

Research Article

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Generalized binomial distribution in photon statistics

Abstract: The photon-number distribution between two parts of a given volume is found for an arbitrary photon statistics. This problem is related to the interaction of a light beam with a macroscopic device, for example a diaphragm, that separates the photon flux into two parts with known probabilities. To solve this problem, a Generalized Binomial Distribution (GBD) is derived that is applicable to an arbitrary photon statistics satisfying probability convolution equations. It is shown that if photons obey Poisson statistics then the GBD is reduced to the ordinary binomial distribution, whereas in the case of Bose-Einstein statistics the GBD is reduced to the Polya distribution. In this case, the photon spatial distribution depends on the phase-space volume occupied by the photons. This result involves a photon bunching effect, or collective behavior of photons that sharply differs from the behavior of classical particles. It is shown that the photon bunching effect looks similar to the quantum interference effect.

Keywords: Bose-Einstein statistics; Polya distribution; photon bunching; quantum interference

PACS: 42.50.-p, 02.50.-r, 42.50.Fx, 42.50.Ar, 05.30.-d

DOI 10.1515/phys-2015-0005

Received April 24, 2014; accepted August 22, 2014

1 Introduction

This article examines the spatial distribution of photons in a light beam with arbitrary photon statistics. If the photons obey the Bose-Einstein (BE) statistics, then the photon spatial distribution is found to exhibit certain features, termed here the 'photon bunching effect', or collective photon behavior that differs sharply from the behavior of classical particles.

In the literature, the term *photon bunching* is sometimes related to the Brown-Twiss effect, which is explained by intensity fluctuations in the light beam [1]. In this paper, the term *photon bunching* is used in an entirely different

sense, in that the effects considered in this article are not associated with intensity fluctuations. As shown in Sections 4-5, the photon bunching effect in the BE statistics is rather similar to the quantum interference effect, which was first observed in [2].

The problem considered in this work is stated in Section 2, which also contains the basic formula for the flux of classical particles obeying Poisson statistics and for the flux of quantum particles obeying BE statistics.

To solve the stated problem, a *Generalized Binomial Distribution* (GBD) is derived in Section 3 for an arbitrary photon statistics. It is shown that in the case of BE statistics, the GBD reduces to the Polya distribution, which predicts the photon bunching effect that is discussed in Section 4.

The equivalence of various statistical problems is discussed in Section 5, suggesting the possibility of applying the results obtained to other equivalent problems involving the interaction of photon flux with a beamsplitter, photodetector, or neutral filter. The main conclusions of this work are summarized in Section 6.

A theoretical approach developed in this paper allows one to study the subtle features of spatial distribution of particles in the BE statistics, as well as in any other photon statistics. In the limit of large phase-space volumes, the results obtained in this work coincide with the known classical solution, while in the limit of small phase volumes the results are found to be consistent with an experiment [3] which examined the phenomenon of quantum interference between two photons incident on the *same* input port of a beamsplitter.

2 Interaction of a light beam with a diaphragm from the viewpoint of photon statistics

2.1 Statement of the problem

Consider a beam of light that has a uniform distribution of statistical properties over its cross-section. Let the beam cross-section s be divided into two *arbitrary* parts, a and b . The photon statistics in part a is measured during a certain sampling time τ , *i.e.* in sampling volume $A = act$,

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where c is the speed of light. Capital characters A , B , and S will be used in this paper to designate sampling volumes corresponding to light beam cross-section areas a , b and s (Figure 1).

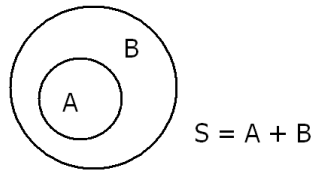


Figure 1: An illustration of a light beam cross-section, representing the interaction of photon flux with a diaphragm when k photons pass through the hole A while $n - k$ photons are absorbed by the screen B . Sampling volumes A and S may correspond to possible detector apertures in a photon statistics measurement.

If a sampling volume is normalized to a coherence volume, then it will correspond to a certain phase-space volume, because a coherence volume corresponds to a single phase-space cell. Therefore, volumes of A , B and $S = A + B$ may be considered as dimensionless phase-space volumes consisting of an arbitrary number of cells.

