DE GRUYTER Nonlinear Engineering 2025; 14: 20250150

Research Article

Kottakkaran Sooppy Nisar*, Chandran Anusha, Chokkalingam Ravichandran, and Sriramulu Sabarinathan

Qualitative analysis on existence and stability of nonlinear fractional dynamic equations on time scales

https://doi.org/10.1515/nleng-2025-0150
received August 18, 2024; accepted May 8, 2025

Abstract: This study explores the qualitative analysis of

$$\mathcal{D}^{\gamma}\mathfrak{h}(\zeta) = \mathcal{L}(\zeta, \mathfrak{h}(\zeta), \mathcal{D}^{\gamma}\mathfrak{h}(\zeta)),$$

where $\mathcal{D}^{\gamma}$ is a Riemann–Liouville nabla (∇) fractional nabla derivative, incorporating integral boundary conditions on arbitrary time scales. We rigorously investigate the existence, uniqueness, and stability of solutions to these equations by employing Krasnoselskii's fixed point theorem and the Banach contraction principle. The incorporation of integral boundary conditions extends the applicability of the results to more general systems. To demonstrate the validity and practicality of the theoretical results, we present an illustrative example supported by MATLAB-generated graphical representations. This work contributes to the broader understanding of fractional dynamic systems and their stability across diverse time scales, offering insights with potential applications in engineering and applied sciences.

Keywords: dynamic equation, Riemann–Liouville nabla-derivative, time scales, fixed point

MSC 2020: 37C25, 26E70, 34N05

*** Corresponding author: Kottakkaran Sooppy Nisar**, Department of Mathematics, College of Science and Humanities in Al Kharj, Prince Sattam bin Abdulaziz University, Al Kharj, 11942, Saudi Arabia; Hourani Center for Applied Scientific Research, Al-Ahliyya Amman University, Amman, Jordan; Research Center of Applied Mathematics, Khazar University, Baku, Azerbaijan, e-mail: n.sooppy@psau.edu.sa
Chandran Anusha: Department of Mathematics, Kongunadu Arts and Science College, Coimbatore, India; Department of Science and Humanities, Karpagam College of Engineering, Coimbatore, India, e-mail: anushachandran2498@gmail.com
Chokkalingam Ravichandran: Department of Mathematics, Kongunadu Arts and Science College, Coimbatore, India, e-mail: ravibirthday@gmail.com
Sriramulu Sabarinathan: Department of Mathematics, SRM Institute of Science and Technology, Kattankulathur, Chennai, India, e-mail: ssabarimaths@gmail.com

1 Introduction

A branch of mathematics that focuses on analyzing the integral and derivative operations for functions with fractional orders is known as fractional calculus, and it has gained momentum in recent years. It will be utilized in simulations and certain practical situations [1–8]. Indeed, it offers a multitude of valuable resources for resolving differential and integral equations, along with numerous other challenges related to specialized mathematical physics functions, including their expansions and advancements in both single and multiple variables [9–13]. In recent times, there has been an application of fractional differential equations (FDEs), for various practical issues, such as developing mathematical equations to describe the operation. In addition, FDEs along the integral boundary conditions are prevalent for various natural phenomena originating from diverse fields including fluid dynamics, chemical kinetics, electronics and bio models. Hence, numerous authors have investigated FDEs using various methodologies. The initial-boundary conditions are addressed using fixed point theorems and nonlinear functional analysis.

In 1990, Hilger proposed the introduction of time scales for combination and expansion upon existing differential equations (DEs), discrete equations, and other systems of DE defined upon a closed subset of real number line that is not empty [14–17]. The formulation of the initial value problem's existence and uniqueness for DE on time scales is presented by Hilger along some practical uses. Combination of separate closed real intervals on time scales functions is a perfect structure to examine population dynamics. For past few years, there has been an increasing focus on time-dependent DEs (e.g., [18–21]). Time scales refer to a unified mathematical framework that integrates both discrete and continuous time, allowing for the examination of dynamic systems across varying types of intervals. This approach generalizes traditional DEs to include models that evolve in either continuous or discrete manners, thus broadening the scope of analysis in fractional dynamic systems. Similarly, integral boundary conditions require that solutions to DEs satisfy particular integral constraints rather than pointwise conditions at the boundaries. This is particularly relevant in real-world applications, where boundary conditions are often influenced by cumulative or averaged effects over time.

