DE GRUYTER OPEN Open Phys. 2015; 13:177–182

Research Article Open Access

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Response of a fractional nonlinear system to harmonic excitation by the averaging method

Abstract: In this work, we consider a fractional nonlinear vibration system of Duffing type with harmonic excitation by using the fractional derivative operator ${}_{-\infty}D_t^{\alpha}$ and the averaging method. We derive the steady-state periodic response and the amplitude-frequency and phase-frequency relations. Jumping phenomena caused by the nonlinear term and resonance peaks are displayed, which is similar to the integer-order case. It is possible that a minimum of the amplitude exists before the resonance appears for some values of the modelling parameters, which is a feature for the fractional case. The effects of the parameters in the fractional derivative term on the amplitude-frequency curve are discussed.

Keywords: fractional calculus; fractional vibration; resonance; averaging method

PACS: 02.60.Cb; 45.10.Hj

DOI 10.1515/phys-2015-0020
Received November 6, 2014; accepted December 11, 2014

1 Introduction

Fractional calculus has been used in the mathematical description of real problems arising in different fields of science including viscoelasticity, anomalous diffusion, acoustics, mechanics, electromagnetism, heat transfer, electrical circuits, signal processing, system identification, control and robotics, chemistry, biology, physics, economy and finance, and so on [1–5]. Scientists and engineers have found the description of some phenomena is more accurate and convenient when the fractional derivative is used.

Viscoelasticity is one of the fields with the most extensive applications of fractional calculus [6–16]. The main reason for theoretical development has been the wide use of polymers in various fields of engineering. In essence, fractional calculus has the ability to model hereditary phenomena with long memory.

Scott-Blair [6–8] proposed a fractional constitutive equation $\sigma(t) = b \cdot {}_0D_t^{\nu}\epsilon(t)$ to characterize a viscoelastic material whose mechanical properties are intermediate between those of a pure elastic solid (Hooke model) and a pure viscous fluid (Newton model). In the monographs [3, 17], this relation was called the Scott-Blair model [6–8]. In [9], a fractional calculus element whose constitutive law holds stress proportional to a fractional derivative of strain is said to be a spring-pot.

Fractional oscillators, fractional vibrations and dynamical systems were discussed and investigated by Caputo [18], Bagley and Torvik [14], Beyer and Kempfle [15], Mainardi [19], Gorenflo and Mainardi [16], and others [20–32].

Achar et al. [20] studied the response characteristics of the fractional oscillator. Li et al. [21] considered the impulse response and the stability behavior of a class of fractional oscillators. Lim et al. [22] established the relationship between fractional oscillator processes and the corresponding fractional Brownian motion processes. Lim and Teo [23] introduced a fractional oscillator process as a solution of a stochastic differential equation with two fractional orders. Wang and Hu [24] investigated stability of a linear oscillator with fractional damping force. Wang and Du [25] considered asymptotic behavior of the linear fractional damped vibration system using the inverse Laplace transform. Shen et al. [26, 27] analyzed dynamical behavior and resonance for a fractional oscillator using the averaging method. Li et al. [28] and Zhang et al. [29] considered fractional dynamical systems and stability. Li and Ma [30] contributed the linearization and stability theorems for the nonlinear fractional system. Tavazoei et al. [31] and Pinto and Machado [32] studied the fractional-order van der Pol oscillator and found multiple limit cycles existing in the system. Meanwhile, the development of theoretical research of linear and nonlinear fractional differen-

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tial equations has proceeded apace in recent decades, including the existence and uniqueness of solution [1, 2, 4], different methods for resolution [4, 5, 33–36] and boundary value problems with various type of boundary conditions [37–39].

Let $f(t)$ be piecewise continuous on $(t_0, +\infty)$ and integrable on any subinterval (t_0, t). Then the Riemann-Liouville fractional integral of $f(t)$ is defined as [1–4]

$$ {}_{t_0}J_t^\alpha f(t) := \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds,\ \alpha > 0, \tag{1}$$

where $\Gamma(\cdot)$ is Euler's gamma function.

