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Detecting extra dimensions by Hydrogen-like atoms

Abstract: We reconsider the idea in spectroscopy of detecting extra dimensions by regarding the nucleus as a homogeneous sphere. In our results, it turns out that the gravitational potential inside the nucleus is much stronger than the potential induced by a particle in the same regime in ref. [16], and thus a more significant correction of the ground state energy of hydrogen-like atoms is obtained, which can be used to determine the existence of ADD's extra dimensions. In order to get a larger order of magnitude for the correction, it is better to apply our theory to high-Z atoms or muonic atoms, where the volume of the nucleus can't be ignored and the relativistic effect is important. Our work is based on the Dirac equation in a weak gravity field, and the result is more precise.

Keywords: extra dimension; gravity; Hydrogen-like atom

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1 Introduction

Gravity is the most mysterious force in modern physics. There are many problems in Einstein's theory of gravity. The most famous one is that the theory is nonlinear, nonrenormalized, and hard to quantize. Although the quantum gravity haven't been obtained, some theories such as string theory and loop gravity are making progress [1, 2]. In addition, some semiclassical quantum theories [3, 4] based on quantum field theory and general relativity also have made some achievements. Corda, for example [5, 6], recently found that a black holes can be considered as a gravitational analogue of the Hydrogen atom. Another famous problem is that the energy scale of gravity, M_{pl} ($M_{pl} = 10^{16}$ TeV is the Planck energy), is far larger than the electroweak scale $M_\star = 1 - 10$ TeV, which is known as the gauge hierarchy problem. Arkanihamed, Dimopoulos, and Dvali [7–9] proposed a brane model (ADD model) to deal with the problem. In the ADD model, all matter and forces except for gravity are confined within a 3-brane whose width is $M_\star^{-1}$, and the extra dimensions are perpendicular to the matter brane and compact. For simplicity we assume that these extra dimensions are circles with the same circumference L and are compactified on an N-dimensional torus T^N. The gravity of a particle with mass m is

$$\phi = \begin{cases} -\dfrac{m}{M_{pl}^2 r}, & r >> L \\ -\dfrac{m}{M_{pl}^2 r}(1 + \alpha e^{-\frac{r}{\lambda}}), & r \sim L \\ -\dfrac{m}{M_\star^{N+2} r^{1+N}}. & r << L \end{cases} \tag{1}$$

where $\alpha, \lambda \sim L$ are related to the structure of the extra dimension space, with $\alpha = N, \lambda = L$ for T^N space. N is the number of the extra dimensions. The Planck energy isn't a fundament constant in the ADD model and is replaced by the electroweak scale $M_\star$:

$$M_{pl}^2 = M_\star^{2+N} L^N. \tag{2}$$

So the energy gap between the gravity and electroweak disappeared. The size of the extra dimension is

$$L \approx 10^{-17+\frac{30}{N}} \text{ cm}, \tag{3}$$

which is large compared with the Planck length. The case $N = 1$ ($L \approx 10^{13}$ cm) can be easily ruled out, since L can be compared with the radius of the solar system and Newtonian gravity works well. The case $N = 2$ ($L \approx 10^{-2}$ cm) may be explored by the torsion pendulum test and it is indicated that $L < 37\ \mu$m and $M_\star > 1.5$ TeV for $N = 2$ [10]. When $N \geq 3$, the extra dimensions are too small to be detected by mechanics methods. Many physicists have tried to detect the existence of the extra dimensions via high energy experiment and astrophysics observation [11–15]. Since spectroscopic experiments can be the most precise way to study physics theories, some physicists [16–18] have also suggested finding the extra dimensions by detecting the shift of the energy levels of atoms. In ref. [16], the nucleus is a point-like source and the electron is nonrelativistic. The results indicated that the shift would be

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more obvious when the charge of the nucleus increased or the electron was replaced by the heavier muon, because the distance between the nucleus and electron (or muon) is shorter and the gravity is stronger. However, it is unreasonable for them to neglect relativistic effects and the volume of the nucleus, since when the charge number of nucleus $Z \sim 100$, the speed of the electron approaches c and the electron is too close to the nucleus. In the subsequent sections, we assume that the nucleus is a homogeneous sphere. The gravity near the nucleus is derived in section 2. Then in section 3 we give the Dirac equation in the gravity field and calculate the correction of ground energy which is significantly stronger than the previous result in ref. [16]. Discussion and conclusion are also given in section 4.