To decode both DEs and difference equations simultaneously, the dynamic equation is employed within a single domain known as time scale $\mathbb{T}$ [22,23]. The innovative and proactive area is broader and highly adaptable compared to the conventional formulation of DE and difference equations, making it as the most suitable approach for precise and flexible mathematical modeling. In the past, it has been commonly believed by researchers that dynamical processes can be categorized as either continuous or discrete [24–26]. Therefore, researchers have used either DEs or difference equations to mathematically represent the dynamics of a model. Specifically, some significant occurrences do not exist solely in a continuous form or solely in a discrete form [27]. For over a 100 years, scientists have continuously been fascinated by the theory of dynamical systems due to its unpredictable nature and wide-ranging applications in various scientific fields. In DE, on nonlocal cauchy problems, several authors performed research. In physics, the nonlocal condition can be utilized more effectively compared to the conventional initial condition. Since dynamic equations unify differential and difference equations within a common framework, it is significant to investigate their existence under nonlocal initial conditions. The fundamental purpose of transitioning from integer to fractional order is to capture the intricate dynamics of nonlocal interactions, long-range memory effects, and time-dependent behaviors, while also providing a robust framework for modeling phenomena such as anomalous diffusion and complex temporal processes [28]. The limitations of traditional integer-order models in capturing memory, hereditary effects, and nonlocal interactions drive the need for fractional-order approaches. Fractional calculus offers a powerful framework to model complex systems with long-term dependencies, making it essential for fields like physics, biology, and engineering [29–34]. The aim of this study is to deepen the understanding of fractional dynamic systems through the use of the Riemann–Liouville (RL) nabla derivative, a powerful tool for capturing complex system behaviors like memory and hereditary effects that traditional integer-order models cannot address. By focusing on the stability, existence, and uniqueness of solutions on arbitrary time scales, this work advances the field of fractional dynamics while incorporating integral boundary conditions to ensure broader real-world applicability. These insights are crucial for improving modeling accuracy in engineering, control systems, and other applied sciences where time-dependent processes play a key role [35–38].

Gogoi et al. [39], by application of the fixed point techniques, discussed the existence of solution of a nonlinear fractional dynamic equation with initial and boundary conditions on time scales. Motivated by the aforementioned work, using time scale we discuss the existence and stability results of a nonlinear fractional dynamic equation along the integral boundary condition,

$$\left.\begin{aligned} &\mathcal{D}^{\gamma}\mathfrak{h}(\zeta) = \mathcal{L}(\zeta, \mathfrak{h}(\zeta), \mathcal{D}^{\gamma}\mathfrak{h}(\zeta)) \quad \zeta \in \mathscr{T} = [0, T]_{\mathbb{T}}, \\ &\gamma \in (0, 1), \\ &\upsilon\mathfrak{h}(0) + \vartheta\mathfrak{h}(T) = \frac{1}{\Gamma(\gamma)}\int_0^T (T - \mathfrak{s})^{\gamma-1} g(\mathfrak{s}, \mathfrak{h}(\mathfrak{s}))\nabla\mathfrak{s}, \end{aligned}\right\} \tag{1.1}$$

where $\mathcal{D}^{\gamma}$ is an RL derivative of fractional order $0 < \gamma < 1$ and $[0, T] \in \mathbb{T}$. $\zeta \in \mathscr{T} = [0, T]_{\mathbb{T}}$, where $\mathscr{T}$ is a time scale interval such that

$$\mathscr{T} = \zeta \in \mathbb{T} : 0 \le \zeta \le T, T \in \mathfrak{R}.$$

2 Preliminaries

In the set of real numbers $\mathfrak{R}$, a closed subset is referred to as a time scale $\mathbb{T}$. The backward jump operator is a function defined as

$$\rho = \sup\{\zeta \in \mathbb{T} : \rho(\zeta) < \zeta\},$$

for $\rho(\zeta) < \zeta$. A point $\zeta \in \mathbb{T}$ is called left scattered point if $\rho(\zeta) < \zeta$. Conversely, if $\rho(\zeta) > \inf\mathbb{T}$ and $\rho(\zeta) = \zeta$, then ζ is called a left dense point in $\mathbb{T}$.

A left dense-continuous function does not exist in $\mathbb{T}$ for any ∇ derivative. An operator $\mathbb{T}_{\kappa}$ for the time scale $\mathbb{T}$ is defined as follows: If $\mathbb{T}_{\kappa} = \mathbb{T}\backslash\{\mathfrak{t}\}$, else $\mathbb{T}_{\kappa} = \mathbb{T}$ is left scattered minimum say in time scale $\mathbb{T}$.

Definition 2.1. [40] If for every left dense point of $\mathscr{T}$, let function be continuous, then a function $\mathcal{G} : \mathscr{T} \to \mathfrak{R}$ is called ld-continuous function, and in the right dense point, the right-sided limit appears.

Space of ld-continuous function is set of every function from $\mathscr{T}$ to $\mathfrak{R}$ and is denoted by $C(\mathscr{T}, \mathfrak{R})$.

Remark 2.2. Space function $C(\mathscr{T}, \mathfrak{R}) = \mathscr{X}$ from a Banach space with the norm is defined as

$$||g|| = \sup_{\zeta\in\mathscr{T}}|g(\zeta)|, \tag{2.1}$$

for $\zeta \in \mathscr{T}$.