Since the light beam is supposed to have a uniform distribution of statistical properties over its cross-section, probabilities α and β for a photon to be in volume A or B , respectively, are given by

$$\alpha = \frac{A}{A+B}, \quad \beta = \frac{B}{A+B}, \quad (1)$$

so that $\alpha + \beta = 1$.

Photon statistics in an arbitrary volume will be discussed in this article. In volume A , the photon statistics is represented by an infinite series $p_n(A)$, where $n = 0, 1, 2, \dots$ and $p_n(A)$ is the probability that n photons are in volume A .

The following problem is considered in this work:

Assume the photon statistics in $S = A + B$ is known and the probabilities α and β for a photon to be in volumes A and B , respectively, are given. What, then, is the probability distribution $W(k, n-k)$ for k photons to be in A and $n-k$ photons to be in B ? Both classical and quantum solutions to this problem are discussed in this work. The results obtained for the quantum statistics, as shown below, include a photon bunching effect that looks similar to the quantum interference effect.

2.2 A flux of classical particles

Let us begin with the assumption that beam S consists of classical non-interacting particles that appear in the se-

lected volume independently of each other, so that there are no correlations between the particles. In this case, statistics $p_n(S)$ in volume $S = A + B$ is related to the relevant statistics in A and B by the classical equation of probability convolution:

$$p_n(A+B) = \sum_{k=0}^n p_k(A) p_{n-k}(B). \quad (2)$$

Equation (2) is known to be applicable to random processes in areas A and B if there are no correlations between these random processes [5]. In addition, (2) takes into account the conservation of energy, or the number of particles $n = k + (n - k)$, if light beam S is separated into two parts A and B .

Classical non-interacting particles also obey Poisson statistics in the arbitrary volume A , so that

$$p_k(A) = \frac{(wA)^k}{k!} e^{-wA}, \quad (3)$$

where w is the average number of particles per unit volume. The Poisson statistics is known to hold for photons in a coherent radiation field [6], *i. e.* in the case of amplitude stabilized laser radiation.

Substituting (3) into (2) yields

$$\frac{(A+B)^n}{n!} = \sum_{k=0}^n \frac{A^k B^{n-k}}{k!(n-k)!}. \quad (4)$$

This equation is an identity, which after multiplying both sides by $n!$ turns into the binomial theorem. Therefore, the Poisson statistics (3) satisfies the probability convolution equation (2).

All the above are well-known facts that are presented here for the sake of convenience in comparing classical and quantum solutions.

2.3 A flux of quantum particles satisfying Bose-Einstein statistics

Now let beam S be thermal radiation, so that the photons in beam S obey BE statistics:

$$p_k = \frac{w^k}{(1+w)^{k+1}}, \quad (5)$$

where w is the degeneracy parameter, or the average number of particles in a coherence volume. Writing photon statistics in this form, one assumes that the sampling volume in the statistics measurement coincides with the coherence volume, *i. e.* a single phase-space cell.

The BE statistics in an arbitrary phase-space volume A is given by:

$$p_k(A) = \frac{A(A+1)\dots(A+k-1)}{k!} \frac{w^k}{(1+w)^{k+A}}. \quad (6)$$

This formula was derived by Leonard Mandel from combinatorial considerations [7]. Actually, (6) is a negative binomial distribution. If $A = 1$ then (6) becomes the usual expression for the BE statistics (5) in a single cell of the phase-space.

Equation (6) can be conveniently presented as

$$p_k(A) = \frac{A^{\bar{k}}}{k!} \frac{w^k}{(1+w)^{k+A}}, \quad (7)$$

where

$$A^{\bar{k}} = A(A+1)\dots(A+k-1) \quad (8)$$

is the rising factorial, or Pochhammer's symbol.

We saw above that the Poisson statistics satisfies the system of probability convolution equations (2). Let us now determine whether the BE statistics satisfies (2). Substituting (7) into (2) after obvious reductions yields

$$\frac{(A+B)^{\bar{n}}}{n!} = \sum_{k=0}^n \frac{A^{\bar{k}}}{k!} \frac{B^{\bar{n-k}}}{(n-k)!}. \quad (9)$$

This is the well-known Vandermonde's identity [8], which is a generalization of the binomial theorem (4) for rising factorials. A proof of this identity is given in Appendix 1.

Thus, the BE statistics (7) also satisfies the probability convolution equations (2). This is because the BE statistics ignores intensity fluctuations in the light beam, and hence does not take into account correlations of photon numbers in volumes A and B , which was proven, among other sources, in [9].