2 The gravity of 3-sphere in 4+n space-time

Since the nucleus is composed of protons and neutrons, which are extremely similar in size and mass, we can treat the nucleus as a homogeneous sphere. The general form of the gravity of a homogeneous sphere is hard to deduce from the formula (1). However, for $N \le 4$, there is an easy way to estimate the gravitational potential near the nucleus ($r << L$). From (3) we see that the nuclear radius (typical scale 10^{-15} m) is much smaller than the extra dimension scale L when $N \le 4$, so the potential near the nucleus ($r << L$) is approximately the potential induced by a sphere in 4+N infinite space-time. In the following sections, our calculations are only for $N = 2, 3, 4$.

Given the existence of extra dimensions, the nucleus may have a mass distribution in them. Since the thickness of the matter brane $M_\star^{-1} = 10^{-17}$ cm is much smaller compared with that of the nucleus, a homogeneous spherical surface with radius R and mass M is not only infinitely thin in the ordinary 3+1 space-time, but also extremely thin in the extra dimensions. When we neglect this thickness, the gravitational potential near such a surface can be obtained by integrating the last formula in (1):

$$\phi = -\frac{M}{2RM_\star^{2+N}(N-1)r}\left[\frac{1}{(r-R)^{N-1}} - \frac{1}{(r+R)^{N-1}}\right]. \quad (4)$$

From (4) we see that the potential is divergent when r approaches the surface. But this divergence can be removed if we reconsider the tiny thickness of the matter brane in extra dimensions. A similar phenomenon appeared in electrostatics where the potential of a charged ball is divergent in the center if we neglect its volume. The radius of the ball, i.e. the charge distribution scale, offers a cut off to obtain a finite result. Likewise, the thickness $M_\star^{-1}$ provides a good cut off of the gravitational potential and smears the singularity in $r = R$. So we introduce the cut off $\phi(r) = \phi(R+M_\star^{-1})$, $R-M_\star^{-1} \le r \le R+M_\star^{-1}$ to formula (4). The gravity of a homogeneous 3-sphere can be obtained easily with (4), when $r > R + M_\star^{-1}$,

$$\phi = \begin{cases} -\dfrac{3M}{2M_\star^4 R^3 r}\left[r\ln\dfrac{r+R}{r-R} - 2R\right] & N=2 \\ -\dfrac{3M}{4M_\star^5 R^3 r}\left[\ln\dfrac{r-R}{r+R} + \dfrac{2Rr}{r^2-R^2}\right] & N=3 \\ -\dfrac{3M}{2(N-1)M_\star^{2+N}R^3 r}\left\{\dfrac{1}{N-3}\left[\dfrac{1}{(r+R)^{N-3}}\right.\right. \\ \left.-\dfrac{1}{(r-R)^{N-3}}\right] + \dfrac{r}{N-2}\left.\left[\dfrac{1}{(r-R)^{N-2}} - \dfrac{1}{(r+R)^{N-2}}\right]\right\}. \\ N \ge 4 \end{cases} \quad (5)$$

When $r < R - M_\star^{-1}$,

$$\phi = \begin{cases} -\dfrac{3M}{2M_\star^4 R^3}\left[\ln M_\star^2(R^2-r^2) - 2\right] & N=2 \\ -\dfrac{3M}{4M_\star^5 R^3 r}\left[\ln\dfrac{R-r}{r+R} - \dfrac{2Rr}{r^2-R^2} + 2M_\star r\right] & N=3 \\ -\dfrac{3M}{2(N-1)M_\star^{2+N}R^3 r}\left\{\dfrac{1}{N-3}\left[\dfrac{1}{(r+R)^{N-3}}\right.\right. \\ \left.-\dfrac{1}{(R-r)^{N-3}}\right] - \dfrac{r}{N-2}\left[\dfrac{1}{(r+R)^{N-2}} + \dfrac{1}{(R-r)^{N-2}}\right] \\ \left.+\dfrac{2rM_\star^{N-2}}{N-2}\right\}. \quad N \ge 4 \end{cases} \quad (6)$$

As the same reason mentioned below (4), we introduce the potential cutoff $\phi(r) = \phi(R + M_\star^{-1})$, $R - M_\star^{-1} \le r \le R + M_\star^{-1}$. Now we need to derive the gravitational potential of a sphere for $r \sim L$ and $r \ge L$, which is different from Equations (5) and (6). Because the nucleus' radius is much smaller than L when $N \le 4$ (see equation (3); for $N = 4, L \sim 10^{-10}$ cm is about 100 times of the nuclear radius), the nucleus can be treated as a point mass when $r \sim L$ or $r \ge L$. Therefore the potential of the nucleus for $r \gg L$ and $r \sim L$ is just the first and second formula in (1).