Definition 2.3. [41] Consider $g(\zeta)$ as a ∇-integrable function defined on $\mathscr{T}$, then

$$\int_0^T g(\varphi)\nabla\varphi = \int_0^\zeta g(\varphi)\nabla\varphi + \int_\zeta^T g(\varphi)\nabla\varphi.$$

Definition 2.4. RL ∇-derivative) [42] Assume ld-continuous function be $g : \mathbb{T}_{\mathfrak{k}^{\mathfrak{m}}} \to \mathfrak{R}$, the RL fractional ∇-derivative of order $\gamma(\geq 0) \in \mathfrak{R}$, is defined as

$$\mathcal{D}^{\gamma}_{0^+}g(\zeta) = \mathcal{D}^{\mathfrak{m}}_{0^+}\mathscr{T}^{\mathfrak{m}-\gamma}_{0^+}g(\zeta), \quad \zeta \in \mathscr{T}.$$

Remark 2.5. From Definition 2.4, we also obtain $\mathcal{D}^{\gamma}_{0^+}g(\zeta) = \mathscr{T}^{\mathfrak{m}-\gamma}_{0^+}\mathcal{D}^{\gamma}_{0^+}$, where $\mathfrak{m} = [\gamma] + 1$.

Definition 2.6. [43] Assume $\mathcal{D} \subset C(T, \mathfrak{R})$ be a set. $\mathcal{D}$ is bounded and equicontinuous simultaneously, then it is relatively compact.

Definition 2.7. [43] Assume a bounded set $\mathfrak{B} \subseteq \mathcal{A}, \mathcal{G}(\mathfrak{B})$ and is relatively compact in $\mathcal{A}$, so a mapping $\mathcal{G} : \mathcal{A} \to \mathfrak{B}$ is completely continuous.

Lemma 2.8. [44] *Consider* $\mathbb{T}$ *be a time scales such that* $\varphi_1, \varphi_2 \in \mathbb{T}$ *with* $\varphi_1 \leq \varphi_2$. *If* $z : \mathfrak{R} \to \mathfrak{R}$ *is nondecreasing continuous function, we have,*

$$\int_{\varphi_1}^{\varphi_2} z(\mathfrak{s})\nabla\mathfrak{s} \leq \int_{\varphi_1}^{\varphi_2} z(\mathfrak{s})d\mathfrak{s}.$$

Lemma 2.9. [45] *Assume* $z : [\mathfrak{a}, \mathfrak{b}]_{\mathbb{T}} \to \mathfrak{R}$ *be an integrable function, we have,* ${}^{\nabla}\mathscr{T}^{\nu_1}_{\mathfrak{a}^+}{}^{\nabla}\mathscr{T}^{\nu_2}_{\mathfrak{a}^+}z = {}^{\nabla}\mathscr{T}^{\nu_1+\nu_2}_{\mathfrak{a}^+}z$ *holds.*

Lemma 2.10. [46] (Krasnoselskii fixed point theorem) *Assume a Banach space S. Consider* $\mathcal{W}$ *be closed, bounded, convex, and nonempty subset. Also* $\mathcal{M}, \mathcal{N}$ *be the operators such that* (*a*) $\mathcal{M}\mathfrak{u} + \mathcal{N}\mathfrak{v} \in \mathcal{W}$ *whenever* $\mathfrak{u}, \mathfrak{v} \in \mathcal{W}$; (b) $\mathcal{M}$ *is continuous and compact*; (*c*) $\mathcal{N}$ *be a contraction mapping. Then, we have* $z \in \mathcal{W}$ *such that* $z = \mathcal{M}z + \mathcal{N}z$.

Lemma 2.11. [47] *For* $\zeta \in \mathscr{T}$,

$$|\mathcal{D}^{\gamma}\mathfrak{i}(\zeta) - \mathcal{L}(\zeta, \mathfrak{i}(\zeta)\omega(\zeta))| \leq \xi,$$

where ξ *be a "+" ve number, that is,* $\xi > 0$.

Definition 2.12. [47] Eq. (1.1) is known as Ulam–Hyer's (UH) stable when there appears a constant $\mathbb{H}_{(\mathcal{E}_{\mathcal{L}},\mathcal{E}_g,\gamma)}\xi > 0$ such that for $\mathfrak{i}$ of Lemma 2.11 and for $\xi > 0$, there appears a unique solution $\mathfrak{h}$ of Eq. (1.1) following with the inequality

$$|\mathfrak{i}(\varphi) - \mathfrak{h}(\varphi)| \leq \mathbb{H}_{(\mathcal{E}_{\mathcal{L}},\mathcal{E}_g,\gamma)}\xi, \quad \varphi \in \mathscr{T}.$$