Along with the BE statistics, the Glauber's statistics [10] for a homogeneously broadened spectral line also satisfies (2), which can be shown by direct verification (see Appendix 2).

Generally speaking, any proper photon statistics² should satisfy the probability convolution equation (2).

In other words, for any photon statistics, photon numbers in volumes A and B are independent random variables, which means that no photon statistics can take into account photon correlations. This conclusion is explained within the density matrix approach by the fact that photon

correlations are described by non-diagonal elements while photon statistics is represented by the diagonal elements of the density matrix. In a semiclassical approach, to account for photon correlations one should examine the intensity correlation function rather than photon statistics. This issue is discussed in detail in [6].

3 Generalized binomial distribution

Dividing the n -th equation in (2) by $p_n(A+B)$ yields

$$\sum_{k=0}^n W(k, n-k) = 1, \quad (10)$$

where

$$W(k, n-k) = \frac{p_k(A) p_{n-k}(B)}{p_n(A+B)}. \quad (11)$$

The idea behind this expression is that $W(k, n-k)$ is the probability that k photons are in volume A on condition that $n-k$ photons are in B . The denominator in (11) guarantees that this probability is correctly normalized according to (10). In other words, eq. (11) gives the probability distribution of photons among parts A and B of volume S given it contains n photons. Equation (11) is valid for an arbitrary photon statistics that satisfies (2).

3.1 Classical statistics

In the classical case, all the probabilities in (11) are given by Poisson statistics (3). Substituting (3) into (11) and taking into account (1) gives

$$W(k, n-k) = \frac{n!}{k!(n-k)!} \alpha^k \beta^{n-k}. \quad (12)$$

This is the binomial distribution that occurs when a flux of classical non-interacting particles is separated into two parts with probabilities α and β .

Obviously, should any *nonpoissonian* statistics be substituted into (11) then the binomial distribution (12) would not be obtained. Therefore, the binomial distribution is valid *only* if particles in beam S obey Poisson statistics. The reverse statement is also true: let X and Y denote numbers of photons in volumes A and B , correspondingly. If X and Y are independent random variables and the conditional distribution of X given $X+Y$ is binomial then both X and Y should obey Poisson statistics (for details see [4]). In other words, the binomial distribution (12) and the Poisson statistics are closely interconnected phenomena – in a beam of photons one is impossible without the other. For

² The term "photon statistics" is applicable only to random processes and does not apply to controlled processes, such as a light beam with given amplitude modulation, sub-Poissonian processes, etc., because in controlled processes photon statistics depends on the choice of starting points for the sampling intervals and, therefore, cannot be uniquely defined.

this reason, it would be a mistake to use the binomial distribution with any non-Poisson statistics. In the next Section the replacement of the binomial distribution is found in the case of BE statistics.

Equation (11), according to its derivation, holds for an arbitrary photon statistics that is a solution of (2). An arbitrary photon statistics inevitably tends towards Poisson statistics in the limit of small photon density because, in this limit, the average distance between photons is much greater than the coherence length. This means that the photons may not take part in quantum interference and should, therefore, behave like classical noninteracting particles that obey Poisson statistics. This conclusion is supported by the fact that the Poisson statistics, like any photon statistics, is a solution of (2). Therefore, for any photon statistics, Eq. (11) will include the binomial distribution as a special case in the limit of small photon density.

Consequently, equation (11) can be regarded as a *Generalized Binomial Distribution*, which is valid for arbitrary statistics $p_k(A)$ that satisfies the probability convolution equations (2).

The binomial distribution (12) is obtained for Poisson statistics on the condition of uniformly distributed radiation statistical properties over the beam cross-section, according to the statement of the problem. For a nonuniform distribution of statistical properties (for example, nonuniform distribution of radiation intensity) the photon density ω will vary across the beam cross-section. In this case, the probabilities α and β for a photon to be in A and B , respectively, will be given by some integral expressions rather than by (1). However, it can be easily shown that this will not change the final result (12) that includes only the probabilities. This conclusion is in line with the results presented in [4].

