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On integral equations with Weakly Singular kernel by using Taylor series and Legendre polynomials

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Abstract: This paper is concerned with the numerical solution for a class of weakly singular Fredholm integral equations of the second kind. The Taylor series of the unknown function, is used to remove the singularity and the truncated Taylor series to second order of $k(x, y)$ about the point (x_0, y_0) is used. The integrals that appear in this method are computed exactly and some of these integrals are computed with the Cauchy principal value without using numerical quadratures. The solution in the Legendre polynomial form generates a system of linear algebraic equations, this system is solved numerically. Through numerical examples, performance of the present method is discussed concerning the accuracy of the method.

Keywords: weakly singular; Fredholm integral equations; Taylor series; Galerkin method; Legendre functions

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1 Introduction

Integral equations with weakly singular kernel have many practical applications in mathematical physics. These equations arise in Dirichlet problems, mathematical problems of radiative equilibrium and radiative heat transfer

problems [1–4]. Furthermore, Badr in [5] has cited important applications of weakly singular integral equations in the fields of fracture mechanics, elastic contact problems, the theory of porous filtering, combined infrared radiation and molecular conduction.

Although, in some special cases, analytical solutions of weakly singular integral equations are obtained [6–8], but generally this is not an easy task. Hence approximate solutions are required.

In this paper, the following type of singular Fredholm integral equation is used

$$\mu(x)\phi(x) + \lambda(x) \int_{-1}^1 \frac{k(x, y)\phi(y)}{(y-x)^\alpha} dy = f(x), \quad (1)$$

$$|x| < 1, \quad 0 < \alpha \leq 1,$$

where $\mu(x) \neq 0$, $\lambda(x) \neq 0$ and $k(x, y)$ is a smooth function and $\mu(x)$, $\lambda(x)$, $f(x)$, $k(x, y) \in L^2[-1, 1]$ are given functions and $\phi(x)$ is the unknown function to be determined. It is assumed that the function $\phi(y)$ has the Taylor series expansion. The main purpose of this paper is to present a numerical method for solving integral Equation (1), which is generalization of the paper [9] for $k(x, y) = 1$.

Until now, some different numerical solutions for weakly singular Fredholm integral equations are considered. In [9] Babolian *et al.* solved (1) with $k(x, y) = 1$, they removed singularity with Taylor expansion of $\phi(y)$ at point x , and then used Legendre functions as basis and computed all definite integrals involved without numerical quadratures. In [10] Lakestani *et al.* utilized Legendre multiwavelets as basis to reduce the solution of Fredholm integro-differential equation to the solution of sparse linear system of algebraic equations. In [11] Jiang *et al.* gave representation of the exact solution of third or first kind weakly singular Fredholm integral equations by a series in the reproducing kernel space. Pedas *et al.* [12], applied fully discrete version of a piecewise polynomial collocation method to solve initial or boundary value problems of Fredholm integro-differential equations with weakly singular kernel.

Generally for finding the numerical solution of an integral equation, first the equation is approximated using an algebraic system of equations and then the solution of this

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system gives an approximate solution of original integral equation.

In this paper by using Taylor expansion of $\phi(y)$ about the point x , the singularity of integral equation(1) is removed and then the truncated Taylor series to second order of $k(x, y)$ about the point (x_0, y_0) is used and some integrals that appear in this method are computed without numerical quadratures to convert the integral Equation (1) approximately to a system of algebraic equations. Finally the Legendre functions are used as basis and with Galerkin method the approximate solution is computed.

The paper is organized as follows Section 2 presents the method, Section 3 uses the Legendre function as the base to implement the method, Section 4 consider the estimation of errors and the convergence of the method and Section 5 contains numerical examples of the applied method. The paper is finished with concluding remarks.

