aligned} \tag{91}$$

where $c_j(0)$, $\bar{c}_j(0)$, $R(0,k)=\frac{b(0,k)}{a(0,k)}$ and $\bar{R}(0,k)=\frac{\bar{b}(0,k)}{\bar{a}(0,k)}$ are the scattering data of Equation (9) *in the cases of $(q(0,x),r(0,x))^T$.*

Proof. It is easy to verify that if $\varphi(x,k)$ is a solution of Equation (9) and satisfies the asymptotic condition (22), then

$$P(x,k)=\varphi_t(x,k)-N\varphi(x,k) \tag{92}$$

is also a solution of Equation (9) and can be represented linearly by $\varphi(x,k)$ and $\widetilde{\varphi}(x,k)$ satisfying Equation (9) but is independent of $\varphi(x,k)$, i.e., there exist two functions $\theta(t,k)$ and $\tau(t,k)$ such that

$$\varphi_t(x,k)-N\varphi(x,k)=\theta(t,k)\varphi(x,k)+\tau(t,k)\widetilde{\varphi}(x,k). \tag{93}$$

Under the above preparation, we first consider $k=\kappa_j(\mathrm{Im}\kappa_j>0)$. Since $\varphi(x,\kappa_j)$ decays exponentially while $\widetilde{\varphi}(x,\kappa_j)$ must increase exponentially as $x\to+\infty$, we can determine $\tau(t,k)=0$. Thus Equation (93) is simplified as:

$$\varphi_t(x,\kappa_j)-N\varphi(x,\kappa_j)=\gamma(t,\kappa_j)\varphi(x,\kappa_j). \tag{94}$$

Left-multiplying Equation (94) by the inner product $(\varphi_2(x,\kappa_j),\varphi_1(x,\kappa_j))$ yields:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\varphi_1(x,\kappa_j)\varphi_2(x,\kappa_j)-[C\varphi_1^2(x,\kappa_j)+B\varphi_2^2(x,\kappa_j)]\\ &\qquad=2\theta(t,\kappa_j)\varphi_1(x,\kappa_j)\varphi_2(x,\kappa_j).\end{aligned} \tag{95}$$

Supposing $\varphi(x,\kappa_j)$ to be the normalization eigenfunction and further integrating Equation (95) with respect to x from $-\infty$ to $+\infty$, then noting that $2\int_{-\infty}^{\infty}c_j^2\varphi_1(x,\kappa_j)\varphi_2(x,\kappa_j)\mathrm{d}x=1$ we have

$$\theta(t,\kappa_j)=-c_j^2\int_{-\infty}^{\infty}[C\varphi_1^2(x,\kappa_j)+B\varphi_2^2(x,\kappa_j)]\mathrm{d}x. \tag{96}$$

For convenience, we rewrite Equation (96) as

$$\theta(t,\kappa_j)=-c_j^2((\varphi_2^2(x,\kappa_j),\varphi_1^2(x,\kappa_j))^T,(B,C)^T), \tag{97}$$

where the following inner product is employed

$$(f(x),g(x))=\int_{-\infty}^{\infty}(f_1(x)g_1(x)+f_2(x)g_2(x))\mathrm{d}x \tag{98}$$

for arbitrary two vectors $f(x)=(f_1(x),f_2(x))^T$ and $g(x)=(g_1(x),g_2(x))^T$.

Using the spectral problem (9), we have

$$\varphi_{1x}(x,\kappa_j)+[i\kappa_j\alpha(t)+\beta(t)]\varphi_1(x,\kappa_j)=q\varphi_2(x,\kappa_j), \quad (99)$$

$$\varphi_{2x}(x,\kappa_j)-[i\kappa_j\alpha(t)+\beta(t)]\varphi_2(x,\kappa_j)=r\varphi_1(x,\kappa_j), \quad (100)$$

from which we derive

$$\begin{aligned}&[\varphi_1(x,\kappa_j)\varphi_2(x,\kappa_j)]_x=\varphi_2(x,\kappa_j)\varphi_{1x}(x,\kappa_j)\\ +\quad &\varphi_{2x}(x,\kappa_j)\varphi_1(x,\kappa_j)=q\varphi_2^2(x,\kappa_j)+r\varphi_1^2(x,\kappa_j)\end{aligned} \quad (101)$$

and then obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}[q\varphi_2^2(x,\kappa_j)+r\varphi_1^2(x,\kappa_j)]\mathrm{d}x\\ =\ &\int_{-\infty}^{\infty}[\varphi_1(x,\kappa_j)\varphi_2(x,\kappa_j)]_x\mathrm{d}x=0.\end{aligned} \quad (102)$$