3.2 Quantum statistics

In the quantum case, all the probabilities in (11) are given by Eq. (7) for the BE statistics in an arbitrary volume. This is true because the BE statistics, as well as the Poisson statistics, satisfies (2), as shown in Section 2.3. Therefore, substituting (7) into (11) yields

$$W(k, n-k) = \frac{p_k(A)p_{n-k}(B)}{p_n(S)} = \frac{n!}{k!(n-k)!} \frac{A^{\bar{k}} B^{\overline{n-k}}}{S^{\bar{n}}} \\ = C_n^k \frac{A(A+1) \dots (A+k-1) B(B+1) \dots (B+n-k-1)}{S(S+1) \dots (S+n-1)}. \quad (13)$$

This probability distribution is known as the Polya distribution. Eq. (13) determines the probabilities of different

photon-number distributions $(k, n-k)$ between volumes A and B if n photons in $S = A + B$ obey the BE statistics. Therefore, (13) is a replacement for the classical binomial distribution in the case where the incident photons obey quantum statistics.

It is known that the Polya distribution takes into account aftereffects that are alien to the Bernoulli process [5]. This property of the Polya distribution is shown below to produce the photon bunching effect.

Given $A = \alpha S$ and $B = \beta S$, eq. (13) may be written using probability α that a photon is in A and probability β that a photon is in B :

$$W(k, n-k) = \frac{(\alpha S)^{\bar{k}} (\beta S)^{\overline{n-k}} n!}{k! (n-k)! S^{\bar{n}}}. \quad (14)$$

This form of the generalized binomial distribution in BE statistics will be convenient for further calculations.

If $S \rightarrow \infty$ then the Polya distribution (14) becomes the classical binomial distribution:

$$\lim_{S \rightarrow \infty} W(k, n-k) = \lim_{S \rightarrow \infty} \frac{(\alpha S)^{\bar{k}} (\beta S)^{\overline{n-k}} n!}{k! (n-k)! S^{\bar{n}}} \\ = \binom{n}{k} \alpha^k \beta^{n-k}, \quad (15)$$

which means that the classical binomial distribution (12) is applicable not only in the case of Poisson statistics (as noted above), but also in the case of BE statistics in the limit of low particle density $n \ll S$. In this limit the mean distance between photons is much greater than the coherence length and photons behave on average as independent classical particles that do not tend to bunch. In the opposite limit as $S \rightarrow 0$ photons show nonclassical properties, which will be discussed below.

Information on the degeneracy parameter w is missing from (14), which means that the probability distribution (14) does not depend on the radiation temperature and the frequency range selected to study photon statistics. However, probabilities $W(k, n-k)$ depend on the phase space volume S occupied by the photons. That is the fundamental difference between (14), which is valid for the BE statistics, and the classical binomial distribution (12), which is valid for Poisson statistics.

The Polya distribution (14) gives the exact quantum statistical solution to the problem of interaction of an arbitrary number of photons of thermal radiation occupying arbitrary phase-space volume S , with a classical device that separates the photon flux into two parts with known probabilities $\alpha = A/S$ and $\beta = B/S$.

In this section it was shown that, in the case of BE statistics, the generalized binomial distribution (11) takes

the form of Polya distribution that is known to occur in random processes that differ from the Bernoulli process [5].

4 Photon bunching in Bose-Einstein statistics

This Section presents several examples of the generalized binomial distribution in the BE statistics. Probabilities $W(k, n-k)$ of different photon-number states $(k, n-k)$ for n photons in volume S are calculated on the basis of (14) in some simple cases, the number of photons in A being designated as k and the number of photons in B being $n - k$.

4.1 One photon

If one photon is in volume $S = A + B$ then the probabilities of photon-number states $(1,0)$ and $(0,1)$ designating a photon in A or in B , respectively, are given by

$$\begin{aligned} W(1, 0) &= \frac{(\alpha S)^1}{1!} \frac{(\beta S)^0}{0!} \frac{1!}{S^1} = \alpha, \\ W(0, 1) &= \frac{(\alpha S)^0}{0!} \frac{(\beta S)^1}{1!} \frac{1!}{S^1} = \beta. \end{aligned} \quad (16)$$

Probabilities (16) turn out to be independent of volume S . One photon is found in A with probability α or in B with probability β as it should be according to (1).