Let $f^{(n)}(t)$ be piecewise continuous on $(t_0, +\infty)$ and integrable on any subinterval (t_0, t). Then the Caputo fractional derivative of $f(t)$ of order α, $n-1 < \alpha < n$, is defined as [1–4]

$$ \begin{gathered} {}_{t_0}D_t^\alpha f(t) := {}_{t_0}J_t^{n-\alpha} f^{(n)}(t), \\ n-1 < \alpha < n,\ n \in \mathbb{N}^+, \end{gathered} \tag{2}$$

In most studies such as those mentioned above, the lower limit t_0 of the integral was taken as 0. In this case, the response includes a decaying component even for harmonic excitation with zero initial conditions, so generally speaking, the solution is not periodic [16, 19, 39, 40]. For harmonic excitation, we are more interested in the steady-state response, or the periodic solution of the fractional vibration system, which is often a desired property in dynamical systems, constituting one of the most important research directions in the theory of dynamical systems.

In this work, we consider the nonlinear fractional vibration system of Duffing type with harmonic excitation using the fractional derivative operator ${}_{-\infty}D_t^\alpha$ and the averaging method. A steady-state periodic response is derived. We show that this operator is convenient and efficient for analysis of the steady-state solution. The amplitude-frequency relation and phase-frequency relations are obtained, where the jump phenomena due to the nonlinear term are displayed. The effects of the parameters in the fractional derivative term on the amplitude-frequency curve are discussed.

2 Response of fractional nonlinear system to harmonic excitation

We consider a fractional nonlinear vibration system of Duffing type with harmonic excitation

$$ m\ddot{x} + kx + c\dot{x} + bx^3 + K \cdot {}_{-\infty}D_t^\alpha x = F\cos(\omega t), \tag{3}$$

where m, k, c, b, F and ω are the system mass, linear stiffness coefficient, linear damping coefficient, nonlinear stiffness coefficient, excitation amplitude and excitation frequency, respectively, and the term $K \cdot {}_{-\infty}D_t^\alpha x$ denotes a force related to the whole deformational history, where $K > 0$ and $0 < \alpha < 1$. For the results and discussion, we also consider the two limiting cases $\alpha = 0$ and $\alpha = 1$.

Following the notation in the averaging method, we introduce

$$ \sqrt{k/m} = \omega_0,\ c/m = 2\varepsilon\mu,\ b/m = \varepsilon\gamma, $$
$$ K/m = \varepsilon\eta,\ F/m = \varepsilon f,\ \omega^2 - \omega_0^2 = \varepsilon\sigma, $$

where ω_0 is the natural frequency for the corresponding linear conservative system, and rewrite Eq. (3) as

$$ \begin{aligned} \ddot{x} + \omega^2 x &= \varepsilon[f\cos(\omega t) + \sigma x - \gamma x^3 - 2\mu\dot{x} \\ &\quad - \eta \cdot {}_{-\infty}D_t^\alpha x]. \end{aligned} \tag{4}$$

Letting $\varepsilon = 0$, we have the derived system's solution

$$ x = a\cos(\omega t - \theta). \tag{5}$$

The first-order derivative and the fractional derivative are calculated to be

$$ \dot{x} = -a\omega\sin(\omega t - \theta), \tag{6}$$

and

$$ \begin{aligned} {}_{-\infty}D_t^\alpha x &= {}_{-\infty}J_t^{1-\alpha}[-a\omega\sin(\omega t - \theta)] \\ &= \frac{-a\omega}{\Gamma(1-\alpha)} \int_{-\infty}^{t} (t-\tau)^{-\alpha} \sin(\omega\tau - \theta)d\tau \\ &= \frac{-a\omega}{\Gamma(1-\alpha)} \int_0^{+\infty} u^{-\alpha} \sin(\omega t - \theta - \omega u)du \\ &= \frac{-a\omega}{\Gamma(1-\alpha)} \int_0^{+\infty} u^{-\alpha}[\sin(\omega t - \theta)\cos(\omega u) \\ &\quad - \cos(\omega t - \theta)\sin(\omega u)]du \\ &= -a\omega^\alpha(\sin(\omega t - \theta)\sin(\frac{\pi\alpha}{2}) \\ &\quad - \cos(\omega t - \theta)\cos(\frac{\pi\alpha}{2})) \\ &= a\omega^\alpha\cos(\omega t - \theta + \frac{\pi\alpha}{2}), \end{aligned} \tag{7}$$

where the Mellin transform formulas for the trigonometric functions $\sin(\omega u)$ and $\cos(\omega u)$

$$ \int_0^{+\infty} \sin(\omega u)u^{p-1}du = \frac{\Gamma(p)\sin(\frac{\pi p}{2})}{\omega^p}, $$
$$ \int_0^{+\infty} \cos(\omega u)u^{p-1}du = \frac{\Gamma(p)\cos(\frac{\pi p}{2})}{\omega^p},\ 0 < p < 1, $$

are used [41, 42]. According to the averaging method, we regard a and θ as slowly varying functions of t and denote $\omega t - \theta = \psi$. From (5) and (6) we calculate

$$\dot{x} = \dot{a}\cos(\psi) - a\sin(\psi)(\omega - \dot{\theta}), \tag{8}$$

$$\ddot{x} = -\dot{a}\omega\sin(\psi) - a\omega\cos(\psi)(\omega - \dot{\theta}). \tag{9}$$