Definition 2.13. [47] Eq. (1.1) is known as generalized UH stable when there appears a constant $\mathbb{H}_{(\mathcal{E}_{\mathcal{L}},\mathcal{E}_g,\gamma)}\xi = 0$ such that for $\mathfrak{i}$ of Lemma 2.11, there appears a unique solution $\mathfrak{h}$ of Eq. (1.1) following with the inequality:

$$|\mathfrak{i}(\varphi) - \mathfrak{h}(\varphi)| \leq \mathbb{H}_{(\mathcal{E}_{\mathcal{L}},\mathcal{E}_g,\gamma)}(\xi), \quad \varphi \in \mathscr{T}.$$

3 Main sequels

For the results of Eq. (1.1), we need some assumptions;

(A1) A function $\mathcal{L}$: $\mathscr{T} \times \mathfrak{R} \to \mathfrak{R}$ is continuous and follows with:
  (i) There appears a constant $\mathcal{M}_{\mathcal{L}} > 0$ such that, $|\mathcal{L}(\zeta, \mathfrak{h})| \leq \mathcal{M}_{\mathcal{L}}(1 + |\mathfrak{h}|)$, for every $\zeta \in \mathscr{T}, \mathfrak{h} \in \mathfrak{R}$.
  (ii) There appears a constant $\mathcal{E}_{\mathcal{L}} > 0$ such that, $|\mathcal{L}(\zeta, \mathfrak{h}) - \mathcal{L}(\zeta, \mathfrak{i})| \leq \mathcal{E}_{\mathcal{L}}|\mathfrak{h} - \mathfrak{i}|$, for every $\zeta \in \mathscr{T}, \mathfrak{h} \in \mathfrak{R}$.

(A2) A function $g : \mathscr{T} \times \mathfrak{R} \to \mathfrak{R}$ is continuous and follows with
  (i) There appears a constant $\mathcal{M}_g > 0$ such that, $|g(\zeta, \mathfrak{h})| \leq \mathcal{M}_g(1 + |\mathfrak{h}|)$, for every $\zeta \in \mathscr{T}, \mathfrak{h} \in \mathfrak{R}$.
  (ii) There appears a constant $\mathcal{E}_g > 0$ such that, $|g(\zeta, \mathfrak{h}) - g(\zeta, \mathfrak{i})| \leq \mathcal{E}_g|\mathfrak{h} - \mathfrak{i}|$, for every $\zeta \in \mathscr{T}, \mathfrak{h} \in \mathfrak{R}$.

(A3) $\mathfrak{k}_1 < 1$, where $\mathfrak{k}_1 < 1 = \frac{T^{\gamma}}{\Gamma(\gamma+1)}\left(\mathcal{M}_{\mathcal{L}} + \frac{\mathcal{M}_{\mathcal{L}}|\vartheta|}{|\upsilon+\vartheta|} + \frac{\mathcal{M}_g}{|\upsilon+\vartheta|}\right)$.

(A4) For $0 < \gamma < 1$, $\mathfrak{h} \in \mathcal{X} \cap \mathcal{L}_{\nabla}(\mathscr{T}, \mathfrak{R})$, then

$\mathcal{L}(\zeta, \mathfrak{h}(\zeta), D^{\gamma}\mathfrak{h}(\zeta))\nabla\zeta = \mathcal{L}(\zeta, \mathfrak{h}(\zeta))\omega(\mathfrak{s})\nabla\zeta$, for every $\zeta \in \mathscr{T}, \mathfrak{h} \in \mathfrak{R}$.

(A5) Assume a set $C = \{\mathfrak{h} = \mathcal{X} : ||\mathfrak{h}|| \leq \upsilon\} \subseteq \mathcal{X}$ and an operator $\Omega : C \to C$, defined as

$$\begin{aligned}\Omega(\mathfrak{h})\zeta = {} & \frac{1}{\Gamma(\gamma)}\int_0^\zeta (\zeta - \mathfrak{s})^{\gamma-1}\mathcal{L}(\mathfrak{s}, \mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s} \\ & - \frac{\vartheta}{(\upsilon + \vartheta)\Gamma(\gamma)}\int_0^T (T - \mathfrak{s})^{\gamma-1}\mathcal{L}(\mathfrak{s}, \mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s} \\ & + \frac{1}{(\upsilon + \vartheta)\Gamma(\gamma)}\int_0^T (T - \mathfrak{s})^{\gamma-1}g(\mathfrak{s}, \mathfrak{h}(\mathfrak{s}))\nabla\mathfrak{s}.\end{aligned}$$

Theorem 3.1. *Let ld-continuous function be* $\mathcal{L} : \mathscr{T} \times \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$. *If a function* $\mathfrak{h}(\zeta)$ *is a solution of the below integral equation, then* $\mathfrak{h}(\zeta)$ *is said to be a solution of Eq.* (1.1).