4.2 Two photons

If there are two photons in S then the probabilities of different photon-number distributions between volumes A and B , according to (14), are:

$$\begin{aligned} W(2, 0) &= \frac{\alpha(\alpha S + 1)}{S + 1} \rightarrow \alpha^2, \\ W(1, 1) &= \frac{2\alpha\beta S}{S + 1} \rightarrow 2\alpha\beta, \\ W(0, 2) &= \frac{\beta(\beta S + 1)}{S + 1} \rightarrow \beta^2. \end{aligned} \quad (17)$$

In this case probabilities $W(k, n-k)$ depend on volume S occupied by the photons. The limit values of corresponding probabilities as $S \rightarrow \infty$ are shown on the right-hand side of (17). These limit values coincide with the well known classical results based on the binomial distribution as it should be according to (15).

Probabilities (17) are shown in Figure 2 as functions of volume S occupied by the photons in the case where

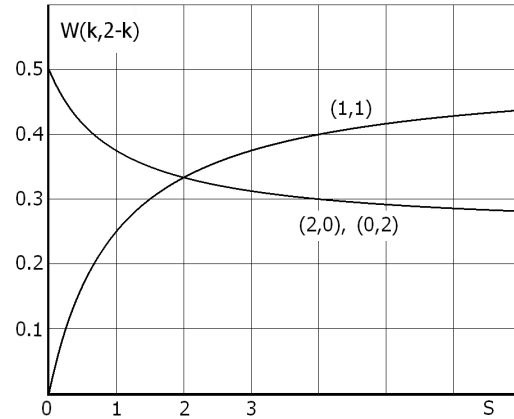


Figure 2: Non-classical probabilities of two-photon distribution among two halves of volume S in the Bose-Einstein statistics. Volume S is given in the units of coherence volume.

$A = B$ ($\alpha = \beta = 0.5$).

Decreasing volume S makes classical probabilities invalid because in quantum statistics, as follows from Figure 2, photons tend to bunch together if they are located in a small phase-space volume. Photon bunching is manifested by an abnormally high probability of states $(2,0)$ and $(0,2)$ that designate both photons occupying only one half of volume S if that volume becomes small, *e. g.* less than 2-3 coherence volumes. Note that $S = s\tau$, therefore, S may be varied by changing the beam cross-section s and/or sampling time τ .

4.3 Three photons

If three photons are in volume S then using (14) one obtains the following probabilities of different photon-number configurations:

$$\begin{aligned} W(3, 0) &= \alpha \frac{(\alpha S + 1)(\alpha S + 2)}{(S + 1)(S + 2)} \rightarrow \alpha^3, \\ W(2, 1) &= 3\alpha\beta \frac{S(\alpha S + 1)}{(S + 1)(S + 2)} \rightarrow 3\alpha^2\beta, \\ W(1, 2) &= 3\alpha\beta \frac{S(\beta S + 1)}{(S + 1)(S + 2)} \rightarrow 3\alpha\beta^2, \\ W(0, 3) &= \beta \frac{(\beta S + 1)(\beta S + 2)}{(S + 1)(S + 2)} \rightarrow \beta^3. \end{aligned} \quad (18)$$

The basic features of the three-photon distribution among volumes A and B are the same as in the previous example of two photons. In the limit of large volume $S \rightarrow \infty$, shown on the right-hand side of (18), photons behave as if they were independent classical particles obeying the binomial distribution. In contrast to such classical behavior, photons occupying a small volume of about several bins exhibit a spatial distribution that deviates markedly from the predictions of the classical binomial distribution.

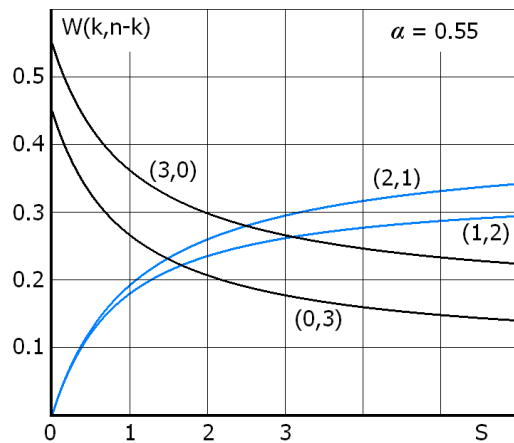


Figure 3: (Color online) Non-classical probabilities of three-photon configurations versus volume S occupied by the photons in a case where the volume is divided into two unequal parts ($\alpha = 0.55 S$). The limit values of $W(k, 3 - k)$ for large S coincide with classical probabilities.