2 Numerical solution

By considering the Fredholm integral Equation (1) and expanding the solution function $\phi(y)$ at point $x \in (0, 1)$, the following function is obtained:

$$\phi(y) = \phi(x) + (y-x)\phi'(x) + \frac{(y-x)^2}{2!}\phi''(x) + \dots + \frac{(y-x)^n}{n!}\phi^{(n)}(x) + \frac{(y-x)^{n+1}}{(n+1)!}\phi^{(n+1)}(\xi_{x,y}) \quad (2)$$

in which $\xi_{x,y}$, is between x and y . By substituting (2) into Equation (1),

$$\mu(x)\phi(x) + \lambda(x) \sum_{k=0}^n \frac{\phi^{(k)}(x)}{k!} \int_{-1}^1 k(x, y)(y-x)^{k-\alpha} dy + E_n(x) = f(x), \quad (3)$$

where

$$\phi^{(0)} = \phi(x),$$

$$E_n(x) = \frac{1}{(n+1)!} \int_{-1}^1 k(x, y)(y-x)^{n+1-\alpha} \phi^{(n+1)}(\xi_{x,y}) dy. \quad (4)$$

By using the truncated Taylor series of $\phi(y)$, the Equation (3) becomes

$$\mu(x)\phi(x) + \lambda(x) \sum_{k=0}^n \frac{\phi^{(k)}(x)}{k!} I_{\alpha,k} \simeq f(x) \quad (5)$$

where

$$I_{\alpha,k} = \int_{-1}^1 k(x, y)(y-x)^{k-\alpha} dy, \\ |x| < 1, \quad 0 < \alpha \leq 1, \quad k = 0, 1, \dots, n.$$

With the Taylor series of $k(x, y)$ about the point (x_0, y_0) the equation becomes

$$\begin{aligned} k(x, y) &= k(x_0, y_0) + (x-x_0)k_x(x_0, y_0) + (y-y_0)k_y(x_0, y_0) \\ &+ \frac{1}{2!} \left[(x-x_0)^2 k_{xx}(x_0, y_0) + (x-x_0)(y-y_0)k_{xy}(x_0, y_0) \right. \\ &\left. + (y-y_0)^2 k_{yy}(x_0, y_0) \right] + \dots \\ &= \left[k(x_0, y_0) + (x-x_0)k_x(x_0, y_0) \right. \\ &\left. + \frac{1}{2!} (x-x_0)^2 k_{xx}(x_0, y_0) + \dots \right] \\ &+ \left[k_y(x_0, y_0) + \left(\frac{2}{1} \right) \frac{1}{2!} (x-x_0)k_{yx}(x_0, y_0) + \right. \\ &\left. + \left(\frac{3}{1} \right) \frac{1}{3!} (x-x_0)^2 k_{yxx}(x_0, y_0) + \dots \right] (y-y_0) \\ &+ \left[\frac{1}{2!} k_{yy}(x_0, y_0) + \left(\frac{3}{2} \right) \frac{1}{3!} (x-x_0)k_{yyx}(x_0, y_0) \right. \\ &\left. + \left(\frac{4}{2} \right) \frac{1}{4!} (x-x_0)^2 k_{yyxx}(x_0, y_0) + \dots \right] (y-y_0)^2 + \dots \end{aligned}$$

Equivalently,

$$k(x, y) = \kappa_0(x) + \kappa_1(x)(y-y_0) + \kappa_2(x)(y-y_0)^2 + \kappa_3(x)(y-y_0)^3 + \dots, \quad (6)$$

where

$$\kappa_i(x) = \sum_{j=i}^{\infty} \binom{j}{i} \frac{\partial^j k(x, y)}{\partial y^i \partial x^{j-i}} \frac{(x-x_0)^{j-i}}{j!}, \quad i = 0, 1, \dots$$

By using the truncated Taylor series of $k(x, y)$ about the point (x_0, y_0) the definite integral, $I_{\alpha,k}$, can be computed exactly:

$$I_{\alpha,k} = \sum_{i=0}^m \kappa_i(x) \int_{-1}^1 (y-y_0)^i (y-x)^{k-\alpha} dy, \\ |x| < 1, \quad 0 < \alpha \leq 1, \quad k = 0, 1, \dots, n.$$

For $k = 0, i = 0$, and $0 < \alpha < 1$, the integral becomes:

$$\begin{aligned} \int_{-1}^1 (y-x)^{-\alpha} dy &= \int_{-1}^x (y-x)^{-\alpha} dy + \int_x^1 (y-x)^{-\alpha} dy \\ &= \frac{1}{1-\alpha} \left[(1-x)^{1-\alpha} - (-1-x)^{1-\alpha} \right]. \end{aligned}$$