$$\begin{aligned}\mathfrak{h}(\zeta) = {} & \int_0^T \mathbb{G}(\zeta, \mathfrak{s})\mathcal{L}(\mathfrak{s}, \mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s}) \\ & + \frac{1}{(\upsilon + \vartheta)\Gamma(\gamma)}\int_0^T (T - \mathfrak{s})^{\gamma-1}g(\mathfrak{s}, \mathfrak{h}(\mathfrak{s}))\nabla\mathfrak{s},\end{aligned}$$

where

$$\mathbb{G}(\zeta,\mathfrak{s})=\begin{cases}\dfrac{(\zeta-\mathfrak{s})^{\gamma-1}}{\Gamma(\gamma)}-\dfrac{\vartheta}{(\upsilon+\vartheta)\Gamma(\gamma)}(T-\mathfrak{s})^{\gamma-1}, & \mathfrak{s}\in(0,\zeta]_{\mathbb{T}}\\ -\dfrac{\vartheta}{(\upsilon+\vartheta)\Gamma(\gamma)}(T-\mathfrak{s})^{\gamma-1}, & \\ \mathfrak{s}\in[\zeta,T]_{\mathbb{T}}. & \end{cases}$$

Proof. RL integral equation defines

$$\mathcal{D}^{\gamma}\mathfrak{h}(\zeta)=\frac{1}{\Gamma(\gamma)}\int_0^{\zeta}(\zeta-\mathfrak{s})^{\gamma-1}\mathfrak{h}(\mathfrak{s})\nabla\mathfrak{s},$$

which implies

$$\mathcal{D}^{\gamma}\mathfrak{h}(\zeta)={}^{\nabla}I^{\gamma}\mathfrak{h}^{\nabla}(\mathfrak{s}).$$

Then, by Lemma 2.9, we have

$${}^{\nabla}I^{\gamma}\mathcal{D}^{\gamma}\mathfrak{h}(\zeta)={}^{\nabla}I^{1}\mathfrak{h}^{\nabla}(\zeta)=\mathfrak{h}(\zeta)-\mathfrak{c}_1\quad\text{where } \mathfrak{c}_1\in\mathfrak{R}.$$

Hence,

$$\begin{aligned}\mathfrak{h}(\zeta)&={}^{\nabla}I^{\gamma}\mathcal{L}(\zeta,\mathfrak{h}(\zeta))\omega(\zeta)+\mathfrak{c}_1\\&=\frac{1}{\Gamma(\gamma)}\int_0^{\zeta}(\zeta-\mathfrak{s})^{\gamma-1}\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}+\mathfrak{c}_1.\end{aligned}$$

From integral boundary condition of Eq. (1.1) we have

$$\mathfrak{c}_1=\frac{1}{(\upsilon+\vartheta)\Gamma(\gamma)}\left(\int_0^{T}(T-\mathfrak{s})^{\gamma-1}g(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\nabla\mathfrak{s}-\vartheta\int_0^{T}(T-\mathfrak{s})^{\gamma-1}\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}\right).$$

Hence, we obtain

$$\begin{aligned}h(\zeta)=&\frac{1}{\Gamma(\gamma)}\int_0^{\zeta}(\zeta-\mathfrak{s})^{\gamma-1}\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}\\&-\frac{\vartheta}{(\upsilon+\vartheta)\Gamma(\gamma)}\int_0^{T}(T-\mathfrak{s})^{\gamma-1}\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}\\&+\frac{1}{(\upsilon+\vartheta)\Gamma(\gamma)}\int_0^{T}(T-\mathfrak{s})^{\gamma-1}g(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\nabla\mathfrak{s}.\end{aligned}$$

Subsequently, we obtain

$$\begin{aligned}\mathfrak{h}(\zeta)=&\int_0^{T}\mathbb{G}(\zeta,\mathfrak{s})\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\\&+\frac{1}{(\upsilon+\vartheta)\Gamma(\gamma)}\int_0^{T}(T-\mathfrak{s})^{\gamma-1}g(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\nabla\mathfrak{s}.\end{aligned}$$ □

Hence, the result follows.

Theorem 3.2. *Assume* (*A*1)–(*A*5) *hold and*

$$\mathcal{M}\mathcal{E}_{\mathcal{L}}+\frac{\mathcal{E}_g T^{\gamma}}{|\upsilon+\vartheta|\Gamma(\gamma+1)}<1, \tag{3.1}$$

Eq. (1.1) *contains unique solution.*

Proof. For $\varsigma=\frac{\mathfrak{k}_1}{1-\mathfrak{k}_1}$, we consider,

$$\mathfrak{B}=\{\mathfrak{h}\in\mathcal{C}(\mathcal{T},\mathfrak{R}):||\mathfrak{h}||_{c}\nabla\le\varsigma\}\subseteq\mathcal{C}(\mathcal{T},\mathfrak{R}).$$