3 The correction of the hydrogen like atom ground energy

From Equation (6) we see that the gravitational potential at the center of a nucleus is $\phi \approx \frac{M}{M_\star^4 R^3} \approx \frac{\rho}{M_\star^4} \approx 10^{-13} \ll 1$, where $\rho \approx 10^{17}$ kg/m^3 is the density of the nucleus. The effect of general relativity is weak, so the Dirac equation coupled with gravity can be obtained in the weak field approximation.

The metric in the weak field approximation is

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu, \tag{7}$$

where $h_{\mu\nu} = 2\delta_{\mu\nu}\phi$. The interaction of the fermion and gravity can be described by the Lagrangian density [19] $-\frac{1}{2}h_{\mu\nu}T^{\mu\nu}$, where $T^{\mu\nu} = \overline{\psi}(i\gamma^\mu\partial^\nu)\psi$ is the fermion's momentum-energy tensor. We have ignored the second and higher order terms of $h_{\mu\nu}$ and ϕ, since they are much smaller than the first order terms of ϕ ($\phi \le 10^{-13}$). Adding the interaction terms to the original Lagrangian density we can obtain the Dirac equation in the gravity field and the Hamiltonian $H = H_0 + H^{'}$, where H_0 is zeroth order in ϕ and $H^{'}$ is the correction due to gravity:

$$H_0 = \alpha \cdot \mathbf{p} + \beta m - \frac{Z\alpha}{r}, \tag{8}$$

$$H^{'} = \phi\left(H_0 + \frac{Z\alpha}{r} + \alpha \cdot \mathbf{p}\right). \tag{9}$$

Replacing the term $\alpha \cdot \mathbf{p}$ in Equation (9) by Equation (8) , we obtain

$$H = \alpha \cdot \mathbf{p} + \beta m - \frac{Z\alpha}{r} + \phi\left(2H_0 + \frac{2Z\alpha}{r} - \beta m\right). \tag{10}$$

The first order correction in the energy is

$$\begin{aligned} \Delta E_n &= \left\langle n \middle| \phi\left(2E_n + \frac{2Z\alpha}{r} - \beta m\right) \middle| n \right\rangle \\ &= \int_0^\infty \left[(g^2 + f^2)\phi(2E_n + \frac{2Z\alpha}{r}) + \phi m(g^2 - f^2)\right] r^2 dr, \end{aligned} \tag{11}$$

where the g, f are radial Dirac eigenfunctions[20]. The correction in the ground energy is

$$\Delta E_{n=1} = \frac{m}{\Gamma(2\gamma+1)} \int_0^\infty \phi\left(3\gamma + \frac{4Z^2\alpha^2}{\rho}\right)\rho^{2\gamma}e^{-\varrho}d\rho, \tag{12}$$

where $\rho = 2Z\alpha mr$, and $\gamma = \sqrt{1 - Z^2\alpha^2}$. The contribution of the inverse-square gravity and the Yukuwa gravity in (1) is

$$\Delta E \approx \frac{Z\alpha Mm^2}{M_{pl}^2} \tag{13}$$

This is far smaller than the contribution of the high dimension gravity near the radius. The calculation indicates that the dominant part of the correction of the ground energy is

$$\Delta E_{n=1} \approx \begin{cases} -\dfrac{3Mm}{M_\star^4 R^3 \Gamma(2\gamma+1)} \ln(M_\star R) \\ \left(\dfrac{3\gamma}{2\gamma+1} + \dfrac{2Z^2\alpha^2}{\gamma R'}\right) R'^{2\gamma+1}, \quad N = 2 \\ -\dfrac{3Mm}{(N-1)(N-2)M_\star^4 R^3 \Gamma(2\gamma+1)} \\ \left(\dfrac{3\gamma}{2\gamma+1} + \dfrac{2Z^2\alpha^2}{\gamma R'}\right) R'^{2\gamma+1}, \quad N = 3, 4 \end{cases} \tag{14}$$

where $R^{'} = 2Z\alpha mR$.