On the other hand, if we rewrite Equation (13) as

$$\begin{pmatrix}B\\C\end{pmatrix}=\sum_{j=1}^{m}\gamma(t)(2\kappa_j)^{m-j}\frac{[\bar{L}-2\beta(t)E]^{j-1}}{\alpha^j(t)}\begin{pmatrix}q\\r\end{pmatrix}+\frac{\alpha'(t)}{\alpha(t)}\begin{pmatrix}xq\\xr\end{pmatrix}, \quad (103)$$

then from Equation (97) we obtain

$$\begin{aligned}\theta(t,\kappa_j)\ =\ &-c_j^2\left(\begin{pmatrix}\varphi_2^2(t,\kappa_j)\\\varphi_1^2(t,\kappa_j)\end{pmatrix},\right.\\ &\left.\sum_{j=1}^{m}\gamma(t)(2\kappa_j)^{m-j}\frac{[\bar{L}-2\beta(t)E]^{j-1}}{\alpha^j(t)}\begin{pmatrix}q\\r\end{pmatrix}\right)\\ -\ &c_j^2\frac{\alpha'(t)}{\alpha(t)}\left(\begin{pmatrix}\varphi_2^2(t,\kappa_j)\\\varphi_1^2(t,\kappa_j)\end{pmatrix},\begin{pmatrix}xq\\xr\end{pmatrix}\right)\\ =\ &-c_j^2\sum_{j=1}^{m}\gamma(t)\frac{(2\kappa_j)^{m-j}}{\alpha^j(t)}(\begin{pmatrix}\varphi_2^2(t,\kappa_j)\\\varphi_1^2(t,\kappa_j)\end{pmatrix},\\ &[\bar{L}-2\beta(t)E]^{j-1}\begin{pmatrix}q\\r\end{pmatrix}\Big)\\ -\ &c_j^2\frac{\alpha'(t)}{\alpha(t)}\left(\begin{pmatrix}\varphi_2^2(t,\kappa_j)\\\varphi_1^2(t,\kappa_j)\end{pmatrix},\begin{pmatrix}xq\\xr\end{pmatrix}\right)\\ =\ &-c_j^2\sum_{j=1}^{m}\gamma(t)\frac{(2\kappa_j)^{m-j}}{\alpha^j(t)}[\bar{L}-2\beta(t)E]^{j-1}\\ \times\ &\left(\begin{pmatrix}\varphi_2^2(t,\kappa_j)\\\varphi_1^2(t,\kappa_j)\end{pmatrix},\begin{pmatrix}q\\r\end{pmatrix}\right)\\ &-c_j^2\frac{\alpha'(t)}{\alpha(t)}\left(\begin{pmatrix}\varphi_2^2(t,\kappa_j)\\\varphi_1^2(t,\kappa_j)\end{pmatrix},\begin{pmatrix}xq\\xr\end{pmatrix}\right).\end{aligned} \quad (104)$$

Further, using the results

$$\left(\begin{pmatrix}\varphi_2^2(t,\kappa_j)\\\varphi_1^2(t,\kappa_j)\end{pmatrix},\begin{pmatrix}xq\\xr\end{pmatrix}\right)=\int_{-\infty}^{\infty}x[q\varphi_2^2(t,\kappa_j)+r\varphi_1^2(t,\kappa_j)]\mathrm{d}x$$

$$=\int_{-\infty}^{\infty}x[\varphi_1(t,\kappa_j)\varphi_2(t,\kappa_j)]_x\mathrm{d}x=-\frac{1}{2c_j^2}, \quad (105)$$

$$\left([\bar{L}^*-2\beta(t)E]^{j-1}\begin{pmatrix}\varphi_2^2(t,\kappa_j)\\\varphi_1^2(t,\kappa_j)\end{pmatrix},\begin{pmatrix}q\\r\end{pmatrix}\right)$$