Define $\Omega:\mathfrak{B}\to\mathfrak{B}$ as

$$\begin{aligned}(\Omega\mathfrak{h})\zeta=&\frac{1}{\Gamma(\gamma)}\int_0^{\zeta}(\zeta-s)^{\gamma-1}\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}\\&-\frac{\vartheta}{(\upsilon+\vartheta)\Gamma(\gamma)}\int_0^{T}(T-s)^{\gamma-1}\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}\\&+\frac{1}{(\upsilon+\vartheta)\Gamma(\gamma)}\int_0^{T}(T-\mathfrak{s})^{\gamma-1}g(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\nabla\mathfrak{s}.\end{aligned}\tag{3.2}$$

Here, Ω: $\mathfrak{B}\to\mathfrak{B}$ is well defined. Then, $\zeta\in\mathcal{T}$ and $\mathfrak{h}\in\mathfrak{B}$ gives

$$\begin{aligned}|(\Omega\mathfrak{h})(\zeta)|=&\left|\frac{1}{\Gamma(\gamma)}\int_0^{\zeta}(\zeta-\mathfrak{s})^{\gamma-1}\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}\right|\\&+\left|\frac{\vartheta}{(\upsilon+\vartheta)\Gamma(\gamma)}\int_0^{T}(T-\mathfrak{s})^{\gamma-1}\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}\right|\\&+\left|\frac{1}{(\upsilon+\vartheta)\Gamma(\gamma)}\int_0^{T}(T-\mathfrak{s})^{\gamma-1}g(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\nabla\mathfrak{s}\right|,\\\le&\frac{\mathcal{M}_{\mathcal{L}}(1+\varsigma)}{\Gamma(\gamma)}\int_0^{\zeta}(\zeta-\mathfrak{s})^{\gamma-1}\nabla\mathfrak{s}\\&+\left(\frac{\mathcal{M}_{\mathcal{L}}(1+\varsigma)|\vartheta|}{|(\upsilon+\vartheta)|\Gamma(\gamma)}+\frac{\mathcal{M}_g(1+\varsigma)}{|(\upsilon+\vartheta)|\Gamma(\gamma)}\right)\int_0^{T}(T-\mathfrak{s})^{\gamma-1}\nabla\mathfrak{s}.\end{aligned}$$

By using Lemma (2.8), we have

$$\begin{aligned}|(\Omega\mathfrak{h})\zeta|\le&\frac{\mathcal{M}_{\mathcal{L}}(1+\varsigma)}{\Gamma(\gamma)}\int_0^{\zeta}(\zeta-\mathfrak{s})^{\gamma-1}d\mathfrak{s}\\&+\left(\frac{\mathcal{M}_{\mathcal{L}}(1+\varsigma)|\vartheta|}{|(\upsilon+\vartheta)|\Gamma(\gamma)}+\frac{\mathcal{M}_g(1+\varsigma)}{|(\upsilon+\vartheta)|\Gamma(\gamma)}\right)\int_0^{T}(T-\mathfrak{s})^{\gamma-1}d\mathfrak{s},\\\le&\frac{\mathcal{M}_{\mathcal{L}}(1+\varsigma)T^{\gamma}}{\Gamma(\gamma+1)}+\left(\frac{\mathcal{M}_{\mathcal{L}}(1+\varsigma)|\vartheta|}{|(\upsilon+\vartheta)|\Gamma(\gamma+1)}+\frac{\mathcal{M}_g(1+\varsigma)}{|(\upsilon+\vartheta)|\Gamma(\gamma+1)}\right)T^{\gamma}.\end{aligned}\tag{3.3}$$

Hence,

$$||\Omega\mathfrak{h}||_c \leq \varsigma.$$

Therefore, $\Omega : \mathfrak{B} \to \mathfrak{B}$ is well defined. Also we show the operator $\Omega : \mathfrak{B} \to \mathfrak{B}$ is contractive and $\zeta \in \mathscr{T}$, we have

$$\begin{aligned}
&|(\Omega\mathfrak{h})(\zeta) - (\Omega\mathfrak{i})(\zeta)| \\
&\leq \frac{1}{\Gamma(\gamma)}\int_0^{\zeta}(\zeta-\mathfrak{s})^{\gamma-1}|\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s}) - \mathcal{L}(\mathfrak{s},\mathfrak{i}(\mathfrak{s}))\omega(\mathfrak{s})|\nabla\mathfrak{s}, \\
&+\frac{|\vartheta|}{(|\upsilon+\vartheta|)\Gamma(\gamma)}\int_0^{T}(T-\mathfrak{s})^{\gamma-1}|\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s}) \\
&\quad - \mathcal{L}(\mathfrak{s},\mathfrak{i}(\mathfrak{s}))\omega(\mathfrak{s})|\nabla\mathfrak{s}, \\
&\quad +\frac{1}{|(\upsilon+\vartheta)|\Gamma(\gamma)}\int_0^{T}(T-\mathfrak{s})^{\gamma-1}|g(\mathfrak{s},\mathfrak{h}(\mathfrak{s})) - g(\mathfrak{s},\mathfrak{i}(\mathfrak{s}))|\nabla\mathfrak{s}, \\
&\leq \frac{\mathcal{E}_{\mathcal{L}}}{\Gamma(\gamma)}\left(\frac{T^{\gamma}}{\gamma} + \frac{T^{\gamma}|\vartheta|}{\gamma|\upsilon+\vartheta|}\right)||\mathfrak{h}-\mathfrak{i}||_c + \frac{T^{\gamma}\mathcal{E}_g}{|\upsilon+\vartheta|\Gamma(\gamma+1)}|\mathfrak{h}-\mathfrak{i}||_c.
\end{aligned}$$