It follows from Eqs. (6) and (8) that

$$\cos(\psi)\dot{a} + a\sin(\psi)\dot{\theta} = 0. \tag{10}$$

Substituting Eqs. (5), (6), (7) and (9) into Eq. (4) leads to

$$-\omega\sin(\psi)\dot{a} + a\omega\cos(\psi)\dot{\theta} = \varepsilon f_1, \tag{11}$$

where

$$\begin{aligned} f_1 &= f\cos(\psi+\theta) + \sigma a\cos(\psi) - \gamma a^3\cos^3(\psi) \\ &\quad +2\mu a\omega\sin(\psi) - \eta a\omega^\alpha\cos(\psi + \frac{\pi\alpha}{2}), \end{aligned} \tag{12}$$

is a periodic function in ψ with period 2π.

Solving the system of equations (10) and (11) we obtain

$$\dot{a} = -\frac{\varepsilon}{\omega} f_1\sin(\psi), \tag{13}$$

$$\dot{\theta} = \frac{\varepsilon}{\omega a} f_1\cos(\psi). \tag{14}$$

The averaging method replaces the right hand sides by the averages in a period as

$$\dot{a} = -\frac{\varepsilon}{\omega}\frac{1}{2\pi}\int_0^{2\pi} f_1\sin(\psi)d\psi, \tag{15}$$

$$\dot{\theta} = \frac{\varepsilon}{\omega a}\frac{1}{2\pi}\int_0^{2\pi} f_1\cos(\psi)d\psi. \tag{16}$$

Calculating the integrals we have

$$\dot{a} = \frac{f\varepsilon}{2\omega}\sin(\theta) - a\varepsilon\mu - \frac{\varepsilon}{2}a\eta\omega^{\alpha-1}\sin(\frac{\pi\alpha}{2}), \tag{17}$$

$$\begin{aligned} \dot{\theta} &= \frac{f\varepsilon}{2\omega a}\cos(\theta) + \frac{\varepsilon\sigma}{2\omega} - \frac{\varepsilon}{2}\eta\omega^{\alpha-1}\cos(\frac{\pi\alpha}{2}) \\ &\quad -\frac{3\varepsilon a^2\gamma}{8\omega}. \end{aligned} \tag{18}$$

Substituting the original coefficients leads to

$$\dot{a} = \frac{F}{2m\omega}\sin(\theta) - \frac{a}{2m}\left(c + K\omega^{\alpha-1}\sin(\tfrac{\pi\alpha}{2})\right), \tag{19}$$

$$\begin{aligned} \dot{\theta} &= \frac{F}{2m\omega a}\cos(\theta) + \frac{\omega}{2} - \frac{3a^2 b}{8m\omega} \\ &\quad -\frac{1}{2m\omega}\left(k + K\omega^\alpha\cos(\tfrac{\pi\alpha}{2})\right). \end{aligned} \tag{20}$$

Denote

$$\tilde{c} = c + K\omega^{\alpha-1}\sin(\frac{\pi\alpha}{2}),\ \tilde{k} = k + K\omega^\alpha\cos(\frac{\pi\alpha}{2}), \tag{21}$$

as the equivalent damping coefficient and the equivalent stiffness coefficient. We note that the two coefficients $\tilde{c}$ and $\tilde{k}$ are related not only to the parameters K and α in the fractional term, but also to the excitation frequency ω. As ω increases, $\tilde{k}$ increases while $\tilde{c}$ decreases.