$$=2[i\kappa_j\alpha(t)+\beta(t)E]^{j-1}\left(\begin{pmatrix}\varphi_2^2(t,\kappa_j)\\\varphi_1^2(t,\kappa_j)\end{pmatrix},\begin{pmatrix}q\\r\end{pmatrix}\right)=0, \quad (106)$$

from Equation (104) we have

$$\theta(t,\kappa_j)=\frac{\alpha'(t)}{2\alpha(t)}, \quad (107)$$

where we have used

$$\bar{L}^*\begin{pmatrix}\varphi_2^2(t,\kappa_j)\\\varphi_1^2(t,\kappa_j)\end{pmatrix}=2i\kappa_j\alpha(t)\begin{pmatrix}\varphi_2^2(t,\kappa_j)\\\varphi_1^2(t,\kappa_j)\end{pmatrix}. \quad (108)$$

Thus, Equation (94) reads

$$\varphi_t(x,\kappa_j)-N\varphi(x,\kappa_j)=\frac{\alpha'(t)}{2\alpha(t)}\varphi(x,\kappa_j). \quad (109)$$

Noting that Equation (11) and

$$\varphi(x,\kappa_j)\to c_j\begin{pmatrix}0\\1\end{pmatrix}e^{[\kappa_j\alpha_t(t)+\beta'(t)]x}, \quad (110)$$

$$\varphi_t(x,\kappa_j)\to c_{jt}\begin{pmatrix}0\\1\end{pmatrix}\mathrm{e}^{[\kappa_j\alpha_t(t)+\beta'(t)]x}$$

$$+xc_j[i\kappa_j\alpha'(t)+i\kappa_{jt}\alpha(t)+\beta'(t)]\begin{pmatrix}0\\1\end{pmatrix}\mathrm{e}^{[i\kappa_j\alpha(t)+\beta(t)]x}. \quad (111)$$

as $x\to+\infty$, then we derive

$$\kappa_{jt}=0,\quad c_{jt}-c_j\frac{1}{2}\gamma(t)(2i\kappa_j)^m=c_j\frac{\alpha'(t)}{2\alpha(t)}. \quad (112)$$

In a similar way, we have

$$\bar{\kappa}_{jt}=0,\quad \bar{c}_{jt}+\bar{c}_j\frac{1}{2}\gamma(t)(2i\bar{\kappa}_j)^m=\bar{c}_j\frac{\alpha'(t)}{2\alpha(t)}, \quad (113)$$

then from Equations (112) and (113) we obtain Equations (87)–(89).

We next consider k as a real continuous spectrum and take a solution $\psi(x,k)$ of Equation (9) satisfying the asymptotic condition (22), then from Equations (92) and (93) we know that

$$Q(x,k)=\psi_t(x,k)-N\psi(x,k) \quad (114)$$

is also a solution of Equation (9) and can be represented linearly by $\psi(x,k)$ and $\widetilde{\psi}(x,k)$ which satisfies Equation (9)

but is independent of $\psi(x,k)$, i.e., there exist two functions $\omega(t,k)$ and $\vartheta(t,k)$ such that

$$\psi_t(x,k) - N\psi(x,k) = \omega(t,k)\psi(x,k) + \vartheta(t,k)\widetilde{\psi}(x,k). \quad (115)$$

Using the asymptotic properties

$$\psi_t(x,k) \to -\frac{1}{2}x[ik\alpha'(t) + ik_t\alpha(t) + \beta'(t)]\begin{pmatrix}1\\0\end{pmatrix}\mathrm{e}^{-[ik\alpha(t)+\beta(t)]x}, \quad (116)$$

$$\psi(x,k) \to \begin{pmatrix}1\\0\end{pmatrix}e^{-[ik\alpha(t)+\beta(t)]x}, \quad (117)$$

as $x \to -\infty$, from Equation (34) and (115) we obtain

$$\vartheta(t,k) = 0, \quad \omega(t,k) = \frac{1}{2}\gamma(t)(2k)^m. \quad (118)$$