Three-photon configuration probabilities (18) are shown as functions of volume S in Figure 3, for a non-symmetrical division of S into two parts ($\alpha = 0.55$). It is obvious from the plots that for small values of S , less than 3-4 coherence volumes, the probabilities $W(k, 3 - k)$ differ markedly from the classical limits that are obtained as $S \rightarrow \infty$. This, again, is the manifestation of photon bunching in quantum statistics.

4.4 Fifty photons

As an example of bunching of a large number of photons in quantum statistics, consider fifty photons in volume S that is divided into two equal parts ($\alpha = \beta = 0.5$). Figure 4 presents probabilities $W(k, n - k)$ that k photons are in part A while $n - k$ photons are in part B given total number of photons in S is $n = 50$. Different curves correspond to different values of phase-space volume S occupied by the photons.

A case with $S = 10^4$ is a good approximation to the limit of $S \rightarrow \infty$ because in this case each photon occupies, on average, a volume of 200 bins, and the probability distribution $W(k, n - k)$, according to (15), looks like the classical binomial distribution.

Fifty photons occupying a single bin (curve $S = 1$) exhibit substantially non-classical properties because the probability distribution in this quantum state displays pronounced maxima at $k = 0$ and $k = n$. This is a manifestation of photon bunching, which is a tendency toward collective behavior in quantum statistics. In this case the probability that a group of photons is separated into two

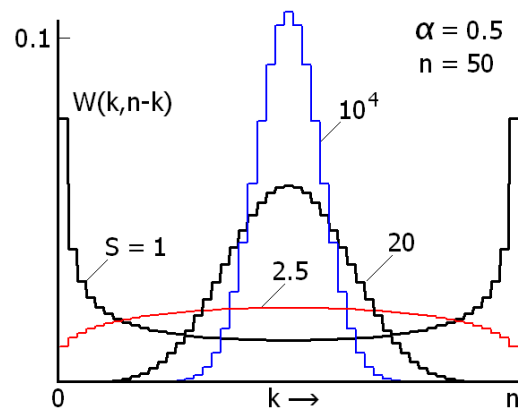


Figure 4: (Color online) Non-classical probability distributions of 50 photons among two equal parts of volume S . Different curves correspond to different phase-space volumes S occupied by the photons obeying the Bose-Einstein statistics. Histogram $S = 10^4$ is close to the classical binomial distribution while other histograms show the deviation of quantum particles' distribution from the classical distribution. Photon bunching, or a tendency toward coalescence, is well manifested if photons are found in a small phase-space volume, for example if $S = 1$.

almost equal parts is minimal, while in the classical case described by the binomial formula this probability is maximal (curve $S = 10^4$). In the limit $S \rightarrow 0$, a photon bunch looks like a single quantum entity that is not inclined to dissolve into smaller groups of photons.

According to the above, in Poisson statistics (coherent radiation) photons behave like classical particles in accordance with the classical binomial distribution, while in quantum statistics (blackbody radiation) photons exhibit different features showing a tendency toward bunching, or collective behavior. In Figure 4, curve $S = 10^4$ actually describes the behavior of classical particles while other curves show the deviation of quantum particles from classical behavior that becomes more pronounced for smaller phase-space volumes occupied by the particles.

It is worthwhile noting that the average number of photons per one cell in BE statistics is defined by the temperature and frequency of radiation:

$$w = \frac{1}{e^{\frac{h\nu}{kT}} - 1}. \quad (19)$$

Therefore, 50 photons in one coherence volume could be observed with notable probability if $kT \approx 50h\nu$, i. e. either in a low-frequency range of blackbody radiation or for a black body of very high temperature.

The Polya distribution (13) and Figures 2-4 present the quantum solution to the problem stated in Section 2.1 regarding the distribution of n photons among volumes A and B in the case of BE statistics.

5 Statistical equivalence of different problems

The problem stated in Section 2.1 is directly related to the interaction of a light beam with a diaphragm because the distribution $W(k, n - k)$ defines the probability that k photons pass through the hole A while $n - k$ photons are absorbed by the screen B (see Figure 1). This problem is also related to the following three problems:

- 1) Photon number distribution at the output ports of a beamsplitter of transmittance $\alpha = A/S$;
- 2) Statistics of photon detection by a photodetector of quantum efficiency α .
- 3) Photon statistics behind a neutral filter of transmittance α .