For $k = 0, i = 0$, and $\alpha = 1$

$$c.p.v \int_{-1}^1 \frac{dy}{(y-x)} = \ln \left(\frac{1-x}{1+x} \right), \quad |x| < 1.$$

For $l > 0$, $k = 0, 1, \dots, n$, and $0 < \alpha \leq 1$,

$$\begin{aligned} \int_{-1}^1 (y-x)^{k-\alpha+l} dy &= \left. \frac{(y-x)^{k-\alpha+l+1}}{k-\alpha+l+1} \right|_{-1}^1 \\ &= \frac{1}{k-\alpha+l+1} \left[(1-x)^{k-\alpha+l+1} - (-1-x)^{k-\alpha+l+1} \right]. \end{aligned} \quad (7)$$

With Equation (7) and the binomial identity:

$$\begin{aligned} &\int_{-1}^1 (y-y_0)^i (y-x)^{k-\alpha} dy \\ &= \int_{-1}^1 (y-x+x-y_0)^i (y-x)^{k-\alpha} dy \\ &= \sum_{l=0}^i \binom{i}{l} (x-y_0)^{i-l} \int_{-1}^1 (y-x)^{k-\alpha+l} dy \\ &= \sum_{l=0}^i \binom{i}{l} (x-y_0)^{i-l} \frac{1}{k-\alpha+l+1} \\ &\quad \left[(1-x)^{k-\alpha+l+1} - (-1-x)^{k-\alpha+l+1} \right] \\ &= \sum_{l=0}^i \binom{i}{l} (x-y_0)^{i-l} \frac{1}{k-\alpha+l+1} \\ &\quad \left[\sum_{j=0}^{\infty} \binom{k-\alpha+l+1}{j} (-x)^j \right. \\ &\quad \left. + (-1)^{k-\alpha+l} \binom{k-\alpha+l+1}{j} (x)^j \right] \\ &= \sum_{l=0}^i \binom{i}{l} (x-y_0)^{i-l} \frac{1}{k-\alpha+l+1} \\ &\quad \left[\sum_{j=0}^{\infty} ((-1)^j + (-1)^{k-\alpha+l}) \binom{k-\alpha+l+1}{j} x^j \right], \end{aligned} \quad (8)$$

Remark 1. In these computations, for $0 < \alpha < 1$, $(-1)^\alpha$ is a complex value equal to

$$\cos(\alpha\pi) + i \sin(\alpha\pi). \quad (9)$$

Now, a new numerical method is proposed to solve Equation (5). Let $\{\phi_n(x)\}_{n=0}^\infty$ be a complete orthogonal basis with respect to the weight function $\omega(x)$ for vector space $L^2[-1, 1]$, which has the property that any $\phi(x) \in L^2[-1, 1]$, can be written uniquely as

$$\phi(x) = \sum_{i=0}^{\infty} \psi_i \phi_i(x). \quad (10)$$

The residual function $r(x)$ of Equation (5) is defined as

$$r(x) := \mu(x)\phi(x) + \lambda(x) \sum_{k=0}^n \frac{\phi^{(k)}(x)}{k!} I_{\alpha,k} - f(x). \quad (11)$$

To solve Equation (5) numerically, $\phi(x)$ is approximated as

$$\phi(x) \simeq \phi_N(x) = \sum_{i=0}^N \psi_i \phi_i(x). \quad (12)$$

By substitution $\phi_N(x)$ from the Equation (12) into Equation (11) the residual function becomes:

$$\begin{aligned} r(x) &= \mu(x) \sum_{i=0}^N \psi_i \phi_i(x) + \lambda(x) \sum_{k=0}^n \frac{I_{\alpha,k}}{k!} \sum_{i=0}^N \psi_i \phi_i^{(k)}(x) - f(x) \\ &= \sum_{i=0}^N \psi_i \left[\mu(x) \phi_i(x) + \lambda(x) \sum_{k=0}^n \frac{I_{\alpha,k}}{k!} \phi_i^{(k)}(x) \right] - f(x). \end{aligned} \quad (13)$$

Using Galerkin method to find ψ_i , $i = 0, 1, \dots, N$:

$$(r(x), \phi_j(x)) = 0, \quad j = 0, 1, \dots, N, \quad (14)$$

where $(r(x), \phi_j(x))$ is the inner product of two functions $r(x)$ and $\phi_j(x)$ with respect to the weight function $w(x)$, i.e.

$$(r(x), \phi_j(x)) = \int_{-1}^1 r(x) \phi_j(x) w(x) dx. \quad (15)$$

So the problem of solving the integral Equation (5), leads to a system of linear equations, which can be solved numerically.