Hence,

$$||(\Omega\mathfrak{h}) - (\Omega\mathfrak{i})||_c \leq \mathcal{E}_{\mathcal{F}}||\mathfrak{h}-\mathfrak{i}||_c,$$

where

$$\mathcal{E}_{\mathcal{F}} = \frac{\mathcal{E}_{\mathcal{L}}T^{\gamma}}{\Gamma(\gamma+1)}\left(1+\frac{|\vartheta|}{|\upsilon+\vartheta|}\right) + \frac{\mathcal{E}_g T^{\gamma}}{|\upsilon+\vartheta|\Gamma(\gamma+1)}.$$

Therefore, Ω has a exact contraction mapping. Applying the Banach contraction theorem, Ω contains unique fixed point and is said to be the solution of Eq. (1.1). □

Theorem 3.3. *Assume* (*A*1) *and* (*A*2) *hold, Eq.* (1.1) *contains atleast one solution, with assumptions is satisfied with* $\mathcal{M}\mathcal{E}_{\mathcal{L}} < 1$.

Proof. To prove the result, we take two maps Ω_1 and Ω_2 such that

$$\begin{aligned}
(\Omega_1\mathfrak{h})(\zeta) &= \frac{1}{\Gamma(\gamma)}\int_0^{\zeta}(\zeta-\mathfrak{s})^{\gamma-1}\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s} \\
&+ \frac{\vartheta}{(\upsilon+\vartheta)\Gamma(\gamma)}\int_0^{T}(T-\mathfrak{s})^{\gamma-1}\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s},
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
(\Omega_2\mathfrak{h})(\zeta) &= \frac{1}{|\upsilon+\vartheta|\Gamma(\gamma)}\int_0^{T}(T-\mathfrak{s})^{\gamma-1}g(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s} \\
&- \frac{\vartheta}{(\upsilon+\vartheta)\Gamma(\gamma)}\int_0^{T}(T-\mathfrak{s})^{\gamma-1}g(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}.
\end{aligned}$$

Here $\Omega = \Omega_1 + \Omega_2$ and the following methods are proved. **Step 1:** Ω_1 is a contraction mapping. Since

$$||(\Omega_1\mathfrak{h})\zeta - (\Omega_1 i)\zeta||_c \leq \frac{\mathcal{E}_{\mathcal{L}}T^{\gamma}}{\Gamma(\gamma+1)}\left(1+\frac{|\vartheta|}{|\upsilon+\vartheta|}\right)||(\mathfrak{h}-\mathfrak{i})||_c.$$

Step 2: For each $h \in \mathfrak{B}$, we know $\Omega = \Omega_1 + \Omega_2$, where $\Omega : \mathfrak{B} \to \mathfrak{B}$. Then, we have $\gamma_1\mathfrak{h} + \gamma_2\mathfrak{h} \in \mathfrak{B}$.

Step 3: Define an operator $\Omega_2 : \mathfrak{B} \to \mathfrak{B}$ as

$$\begin{aligned}
(\Omega_2\mathfrak{h})(\zeta) &= \frac{1}{\Gamma(\gamma)}\int_0^{\zeta}(\zeta-\mathfrak{s})^{\gamma-1}\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s} \\
&- \frac{\vartheta}{(\upsilon+\vartheta)\Gamma(\gamma)}\int_0^{T}(T-\mathfrak{s})^{\gamma-1}\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s} \\
&+ \frac{1}{|\upsilon+\vartheta|\Gamma(\gamma)}\int_0^{T}(T-\mathfrak{s})^{\gamma-1}g(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}.
\end{aligned} \tag{3.5}$$

Here, Ω_2: $\mathfrak{B} \to \mathfrak{B}$ is well defined. Then, for $\zeta \in \mathscr{T}$ and $\mathfrak{h} \in \mathfrak{B}$,