In all of the above problems, a photon flux is separated into two parts with given probabilities α and $\beta = 1 - \alpha$. Although very different physical processes are involved in these problems, with regard to photon statistics they are equivalent and described by the same formulas provided the light beam has the same properties in all problems. This is because in statistics it is *the probability of an event that is important, not the nature of the event*.

For example, the photon statistics in part A of light beam S (Figure 1) should coincide with photon statistics produced by the same light beam S after a beamsplitter of transmittance $\alpha = A/S$, because the probability that a photon is in volume A equals the probability that a photon is transmitted through the beamsplitter. Likewise, if the probability of photodetection is α , the same as the probability of a photon passing through the beamsplitter, then the statistics of photo-absorptions should be the same as the photon statistics behind the beamsplitter. A neutral filter actually may be considered as a beamsplitter if the reflected radiation is absorbed.

Consequently, the above problems are *equivalent* with respect to photon statistics, provided the light beams involved in these problems have identical properties, *i. e.* in the different problems the photons are in the same quantum state. Therefore, any statistical result obtained for one problem is sure to be valid for the other equivalent problems as well. In particular, the Polya distribution (14) and photon bunching effects shown in Figures 2-4 must hold in all of the above problems if thermal photons are involved.

Given such an interpretation of the results, Figure 2 shows the nonclassical probabilities of photon-number states at the output ports of a beamsplitter versus the volume occupied by the two photons before interacting with the beamsplitter, provided that the photons obey the BE statistics and have entered the same input port of the

beamsplitter. Such photon behavior as that shown in Figure 2 is not unusual: indeed, similar results were observed in a quantum interference experiment [3], in which two indistinguishable photons entered the *same* input port of a beamsplitter. The probability of output states (2,0) and (0,2) was measured in this experiment against the distance between two input photons and was found to be maximum for zero delay, just as indicated in Figure 2 (for details refer to [3]). Regardless of the fact that non-thermal photons were used in this experiment, the experimental results match well the curves presented in Figure 2. Qualitative coincidence of experimental and theoretical curves indicates that the photon bunching effect considered in this work does really exist and turns out to be analogous to the quantum interference effect that was examined in the above experiment.

The bunching of photons means that the photon statistics behind a beamsplitter should, in a general case, deviate from the classical binomial distribution, which holds only if the incident photons obey the Poisson statistics. That conclusion was proven in deriving eq. (12). A similar result was obtained using a different theoretical method in [11, 12] and later confirmed experimentally in [13] for a system of multiplexed on-off detectors.

The applicability of the results shown in Figure 2 to the equivalent problems of photon interaction with a diaphragm, photodetector, and neutral filter suggests that the quantum interference phenomenon (or photon bunching effect) may also manifest itself in situations where no beamsplitters are involved. This conclusion is based on the fact that the photon bunching effect is a property of photons in a certain quantum state rather than the property of a macroscopic device.

6 Conclusions

The major results of this work are derived from two basic facts: firstly, the probability convolution equations (2) for the interaction of a light beam with a macroscopic device,

$$p_n(A + B) = \sum_{k=0}^n p_k(A)p_{n-k}(B)$$

and, secondly, Mandel's formula (6) for the Bose-Einstein statistics in an arbitrary phase-space volume A

$$p_k(A) = \frac{A(A+1)\dots(A+k-1)}{k!} \frac{w^k}{(1+w)^{k+A}},$$

where w is the degeneracy parameter of photon gas.

It is shown that the Mandel's formula satisfies the convolution equations, which immediately yields the proba-

bility (13) of different photon-number states $(k, n - k)$ after the interaction of photon flux with a macroscopic device:

$$W(k, n - k) = \frac{p_k(A)p_{n-k}(B)}{p_n(S)} = C_n^k \frac{A^{\bar{k}} B^{\overline{n-k}}}{(A + B)^{\bar{n}}}.$$

The last formula is the Polya distribution that, according to the analysis presented in Sections 4-5, describes the photon bunching effect (which looks like quantum interference) in the BE statistics.

Therefore, the main results of this work have been obtained as a direct consequence of first principles, *i. e.* the quantum statistics. No additional assumptions were made and no approximate methods were applied to obtain the results. For this reason, the Polya distribution (13) is the exact quantum statistical solution of the stated problem.