3 Implementation of the method with Legendre polynomials

In this section, Legendre polynomials are used as a basis. Let $\phi_n(x) = P_n(x)$, where $P_n(x)$ is the Legendre polynomial of degree n . These polynomials are orthogonal over the interval $[-1, 1]$ with respect to the weight function $w(x) = 1$ and

$$\int_{-1}^1 P_n(x) P_m(x) dx = \frac{2}{2n+1} \delta_{m,n}, \quad (16)$$

where

$$\delta_{m,n} = \begin{cases} 1 & m = n, \\ 0 & m \neq n. \end{cases} \quad (17)$$

When $\alpha = 1$, Equation (1) turns into the following integral equation

$$\mu(x)\phi(x) + \lambda(x) \int_{-1}^1 \frac{k(x,y)\phi(y)}{y-x} dy = f(x), \quad |x| < 1. \quad (18)$$

By substituting $\phi(x) \simeq \phi_N(x) = \sum_{j=0}^N \psi_j P_j(x)$ into (18)

$$\mu(x) \sum_{j=0}^N \psi_j P_j(x) + \lambda(x) \sum_{j=0}^N \psi_j \int_{-1}^1 \frac{k(x, y) P_j(y)}{y - x} dy \simeq f(x), \quad |x| < 1, \quad (19)$$

and using the truncated Taylor series of $k(x, y)$ about the point (x_0, y_0) in Eq(6) the equation becomes:

$$\sum_{j=0}^N \psi_j \left[\mu(x) P_j(x) + \lambda(x) \sum_{l=0}^m \kappa_l(x) \int_{-1}^1 \frac{P_j(y)(y - y_0)^l}{y - x} dy \right] = f(x), \quad |x| < 1. \quad (20)$$

Applying the expansion of $(y - y_0)^l$, based on Legendre polynomials, gives:

$$(y - y_0)^l = \sum_{k=0}^l c_k P_k(y). \quad (21)$$

Using Equation (21) in Equation (20):

$$\sum_{j=0}^N \psi_j \left[\mu(x) P_j(x) + \lambda(x) \sum_{l=0}^m \kappa_l(x) \sum_{k=0}^l c_k \int_{-1}^1 \frac{P_k(y) P_j(y)}{y - x} dy \right] = f(x), \quad |x| < 1. \quad (22)$$

Using the famous relation [13–16]

$$\int_{-1}^1 \frac{P_k(y) P_j(y)}{y - x} dy = -2 P_k(x) Q_j(x) \quad k \leq j, \quad (23)$$

where $Q_j(x)$ is the Legendre function of the second kind [17]. By substituting Equation (23) in Equation (22):

$$\sum_{j=0}^N \psi_j \left[\mu(x) P_j(x) + \lambda(x) \sum_{l=0}^m \kappa_l(x) \left(\sum_{k \leq j} -2 c_k P_k(x) Q_j(x) + \sum_{k > j} -2 c_k P_j(x) Q_k(x) \right) \right] = f(x), \quad |x| < 1. \quad (24)$$

For determining coefficients ψ_j both sides are multiplied by Equation (24) by $P_i(x)$, $i = 0, \dots, N$ and integrate from 1 to -1 . This process leads to

$$\sum_{j=0}^N A_{ij} \psi_j = b_i, \quad i = 0, \dots, N,$$

where for $\alpha = 1$ and $i, j = 0, 1, \dots, N$,

$$A_{ij} = \int_{-1}^1 \left[\mu(x) P_j(x) + \lambda(x) \sum_{l=0}^m \kappa_l(x) \left(\sum_{k \leq j} -2 c_k P_k(x) Q_j(x) + \sum_{k > j} -2 c_k P_j(x) Q_k(x) \right) \right] P_i(x) dx, \quad i = 0, \dots, N,$$

and for $0 < \alpha < 1$ and $i, j = 0, 1, \dots, N$,

$$A_{ij} = \int_{-1}^1 \left[\mu(x) P_j(x) + \lambda(x) \sum_{k=0}^n \frac{I_{\alpha, k}}{k!} P_j^{(k)}(x) \right] P_i(x) dx, \quad i = 0, \dots, N,$$

$$b_i = \int_{-1}^1 f(x) P_i(x) dx, \quad i = 0, \dots, N.$$

In both cases $\alpha = 1$ and $0 < \alpha < 1$, A_{ij} and b_i are estimated using the Gauss–Legendre quadrature rule.