Substituting Equations (37) and (118) into Equation (115) yields

$$\begin{aligned} & [a(t,k)\bar{\varphi}(x,k) + b(t,k)\varphi(x,k)]_t \\ - \; & N[a(t,k)\bar{\varphi}(x,k) + b(t,k)\varphi(x,k)] \\ = \; & \frac{1}{2}\gamma(t)(2k)^m[a(t,k)\bar{\varphi}(x,k) + b(t,k)\varphi(x,k)], \end{aligned} \quad (119)$$

then letting $x \to +\infty$ and using

$$\begin{aligned} \varphi(x,k) &\to \begin{pmatrix}0\\1\end{pmatrix}\mathrm{e}^{[ik\alpha(t)+\beta(t)]x}, \\ \bar{\varphi}(x,k) &\to \begin{pmatrix}1\\0\end{pmatrix}\mathrm{e}^{-[ik\alpha(t)+\beta(t)]x}, \end{aligned} \quad (120)$$

from Equation (119) we obtain

$$a_t(t,k) = 0, \quad b_t(t,k) = \gamma(t)(2k)^m b(t,k). \quad (121)$$

Similarly, we have

$$\bar{a}_t(t,k) = 0, \quad \bar{b}_t(t,k) = -\gamma(t)(2k)^m \bar{b}(t,k). \quad (122)$$

Solving Equations (121) and (122) directly gives Equations (90) and (91). We finish the proof. □

3.4 Exact solutions

According to Equations (73), (74) and Theorem 4, we have the following Theorem 5.

Theorem 5. *Given the scattering data* (63) *and* (64) *determined by Equations* (87)–(91) *for the spectral problem* (9), *we can obtain exact solution of the AKNS hierarchy* (1)*:*

$$q = -2K_1(t,x,x), \quad (123)$$

$$r = \frac{K_{2x}(t,x,x)}{K_1(t,x,x)}, \quad (124)$$

where $K(t,x,y) = (K_1(t,x,y), K_2(t,x,y))^T$ *satisfies the Gel'fand-Levitan-Marchenko (GLM) integral equation:*

$$K(t,x,y) - \begin{pmatrix}0\\1\end{pmatrix}\bar{F}(t,x+y) + \begin{pmatrix}0\\1\end{pmatrix}\int_x^\infty F(t,z+x)\bar{F}(t,z+y)\mathrm{d}z$$

$$+ \int_x^\infty K(t,x,s)\int_x^\infty F(t,z+s)\bar{F}(t,z+y)\mathrm{d}z\mathrm{d}s = 0, \quad (125)$$

with

$$F(t,x) = \frac{1}{2\pi}\int_{-\infty}^\infty R(t,k)\mathrm{e}^{[ik\alpha(t)+\beta(t)]x}\mathrm{d}k + \sum_{j=1}^n c_j^2\mathrm{e}^{[i\kappa_j\alpha(t)+\beta(t)]x}, \quad (126)$$

$$\bar{F}(t,x) = \frac{1}{2\pi}\int_{-\infty}^\infty \bar{R}(t,k)\mathrm{e}^{[ik\alpha(t)+\beta(t)]x}\mathrm{d}k + \sum_{j=1}^{\bar{n}} \bar{c}_j^2\mathrm{e}^{[i\bar{\kappa}_j\alpha(t)+\beta(t)]x}. \quad (127)$$

3.5 Soliton Solutions

In order to give explicit form of solutions (123) and (124), we consider the reflectionless potentials $q(x,t)$ and $r(x,t)$, namely $R(t,k) = \bar{R}(t,k) = 0$. In this reflectionless case, the GLM integral equation (125) can be solved exactly. If we let E be a $\bar{n}\times\bar{n}$ unit matrix and take

$$\begin{aligned} W(t,x) &= E + P(t,x)P^T(t,x), \\ P(t,x) &= \left(\frac{c_j(t)\bar{c}_m(t)}{\alpha(t)(\kappa_j - \bar{\kappa}_m)}\mathrm{e}^{i\alpha(t)(\kappa_j-\bar{\kappa}_m)x}\right)_{\bar{n}\times n}, \end{aligned} \quad (128)$$

$$\begin{aligned} \Lambda = \Big(& c_1\mathrm{e}^{-[i\kappa_1\alpha(t)+\beta(t)]x}, c_2\mathrm{e}^{-[i\kappa_2\alpha(t)+\beta(t)]x}, \\ & \cdots, c_n\mathrm{e}^{-[i\kappa_n\alpha(t)+\beta(t)]x}\Big)^T, \end{aligned} \quad (129)$$