For the steady-state solution, we set $\dot{a} = 0$ and $\dot{\theta} = 0$, i.e.

$$\frac{F}{2m\omega}\sin(\theta) - \frac{a}{2m}\left(c + K\omega^{\alpha-1}\sin(\frac{\pi\alpha}{2})\right) = 0, \tag{22}$$

$$\begin{aligned} &\frac{F}{2m\omega a}\cos(\theta) + \frac{\omega}{2} - \frac{3a^2 b}{8m\omega} \\ &-\frac{1}{2m\omega}\left(k + K\omega^\alpha\cos(\frac{\pi\alpha}{2})\right) = 0. \end{aligned} \tag{23}$$

From Eqs. (22) and (23), the amplitude is determined by the implicit equation

$$\begin{aligned} &a^2\Big[\left(c\,\omega + K\omega^\alpha\sin(\tfrac{\pi\alpha}{2})\right)^2 + \\ &\left(k + K\omega^\alpha\cos(\tfrac{\pi\alpha}{2}) - m\omega^2 + \tfrac{3}{4}a^2 b\right)^2\Big] = F^2, \end{aligned} \tag{24}$$

and the tangent of the phase is solved to be

$$\tan(\theta) = \frac{c\,\omega + K\omega^\alpha\sin(\frac{\pi\alpha}{2})}{k + K\omega^\alpha\cos(\frac{\pi\alpha}{2}) - m\omega^2 + \frac{3}{4}a^2 b}. \tag{25}$$

Let $\omega \to 0$ in Eq. (24), we obtain

$$\begin{aligned} a(k + \tfrac{3}{4}a^2 b) = F, &\quad \text{for} \quad 0 < \alpha \le 1, \\ a(k + K + \tfrac{3}{4}a^2 b) = F, &\quad \text{for} \quad \alpha = 0. \end{aligned} \tag{26}$$

We notice that the limit of the amplitude a as $\omega \to 0$ has a jump at $\alpha = 0$ if $K > 0$.

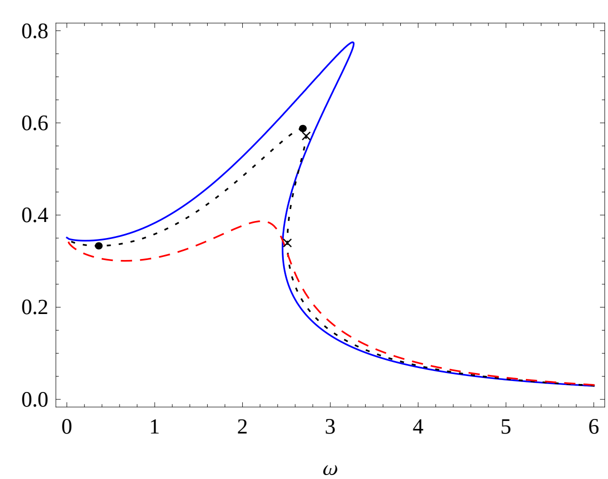

Figure 1: a versus ω for $\alpha = 0.5$ and different K: $K = 0.5$ (solid line), $K = 1$ (dot line) and $K = 2$ (dash line).

We take $m = k = F = 1$, $c = 0.2$ and $b = 20$, and plot the curves of the amplitude a versus ω for specified values of K and α in Figs. 1 and 2. Jumping phenomena caused by the nonlinear term and resonance peaks

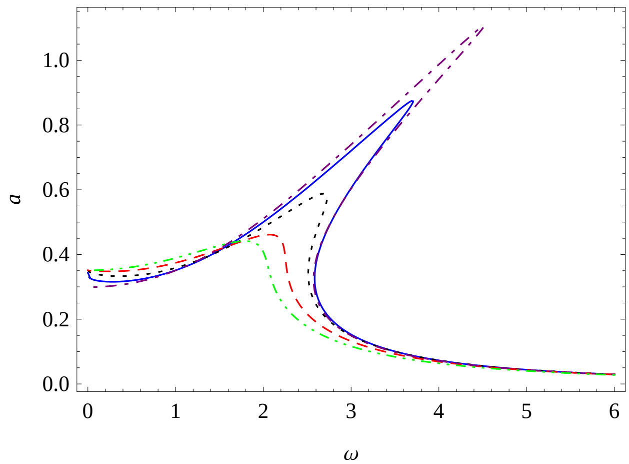

Figure 2: a versus ω for $K = 1$ and different α: 0 (dot-dash line), 0.2 (solid line), 0.5 (dot line), 0.8 (dash line) and 1 (dot-dot-dash line).

are clearly displayed in these figures, which is similar to the integer-order case. Also it is possible that a minimum for the amplitude a exists before the resonance appears for some values of the modelling parameters, which is a feature for the fractional case. From Figs. 1 and 2, we observe that as the parameter K or α in the fractional derivative term increases, the peak value and the degree of inclination of the resonance peak on the amplitude-frequency curve decrease. That is, the two extreme points of the amplitude a gradually approach and the two singular points eventually disappear.