4 Error estimate and convergence

In this section, error analysis and convergence of proposed method based on Legendre polynomials is discussed.

Theorem 1. Let $\phi(x) \in L^2[-1, 1]$, be the exact solution of Equation (1), $\phi_{N,m}(x) = \sum_{i=0}^N \psi_{i,m} P_i(x)$ be the solution of equation

$$\mu(x) \phi_{N,m}(x) + \lambda(x) \int_{-1}^1 \frac{k_m(x, y) \phi_{N,m}(y)}{(y - x)^\alpha} dy = f(x), \quad (25)$$

in which $k_m(x, y)$ is the truncated Taylor series to second order of $k(x, y)$, of order m , about the point (x_0, y_0) , $\|(x, y) - (x_0, y_0)\|_2 < r$ where r is the radius of convergence of Taylor series, $\phi_N(x) = \sum_{i=0}^N \psi_i P_i(x)$ is the solution of equation

$$\mu(x) \phi_N(x) + \lambda(x) \int_{-1}^1 \frac{k(x, y) \phi_N(y)}{(y - x)^\alpha} dy = f(x), \quad (26)$$

then

1. $\lim_{N, m \rightarrow \infty} \|\phi(x) - \phi_{N,m}(x)\|_2 = 0$,
2. $\lim_{N \rightarrow \infty} \|E_N(x)\|_2 = 0$,

Proof. To prove the first part consider the following

$$\begin{aligned} & \|\phi(x) - \phi_{N,m}(x)\|_2 \\ &= \|\phi(x) - \phi_N(x) + \phi_N(x) - \phi_{N,m}(x)\|_2 \\ &\leq \|\phi(x) - \phi_N(x)\|_2 + \|\phi_N(x) - \phi_{N,m}(x)\|_2, \end{aligned}$$

and

$$\begin{aligned} k(x, y) &= k(x_0, y_0) + (x - x_0) k_x(x_0, y_0) \\ &+ (y - y_0) k_y(x_0, y_0) + \frac{1}{2!} \left[(x - x_0)^2 k_{xx}(x_0, y_0) \right. \\ &+ (x - x_0)(y - y_0) k_{xy}(x_0, y_0) \\ &+ (y - y_0)^2 k_{yy}(x_0, y_0) \left. \right] + \dots \\ &= k_m(x, y) + Er_m(x, y), \end{aligned}$$

where

$$Er_m(x, y) = \frac{1}{(m+1)!} \sum_{l=0}^{m+1} \binom{m+1}{l} \frac{\partial^{m+1} k(\xi_x, \xi_y)}{\partial y^{m+1-l} \partial x^l} (x-x_0)^l (y-y_0)^{m+1-l},$$

and using the Taylor series for $k(x, y)$:

$$\lim_{m \rightarrow \infty} Er_m(x, y) = 0, \quad (27)$$

subtracting Equation (25) from Equation (26)

$$\begin{aligned} & \mu(x)(\phi_N(x) - \phi_{N,m}(x)) \\ & + \lambda(x) \int_{-1}^1 \frac{Er_m(x, y)(\phi_N(y) - \phi_{N,m}(y))}{(y-x)^\alpha} dy = 0, \end{aligned}$$

by Equation (27), since $\mu(x) \neq 0$,

$$\lim_{m, N \rightarrow \infty} \|(\phi_N(x) - \phi_{N,m}(x))\|_2 = 0. \quad (28)$$