$$\begin{aligned} \bar{\Lambda} = \Big(& \bar{c}_1\mathrm{e}^{-[i\bar{\kappa}_1\alpha(t)+\beta(t)]x}, \bar{c}_2\mathrm{e}^{-[i\bar{\kappa}_2\alpha(t)+\beta(t)]x}, \\ & \cdots, \bar{c}_{\bar{n}}\mathrm{e}^{-[i\bar{\kappa}_{\bar{n}}\alpha(t)+\beta(t)]x}\Big)^T, \end{aligned} \quad (130)$$

then it is easy to obtain

$$K_1(x,y,t) = -\mathrm{tr}(W^{-1}(t,x)\bar{\Lambda}(t,x)\bar{\Lambda}^T(t,y)), \quad (131)$$

$$K_2(x,y,t) = -i\mathrm{tr}(W^{-1}(t,x)P(t,x)\Lambda(t,x)\bar{\Lambda}^T(t,y)), \quad (132)$$

where $\mathrm{tr}(A)$ means the trace of a given matrix A, $W^{-1}(t,x)$ is the inverse matrix of $W(t,x)$.

Substituting Equations (131) and (132) into Equations (123) and (124), we can obtain n-soliton solutions of the mixed spectral AKNS hierarchy (1):

$$q = 2\mathrm{tr}(W^{-1}(t,x)\bar{\Lambda}(t,x)\bar{\Lambda}^T(t,x)), \quad (133)$$

$$r = -\frac{[\mathrm{tr}(W^{-1}(t,x)E(t,x)E_x^T(t,x))]_x}{\mathrm{tr}(W^{-1}(t,x)\bar{\Lambda}(t,x)\bar{\Lambda}^T(t,x))}. \quad (134)$$

In particular, when $n = \bar{n} = 1$, Equations (133) and (134) give the single-soliton solutions:

$$q = \frac{2\bar{c}_1^2(0)\mathrm{e}^{-2[i\alpha(t)\bar{\kappa}_1+\beta(t)]x+(2i\kappa_1)^m\int_0^t\gamma(s)\mathrm{d}s+\ln|\alpha(t)|}}{1+\left[\frac{c_1(0)\bar{c}_1(0)}{\alpha(t)(\kappa_1-\bar{\kappa}_1)}\right]^2\mathrm{e}^{2i\alpha(t)(\kappa_1-\bar{\kappa}_1)x+[(2i\kappa_1)^m-(2i\bar{\kappa}_1)^m]\int_0^t\gamma(s)\mathrm{d}s+\ln|\alpha(t)|}}, \quad (135)$$

$$r = \frac{2c_1^2(0)\mathrm{e}^{2[i\alpha(t)\bar{\kappa}_1+\beta(t)]x+(2i\kappa_1)^m\int_0^t\gamma(s)\mathrm{d}s+\ln|\alpha(t)|}}{1+\left[\frac{c_1(0)\bar{c}_1(0)}{\alpha(t)(\kappa_1-\bar{\kappa}_1)}\right]^2\mathrm{e}^{2i\alpha(t)(\kappa_1-\bar{\kappa}_1)x+[(2i\kappa_1)^m-(2i\bar{\kappa}_1)^m]\int_0^t\gamma(s)\mathrm{d}s+\ln|\alpha(t)|}}. \quad (136)$$

Clearly, solutions (133) and (134) have three functions $\alpha(t)$, $\beta(t)$ and $\gamma(t)$, the arbitrariness of which provide enough freedom to construct enriched local structures of solutions. In Figures 1 and 2, the single-soliton solutions (135) and (136) are shown by selecting $c_1(0) = i$, $\bar{c}_1(0) = 1$, $\kappa_1 = 0.5i$, $\bar{\kappa}_1 = i$, $\alpha(t) = 0.2\mathrm{sech}(0.2t)$, $\beta(t) = 0.01t$, $\gamma(t) = 0.01t^3$. In Figures 3 and 4, we select $c_1(0) = 1$, $\bar{c}_1(0) = 2i$, $c_2(0) = 3$, $\bar{c}_2(0) = 4i$, $\kappa_1(0) = 0.5i$, $\bar{\kappa}_1(0) = i$, $\kappa_2(0) = 2i$, $\bar{\kappa}_2(0) = -3i$, $\alpha(t) = 0.2\mathrm{sech}(0.2t^3)$, $\beta(t) = 0.01t$ and $\gamma(t) = -0.01t^3$ to show the double-soliton solutions determined by (133) and (134) when $n = \bar{n} = 2$.