In order to determine the frequency and the amplitude at resonance we calculate the derivative $\frac{da}{d\omega}$ from Eq. (24). Then by setting $\frac{da}{d\omega} = 0$ we obtain

$$\begin{aligned}&\omega\left(c + K\omega^{\alpha-1}\sin(\frac{\pi\alpha}{2})\right)^2 + \\ &\omega^{\alpha}K(\alpha-1)\left(c + K\omega^{\alpha-1}\sin(\frac{\pi\alpha}{2})\right)\sin(\frac{\pi\alpha}{2}) + \\ &\left(k + K\omega^{\alpha}\cos(\frac{\pi\alpha}{2}) - m\omega^2 + \frac{3}{4}a^2 b\right)\times \\ &\left(K\alpha\omega^{\alpha-1}\cos(\frac{\pi\alpha}{2}) - 2m\omega\right) = 0. \qquad (27)\end{aligned}$$

The frequency and amplitude at resonance, and the minimum of the amplitude a, when they exist, are determined by Eqs. (24) and (27).

With increasing values of the two parameters K and α, the two extreme points on the amplitude-frequency curve gradually approach and eventually disappear. We checked that if we take $K = 5, \alpha = 0.5$ or $K = 3, \alpha = 0.8$ in Fig. 1 or Fig. 2, the two extreme points of the amplitude a disappear and the amplitude-frequency curve becomes monotonically decreasing.

For the singular points on the amplitude-frequency curve, we obtain them by setting $\frac{da}{d\omega} = \infty$. This leads to the equation

$$\begin{aligned}&2\left(c\,\omega + K\omega^{\alpha}\sin(\tfrac{\pi\alpha}{2})\right)^2 + \\ &2\left(k + K\omega^{\alpha}\cos(\tfrac{\pi\alpha}{2}) - m\omega^2 + \tfrac{3}{4}a^2 b\right)^2 + \\ &3a^2 b\left(k + K\omega^{\alpha}\cos(\tfrac{\pi\alpha}{2}) - m\omega^2 + \tfrac{3}{4}a^2 b\right) = 0. \qquad (28)\end{aligned}$$

Eqs. (24) and (28) determine the the singular points (ω, a) on the amplitude-frequency curves.

We take $m = k = F = 1$, $c = 0.2$, $b = 20$, $\alpha = 0.5$ and $K = 1$. Solving Eqs. (24) and (27) we obtain the minimum and the maximum on the amplitude-frequency curve, corresponding to the points $(0.36057788, 0.33319415)$ and $(2.6844531, 0.58790828)$, respectively, which are denoted by the two heavy dots on the dot line in Fig. 1. Solving Eqs. (24) and (28), we obtain the two singular points $(2.5114361, 0.34223542)$ and $(2.7257321, 0.57486028)$ on the amplitude-frequency curve, which are denoted by the two crosses on the dot line in Fig. 1.

The amplitude-frequency and the phase-frequency relations (24) and (25) include the following three special cases.

(i) Fractional linear case

The linear case is given as a special case of $b = 0$. The amplitude and the tangent of the phase are

$$a = F\Big/\left(\left(c\,\omega + K\omega^{\alpha}\sin(\frac{\pi\alpha}{2})\right)^2 + \left(k + K\omega^{\alpha}\cos(\frac{\pi\alpha}{2}) - m\omega^2\right)^2\right)^{1/2}, \qquad (29)$$

$$\tan(\theta) = \frac{c\,\omega + K\omega^{\alpha}\sin(\frac{\pi\alpha}{2})}{k + K\omega^{\alpha}\cos(\frac{\pi\alpha}{2}) - m\omega^2}. \qquad (30)$$

Also the limit of the amplitude a as $\omega \to 0$ is discontinuous at $\alpha = 0$, i.e.

$$\lim_{\omega\to 0} a = \begin{cases} F/k, & 0 < \alpha \le 1, \\ F/(k+K), & \alpha = 0. \end{cases} \qquad (31)$$

The resonance frequency and the resonance amplitude are determined by Eq. (29) and the equation

$$\begin{aligned}&\omega\left(c + K\omega^{\alpha-1}\sin(\frac{\pi\alpha}{2})\right)^2 + \\ &\omega^{\alpha}K(\alpha-1)\left(c + K\omega^{\alpha-1}\sin(\frac{\pi\alpha}{2})\right)\sin(\frac{\pi\alpha}{2}) + \\ &\left(k + K\omega^{\alpha}\cos(\frac{\pi\alpha}{2}) - m\omega^2\right)\times \\ &\left(K\alpha\omega^{\alpha-1}\cos(\frac{\pi\alpha}{2}) - 2m\omega\right) = 0, \qquad (32)\end{aligned}$$

which also determine the minimum on the amplitude-frequency curve if it exists.