On the other hand the sequence $\{P_n(x)\}_{n=0}^\infty$ of Legendre polynomials is a complete orthogonal basis with respect to the weight function $\omega(x) = 1$ for vector space $L^2[-1, 1]$, so $\phi(x) \in L^2[-1, 1]$, can be written uniquely as a linear combination of basis functions

$$\phi(x) = \sum_{i=0}^{\infty} \psi_i P_i(x),$$

where

$$\psi_i = \frac{2i+1}{2} \int_{-1}^1 \phi(x) P_i(x) dx,$$

Therefore,

$$\phi(x) - \phi_N(x) = \sum_{i=N+1}^{\infty} \psi_i P_i(x),$$

by orthogonality of Legendre functions,

$$\|\phi(x) - \phi_N(x)\|_2^2 = \sum_{i=N+1}^{\infty} \frac{2}{2i+1} \psi_i^2,$$

thus, Parseval's identity holds and from [17] this property in Hilbert space $L^2[-1, 1]$ ensures

$$\lim_{N \rightarrow \infty} \|\phi(x) - \phi_N(x)\|_2 = 0, \quad (29)$$

with (28) and (29) the first part of the proof is completed.

From Equation (4) and Cauchy Schwarz inequality

$$\begin{aligned} |E_n(x)|^2 &= \left| \frac{1}{(n+1)!} \int_{-1}^1 k(x, y)(y-x)^{n+1-\alpha} \phi^{(n+1)}(\xi_{x,y}) dy \right|^2 \\ &\leq \frac{M^2}{((n+1)!)^2} \|k(x, y)\|_2^2 \|(y-x)^{n+1-\alpha}\|_2^2 \\ &\leq \frac{M^2 K^2 2^{2n-2\alpha+3}}{((n+1)!)^2}, \end{aligned}$$

where $M = \max_{x \in [-1, 1]} \phi^{(n+1)}(x)$, $K = \|k(x, y)\|_2$ and the proof is complete. $\square$

5 Numerical examples

This section contains examples to demonstrate numerically the effectiveness of the proposed method. The error is defined as $Er_N = |\phi(x) - \phi_N(x)|$, where $\phi_N(x) = \sum_{j=0}^N \psi_j P_j(x)$. Also, m is the number of terms of the Taylor series of $k(x, y)$ and n is the number of terms of the Taylor series for $\phi(x)$.

Example 1. Consider the following integral equation

$$(1+x^2)\phi(x) + (1+x^2) \int_{-1}^1 \frac{(x^2+y)\phi(y)}{(y-x)} dy = f(x), \quad (30)$$

where

$$f(x) = x^3 + x(1+x) \left[x^3 \ln\left(\frac{1-x}{1+x}\right) + 2x^2 + \frac{4-\pi}{2} \right],$$

is chosen such that

$$\phi(x) = \frac{x^3}{1+x^2},$$

is the exact solution. The numerical results are given in Table 1.

In Example 1 for $m = 2$, as n increases the solutions become more accurate.

Example 2. Consider the following integral equation

$$\sin(x)\phi(x) + \cos(x) \int_{-1}^1 \frac{e^{x+y}\phi(y)}{(y-x)^{\frac{1}{5}}} dy = f(x), \quad (31)$$

where

$$\begin{aligned} f(x) &= x e^{-x} \sin(x) + \frac{5}{36} e^x \cos(x) ((4-5x)(-1-x)^{4/5} \\ &\quad + (1-x)^{4/5}(4+5x)), \end{aligned}$$

is chosen such that

$$\phi(x) = x e^{-x},$$

is the exact solution. The numerical results are given in Tables 2, 3. In Table 2 for fix $n = 4$ and increasing $m = 2, 5, 10$ absolute errors converge to zero, but the rate of convergence is low. In Table 3 for fix $n = 8$ and increasing $m = 2, 5, 10$, absolute errors tend to zero, the rate of convergence is faster than when $n = 4$. Generally when n, m , increase absolute errors tends to zero.