4 Conclusions

In summary, starting from the related isospectral problem we have derived a new and more general AKNS hierarchy (1), called the mixed spectral AKNS hierarchy, which includes the known constant-coefficient isospectral AKNS hierarchy (3) and a new variable-coefficient nonisospectral AKNS hierarchy (4) as special cases. Since the obtained AKNS hierarchy (1) contains nonisospectral equations (7) and (8), the AKNS spectral problem (9) being nonisospectral is not a necessary condition to construct nonisospectral AKNS hierarchy (1). In order to solve the mixed spectral AKNS hierarchy (1), we utilized the IST. As a result, the uniform formulae (123) and (124) of exact solutions of the mixed spectral AKNS hierarchy (1) are obtained. In the case of reflectionless potentials, the obtained exact solutions (123) and (124) are reduced to n-soliton solutions (133) and (134). Usually, the procedures of the IST for solving nonlinear PDEs are analogous, the steps of which can be outlined as follows: the first step is to solve the initial-value problem of linear spectral problem for the required scattering data; the second step is to determine the time dependence of scattering data via the time evolution equation of eigenfunction associated with the linear spectral problem; the last step is to reconstruct the potential function by the time dependence of scattering data obtained in the second step. To make the procedure of the IST self-contained, these three steps need to be included. Though some of the obtained results are similar to those given in Ref. [30], there are substantial differences. For example, Equations (39) and (40) are similar to Equations (3.4a) and (3.4b) in Ref. [30], but they are different because of the different eigenfunctions (33)–(36) in the Wronskians. Especially, the mixed spectral AKNS hierarchy (1), exact solutions (123) and (124), and n-soliton solutions (133) and (134) obtained in this paper cannot be obtained by the work of Ref. [30]. For the convenience of subsequent discussions, we have done some similar but necessary expressions in advance. Such similar expressions also provide convenience in comparing our results and those in references. In Ref. [30], a nonisospectral AKNS hierarchy is derived from the AKNS nonisospectral problem with $\eta_t = \frac{1}{2}(2\eta)^{2n}$ and then the nonisospectral AKNS hierarchy is solved by means of the IST, where

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L^n \begin{pmatrix} -xq \\ xr \end{pmatrix}, \quad (n = 0, 1, 2, \cdots), \quad (137)$$

$$\begin{aligned} M &= \begin{pmatrix} -\eta & q \\ r & \eta \end{pmatrix}, \\ N|_{(q,r)=(0,0)} &= \begin{pmatrix} -\frac{1}{2}(2\eta)^n x & 0 \\ 0 & \frac{1}{2}(2\eta)^m x \end{pmatrix}, \end{aligned} \quad (138)$$

$$\begin{aligned} A &= \partial^{-1}(r,q)\begin{pmatrix} -B \\ C \end{pmatrix} - \frac{1}{2}(2\eta)^n x, \\ \begin{pmatrix} B \\ C \end{pmatrix} &= \sum_{m=1}^{n}(2ik_j)^{n-m}\bar{L}^{m-1}\begin{pmatrix} xq \\ xr \end{pmatrix}, \end{aligned} \quad (139)$$

which differ from the ones in Equations (1), (9), (11), (12) and (106) of this paper.