The main conclusions of this work may be summarized as follows:

1. It is shown that the Bose-Einstein statistics satisfies the probability convolution equations, which is another proof of the well-established fact that photon statistics is unable to take into account intensity fluctuations and related photon correlations.
2. For probabilities of final photon-number states (k, n) arising if the photon flux is separated into two parts a *generalized binomial distribution* is obtained

$$W(k, n) = \frac{p_k(A)p_n(B)}{p_{k+n}(A + B)}$$

that is applicable for arbitrary photon statistics $p_k(A)$ satisfying the probability convolution equations.

3. In the case of Bose-Einstein statistics, the generalized binomial distribution takes the form of the Polya distribution, which presents the exact quantum solution of the problem of interaction of an arbitrary number of thermal photons occupying an arbitrary phase-space volume with a macroscopic device that separates the photon flux into two parts with known probabilities.
4. Due to the statistical equivalence of different problems, the theoretical results obtained for the interaction of photons of arbitrary statistics with a diaphragm can be applied to the interaction of photons with a beamsplitter, neutral filter, or photodetector.
5. It is shown that the classical binomial distribution correctly describes the probabilities of output photon-number states after the interaction of photons with the macroscopic device in two cases only: (a) if the photons obey Poisson statistics; or (b) if the average distance between photons greatly exceeds the coherence length (in this case any photon statistics tends toward Poisson statistics). Therefore, these two conditions determine the domain of applicability of the binomial formula.

6. It follows from Figures 2-4 that photon bunching, or quantum interference in the BE statistics, becomes notable if the mean occupation number is larger than unity $\omega \gtrsim 1$.

Acknowledgement: The author would like to thank V.D. Ivanov and V.A. Kozminykh of Moscow Institute for Physics and Technology, as well as A.V. Masalov of the Lebedev Physical Institute for stimulating discussions.

Appendix 1: Proof of identity (9)

Equation (9) includes the numerical sequence

$$f_k(A) = \frac{A^{\bar{k}}}{k!}, \quad (20)$$

so that (9) can be written as

$$f_n(A + B) = \sum_{k=0}^n f_k(A) f_{n-k}(B). \quad (21)$$

The generating function of sequence (20) is

$$F(A) = \frac{1}{(1 - z)^A}, \quad (22)$$

which can be verified by direct expansion of (22) in a Maclaurin series:

$$\frac{1}{(1 - z)^A} = \sum_{k=0}^{\infty} \frac{A^{\bar{k}}}{k!} z^k = \sum_{k=0}^{\infty} f_k(A) z^k. \quad (23)$$

Identity (21) is now validated by the following obvious relation between the generating functions:

$$\frac{1}{(1 - z)^{A+B}} = \frac{1}{(1 - z)^A} \frac{1}{(1 - z)^B}, \quad (24)$$

Q.E.D.

Appendix 2: The Glauber's statistics satisfies the probability convolution equation

Let $P(A)$ be the generating function (GF) of statistics $p_k(A)$ in an arbitrary phase-space volume A . If this statistics satisfies eq. (2) then its GF obeys [5]

$$P(A + B) = P(A)P(B). \quad (25)$$

The functional equation (25) in P has the solution

$$P(A) = P(1)^A, \quad (26)$$

where $P(1)$ is the GF of photon statistics in a unit volume $A = 1$. Obviously, any statistics satisfies (2) if its GF satisfies (26).

Recalling that by definition $A = ac\tau$, eq. (26) may be written as

$$P(A) = F(z)^\tau, \quad (27)$$

where $F(z) = P(1)^{ac}$.

The photon statistics obtained by Glauber [10] for a Lorentzian spectral line has GF (in Glauber's notations)

$$Q(\lambda, \tau) = \exp \left\{ - \left[\left(\gamma^2 + 2\gamma W\lambda \right)^{1/2} - \gamma \right] \tau \right\}, \quad (28)$$

where γ is the half-width of the spectral line, W is the average number of photons per second, τ is the sampling time in the photon statistics measurement, and $\lambda = 1 - z$.

So, the GF of Glauber's statistics (28) has the form $Q(\lambda, \tau) = F(z)^\tau$, which coincides with (27). For this reason, Glauber's statistics satisfies (2), Q.E.D.

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