Table 1: The Er_n of Example 1 for $n = 4, 8, 16$ and $m = 2$.

x	$Er_4(x)$	$Er_8(x)$	$Er_{16}(x)$
-1	2.66166×10^{-2}	1.13706×10^{-3}	1.40121×10^{-6}
-0.8	1.10254×10^{-2}	3.1887×10^{-4}	1.54474×10^{-7}
-0.6	1.14159×10^{-4}	1.62844×10^{-4}	4.5442×10^{-7}
-0.4	2.142×10^{-2}	2.87953×10^{-4}	3.81212×10^{-7}
-0.2	2.43581×10^{-2}	9.19528×10^{-4}	2.02625×10^{-7}
0	4.47326×10^{-3}	2.42039×10^{-4}	2.51121×10^{-7}
0.2	1.80403×10^{-2}	8.95789×10^{-4}	3.1924×10^{-7}
0.4	2.17433×10^{-2}	1.48904×10^{-4}	6.16621×10^{-8}
0.6	7.35652×10^{-3}	2.46889×10^{-4}	1.86428×10^{-7}
0.8	2.81081×10^{-3}	2.31861×10^{-4}	7.98199×10^{-8}
1	2.11425×10^{-2}	4.63129×10^{-4}	2.17689×10^{-7}

Table 2: The Er_4 of Example 2 for $m = 2, 5, 10$.

x	$m = 2$	$m = 5$	$m = 10$
0	8.24039×10^{-2}	3.22493×10^{-3}	3.05565×10^{-3}
0.2	7.89611×10^{-2}	2.51934×10^{-3}	2.08542×10^{-3}
0.4	4.98806×10^{-2}	3.1168×10^{-3}	2.86393×10^{-3}
0.6	4.46485×10^{-2}	1.99465×10^{-3}	1.83149×10^{-3}
0.8	5.07106×10^{-2}	1.42699×10^{-3}	1.26791×10^{-3}
1	6.2562×10^{-2}	1.8699×10^{-3}	2.01917×10^{-3}

Table 3: The Er_8 of Example 2 for $m = 2, 5, 10$.

x	$m = 2$	$m = 5$	$m = 10$
0	1.20729×10^{-1}	6.55561×10^{-4}	7.39673×10^{-8}
0.2	9.18806×10^{-2}	5.05137×10^{-4}	1.77655×10^{-7}
0.4	5.17136×10^{-2}	2.67498×10^{-4}	6.42829×10^{-8}
0.6	3.23127×10^{-2}	1.05396×10^{-4}	4.18855×10^{-8}
0.8	3.98924×10^{-2}	6.16701×10^{-5}	5.86223×10^{-8}
1	5.95749×10^{-2}	3.15855×10^{-4}	1.34724×10^{-7}

Table 4: The MaxError of Example 4 for $m = 2, 5, 10$.

n	$n = 16$	$n = 32$
Our method for $m = 1$	1.59×10^{-16}	
Lakestani's method[10]	1.27×10^{-6}	2.71×10^{-8}
Product integration method [18]	1.5×10^{-5}	9.39×10^{-7}
Lagrangian interpolate [18]	2.12×10^{-5}	1.94×10^{-6}

Example 3. Consider the following integral equation

$$\sinh(x)\phi(x) + \cosh(x) \int_{-1}^1 \frac{(1 + xy + 2y^3)\phi(y)}{(y - x)^{\frac{3}{5}}} dy = f(x), \quad (32)$$

where $f(x)$, is chosen such that

$$\phi(x) = x^5,$$

is the exact solution of Equation (32), the numerical result for $n = m = 5$, is

$$\phi_5(x) = -1.3526 \times 10^{-10} + 2.45121 \times 10^{-9}x - 8.09197 \times 10^{-9}x^2 + 1.09271 \times 10^{-8}x^3 + 3.1628 \times 10^{-9}x^4 + x^5.$$

Generally when $k(x, y)$, is a polynomial of degree ω , with $m = \omega$, approximate solution is very close to the exact solution.

Example 4. Consider the following integral equation [10, 18],

$$\phi(x) - \int_0^1 |x - y|^{\frac{1}{2}} \phi(y) dy = f(x), \quad (33)$$

where $f(x)$, is chosen such that $\phi(x) = x$, is the exact solution of Equation (33). Table 4 shows the absolute errors which computed by using proposed method. Also, Comparison is made with the result presented in [10, 18].

There the maximum error is defined as $MaxError = \max|\phi(x) - \phi_N(x)|$, to calculate the errors in Table 4.

6 Conclusion

This paper proposes a new method to solve Fredholm integral equations of the second kind with weakly singular

kernel. This method uses the Taylor series and Legendre functions of the second kind to remove singularity. Legendre polynomials were used as basis. Several test examples are used to observe the efficiency and applicability of the new method.

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