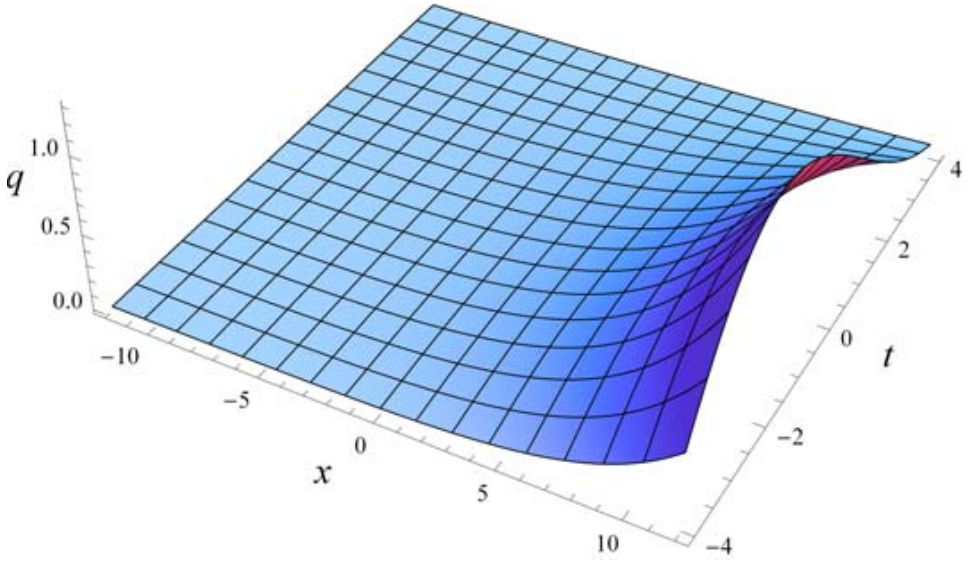

Figure 1: Spatial structure of single-soliton solution (135).

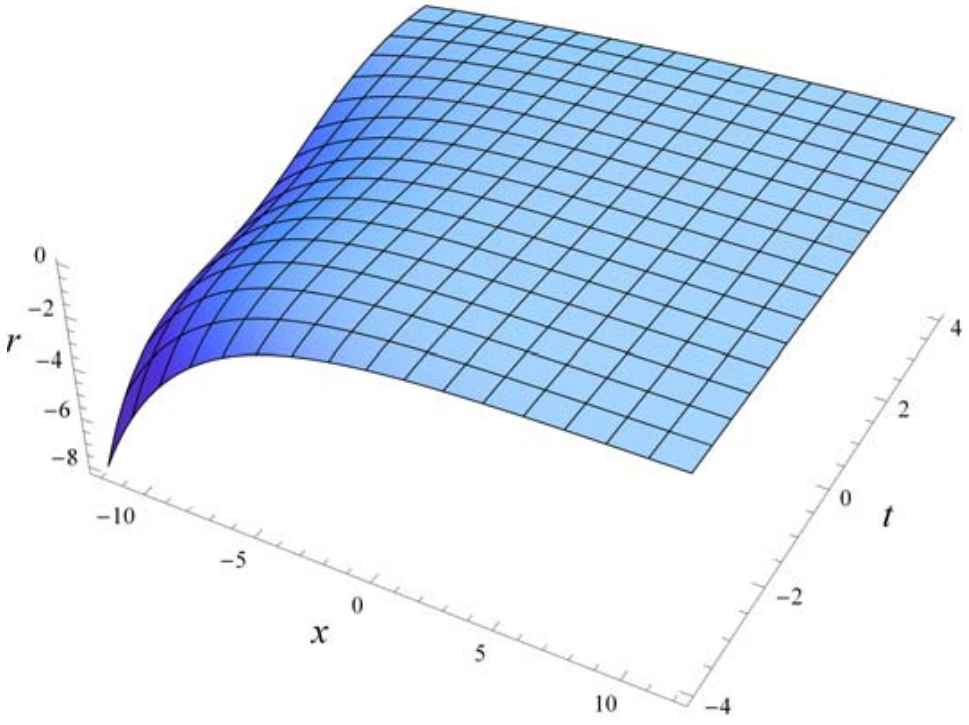

Figure 2: Spatial structure of single-soliton solution (136).