To show that our results will reduce to Newtonian theory in the absence of extra dimensions, there is a self-consistent verification: when the size of extra dimension L approaches zero, the theory would also reduce to Newtonian theory. From (2) we see that $M_\star$ would be infinitely large in this case, and the correction of gravity from extra dimensions would approach zero.

The Table 1 shows Pb^{+81}, hydrogen, and Pb muonic atoms' frequency shifts when $M_\star \approx 1$ TeV, and the third column display LUO's results in ref. [16].

4 Discussion and conclusion

From Table 1 we see that our corrections are dramatically different from the results in the ref [16]. On the one hand, our results are significantly larger. On the other hand, in the ref [16], the frequency shift decreases dramatically when N is increasing, but our results aren't sensitive to the number of extra dimensions.

The reasons for these differences can be found by comparing the gravity of a sphere (5), (6) and the gravity of a point mass (1), which was used in ref [16–18]. In ref [16], LUO introduced the cut off $R = A^{\frac{1}{3}}r_0$ (the radius of

Table 1: The shift of the frequencies. In ref [16] $\Delta\nu$ was not obtained for $N = 4$

	N	$\Delta\nu = \frac{\Delta E}{h}$	$\Delta\nu$ in ref [16]
Pb^{+81}	2	2.3×10^4Hz	10^0Hz
	3	1.1×10^3Hz	10^{-5}Hz
	4	3.7×10^2Hz	
H	2	2.1×10^{-6}Hz	10^{-8}Hz
	3	1.2×10^{-7}Hz	10^{-13}Hz
	4	4.1×10^{-8}Hz	
$Pb^{+82} + muon$	2	1.4×10^{11}Hz	10^9Hz
	3	6.6×10^9Hz	10^4Hz
	4	2.2×10^9Hz	

a nucleus) to the gravity of a particle, so they only considered the contribution outside of the nucleus. Our results, however, show that the interior of the nucleus plays a more significant role in the correction, therefore our results are much larger. In addition, the gravity of a particle decreases more rapidly with increasing N, so their results were highly sensitive to N. In our derivation, the wave-function of the ground state is a constant around the nucleus and the potential inside the nucleus has the same order of magnitude for different N (It can be found in (6),$\phi \sim -\frac{3M}{(N-1)(N-2)M_\star^4 R^3}$). So our correction (14) isn't so sensitive to N in Table 1.

Table 1 also indicates that the spectroscopy of high Z atoms and muonic atoms may serve as a new method to detect extra dimensions. The previous results is based on $N = 2, 3, 4$, but it can also be generalized to $N \geq 5$. The reason can also be found in (6), where the gravitational potential inside the sphere is $\phi \sim -\frac{3M}{(N-1)(N-2)M_\star^4 R^3}$, thus it is proportional to nuclear density. If the nuclear radius is bigger than the extra dimension scale L, we can divide the nucleus into a number of smaller parts compared with extra dimensions, within which the potentials are of the same form $\phi \sim -\frac{3M}{(N-1)(N-2)M_\star^4 R^3}$. So the correction is almost of the same order when $N \geq 3$. Equation (14) provides the threshold to detect extra dimensions. If experiments can reach this precision, we will find the extra dimension or rule out the ADD model.

There are some uncertain things. The first one comes from the uncertainty of $M_\star \approx 1 \sim 10\,\mathrm{TeV}$. Because $\Delta\nu = \frac{\Delta E}{h} \propto M_\star^{-4}$, the order of $\Delta\nu$'s magnitudes will change greatly. The other uncertainty is that we can't determine the number of extra dimensions by (14). We must consider a smaller term in the derivation which we didn't give in (14):

$$\Delta E'_{n=1} \approx -\frac{3(N-3)Mm\left(3\gamma + \frac{4Z^2\alpha^2}{R'}\right)}{(N-1)(N-2)M_\star^4 R^3 \Gamma(2\gamma+1)}\left(\frac{2Z\alpha m}{M_\star}\right)^{N-3} \times \ln\frac{M_\star}{2Z\alpha m}, \tag{15}$$

where $N = 3, 4$. This decreases with the increasing N:

$$\frac{\Delta E'_{n=1}}{\Delta E_{n=1}} \approx \begin{cases} 10^{-3} & N = 3 \\ 10^{-6} & N = 4. \end{cases} \tag{16}$$

So more rigorous accuracy is needed to detect the number of extra dimensions.

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