In Figs. 3 and 4, we plot the curves of the amplitude a versus ω for the linear case, where the parameters are

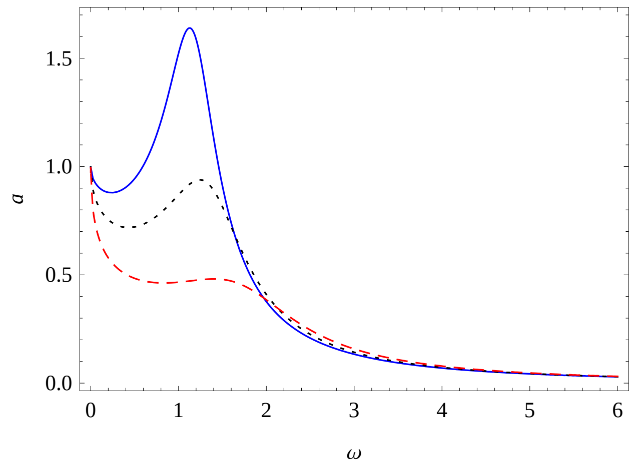

Figure 3: a versus ω in the linear case for $\alpha = 0.5$ and different K: $K = 0.5$ (solid line), $K = 1$ (dot line) and $K = 2$ (dash line).

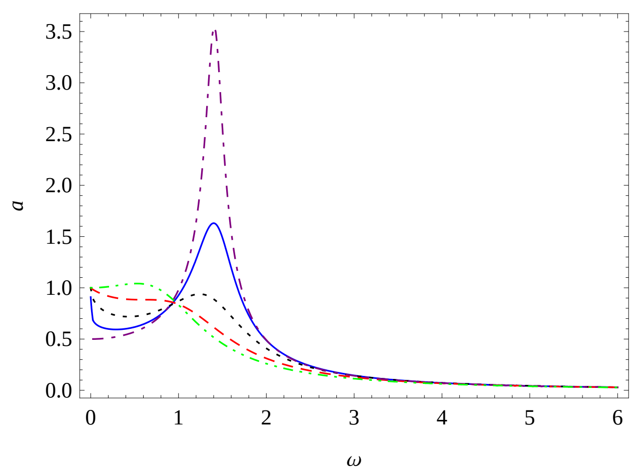

Figure 4: a versus ω in the linear case for $K = 1$ and different α: $\alpha = 0$ (dot-dash line), 0.2 (solid line), 0.5 (dot line), 0.8 (dash line) and 1 (dot-dot-dash line).

taken as $m = k = F = 1$ and $c = 0.2$. The jumping phenomena disappear for the linear case.

(ii) Integer-order nonlinear case

Letting $K = 0$ leads to the integer-order case. The amplitude is determined by the implicit equation

$$a^2\left[(c\,\omega)^2 + \left(k - m\omega^2 + \frac{3}{4}a^2 b\right)^2\right] = F^2, \tag{33}$$

and the tangent of the phase is

$$\tan(\theta) = \frac{c\,\omega}{k - m\omega^2 + \frac{3}{4}a^2 b}. \tag{34}$$

(iii) Integer-order linear case

The integer-order linear case corresponds to $b = K = 0$. The amplitude and the tangent of the phase are

$$a = \frac{F}{\sqrt{(c\,\omega)^2 + (k - m\omega^2)^2}}, \tag{35}$$

$$\tan(\theta) = \frac{c\,\omega}{k - m\omega^2}. \tag{36}$$

3 Conclusions

We considered a fractional nonlinear vibration system of Duffing type with harmonic excitation using the fractional derivative operator ${}_{-\infty}D_t^{\alpha}$ and the averaging method. We derived the steady-state periodic response and the amplitude-frequency and phase-frequency relations, where the jump phenomena due to the nonlinear term were displayed. The effects of the parameters in the fractional derivative term on the amplitude-frequency curve were discussed. It is possible that a minimum of the amplitude a exists before the resonance appears for some values of the modelling parameters, which is a feature for the fractional case.

Acknowledgement: This work was supported by the Natural Science Foundation of Shanghai (No.14ZR1440800), the Innovation Program of Shanghai Municipal Education Commission (No.14ZZ161) and the National Natural Science Foundation of China (No.11201308).

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