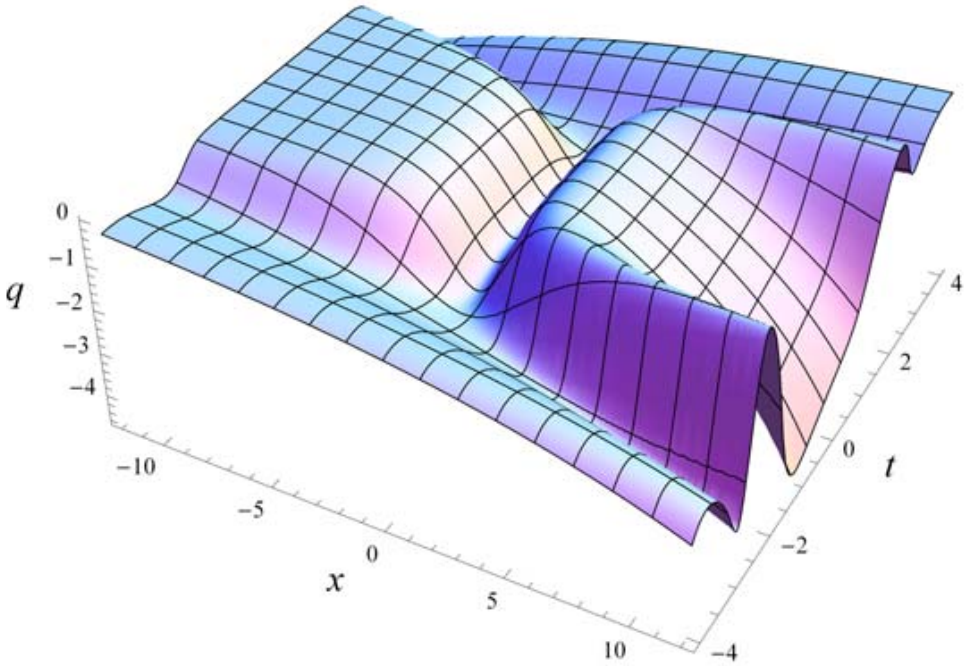

Figure 3: Spatial structure of double-soliton solution determined by (133).

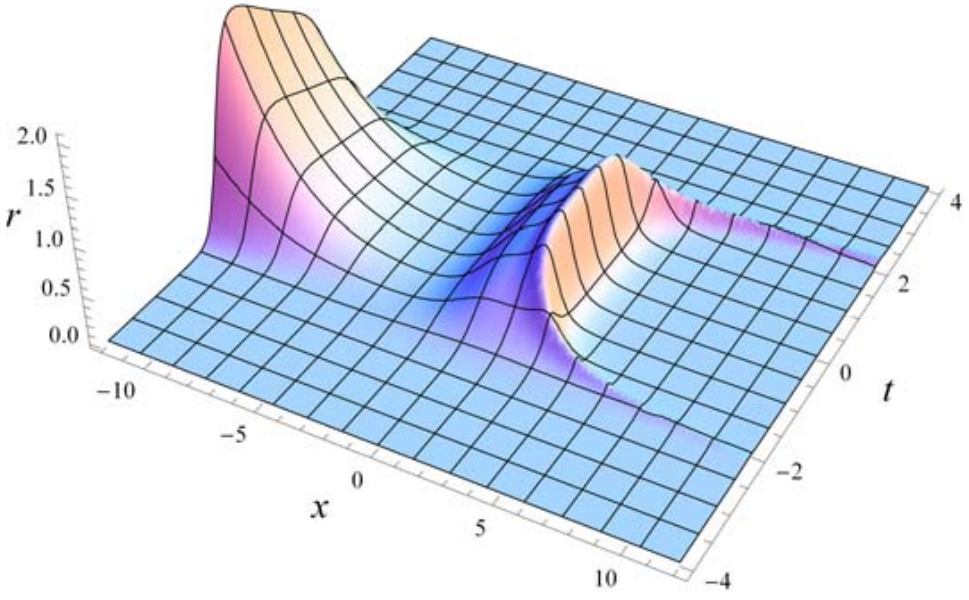

Figure 4: Spatial structure of double-soliton solution determined by (134).

More importantly, the mixed spectral AKNS hierarchy (1) obtained in this paper is resulted from the AKNS isospectral problem (9), i.e., $\eta_t = 0$. Other main differences, such as Lemmata 1 and 2, Theorems 1–5, solutions (120), (121), (133) and (134), caused by the coefficient functions $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are omitted here for simplicity. To the best of our knowledge, the mixed spectral AKNS hierarchy (1) and its solutions (123), (124), (133) and (134) obtained in this paper have not been reported in literature. How to extend the method used in this paper for some other variable-coefficient mixed hierarchies of isospectral equations and nonisospectral equations is worthy of study. This is our task in the future.

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