$$\begin{aligned}
|(\Omega_2\mathfrak{h})(\zeta)| &= \left|\frac{1}{\Gamma(\gamma)}\int_0^{\zeta}(\zeta-\mathfrak{s})^{\gamma-1}\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}\right| \\
&+ \left|\frac{\vartheta}{(\upsilon+\vartheta)\Gamma(\gamma)}\int_0^{T}(T-\mathfrak{s})^{\gamma-1}\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}\right| \\
&+ \left|\frac{1}{|\upsilon+\vartheta|\Gamma(\gamma)}\int_0^{T}(T-\mathfrak{s})^{\gamma-1}g(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}\right|, \\
&\leq \frac{\mathcal{M}_{\mathcal{L}}(1+\varsigma)}{\Gamma(\gamma)}\int_0^{\zeta}(\zeta-\mathfrak{s})^{\gamma-1}\nabla\mathfrak{s} \\
&+ \left(\frac{\mathcal{M}_{\mathcal{L}}(1+\varsigma)|\vartheta|}{|(\upsilon+\vartheta)|\Gamma(\gamma)} + \frac{\mathcal{M}_g(1+\varsigma)}{|(\upsilon+\vartheta)|\Gamma(\gamma)}\right)\int_0^{T}(T-\mathfrak{s})^{\gamma-1}\nabla\mathfrak{s}.
\end{aligned}$$

By using Lemma (2.8), we obtain

$$\begin{aligned}
|(\Omega_2\mathfrak{h})(\zeta)| &\leq \frac{\mathcal{M}_{\mathcal{L}}(1+\varsigma)}{\Gamma(\gamma)}\int_0^{\zeta}(\zeta-\mathfrak{s})^{\gamma-1}d\mathfrak{s} \\
&+ \left(\frac{\mathcal{M}_{\mathcal{L}}(1+\varsigma)|\vartheta|}{|(\upsilon+\vartheta)|\Gamma(\gamma)} + \frac{\mathcal{M}_g(1+\varsigma)}{|(\upsilon+\vartheta)|\Gamma(\gamma)}\right) \\
&\times \int_0^{T}(T-\mathfrak{s})^{\gamma-1}d\mathfrak{s}, \\
&\leq \frac{\mathcal{M}_{\mathcal{L}}(1+\varsigma)T^{\gamma}}{\Gamma(\gamma+1)} + \left(\frac{\mathcal{M}_{\mathcal{L}}(1+\varsigma)|\vartheta|}{|(\upsilon+\vartheta)|\Gamma(\gamma+1)}\right. \\
&\left.+ \frac{\mathcal{M}_g(1+\varsigma)}{|(\upsilon+\vartheta)|\Gamma(\gamma+1)}\right)T^{\gamma}.
\end{aligned} \tag{3.6}$$

Hence,

$$||\Omega_2\mathfrak{h}||_c \le \varsigma.$$

Step 4: To prove the operator Ω is continuous, consider $\mathfrak{h}_n$ is a sequence such that, $\mathfrak{h}_n \to \mathfrak{h}_n$ in $C(\mathscr{T}, \mathfrak{R})$, for any $\zeta \in \mathscr{T}$, we have

$$\begin{aligned}
&|(\Omega_2\mathfrak{h}_n)(\zeta) - (\Omega_2\mathfrak{h})(\zeta)|\\
&\le \frac{1}{\Gamma(\gamma)}\int_0^\zeta(\zeta-\mathfrak{s})^{\gamma-1}|\mathcal{L}(\mathfrak{s},\mathfrak{h}_n(\mathfrak{s}))\omega(\mathfrak{s}) - \mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})|\nabla\mathfrak{s}\\
&\quad + \frac{|\vartheta|}{(|\upsilon+\vartheta|)\Gamma(\gamma)}\int_0^T(T-\mathfrak{s})^{\gamma-1}|\mathcal{L}(\mathfrak{s},\mathfrak{h}_n(\mathfrak{s}))\omega(\mathfrak{s})\\
&- \mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})|\nabla\mathfrak{s}\\
&\quad + \frac{1}{|(\upsilon+\vartheta)|\Gamma(\gamma)}\int_0^T(T-\mathfrak{s})^{\gamma-1}|g(\mathfrak{s},h_n(\mathfrak{s})) - g(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))|\nabla\mathfrak{s}.
\end{aligned}$$

Since functions $\mathcal{L}$ and g are continuous with respect to $\mathfrak{h}$, then we have $||(\Omega_2\mathfrak{h}_n) - (\Omega_2\mathfrak{h})||_c \to 0$ as $\mathfrak{n} \to 0$.

Therefore, Ω_2 is continuous.

Step 5: Let $\vartheta_1, \vartheta_2 \in \mathscr{T}$ such that $\vartheta_1 < \vartheta_2$, then we have

$$\begin{aligned}
&|(\Omega_2\mathfrak{h})(\vartheta_2) - (\Omega_2\mathfrak{h})(\vartheta_1)|\\
&\le \left|\frac{1}{\Gamma(\gamma)}\int_0^{\vartheta_1}((\vartheta_2-\mathfrak{s})^{\gamma-1} - (\vartheta_1-\mathfrak{s})^{\gamma-1})\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}\right|\\
&\quad + \left|\frac{1}{\Gamma(\gamma)}\int_0^{\vartheta_1}((\vartheta_2-\mathfrak{s})^{\gamma-1} - (\vartheta_1-\mathfrak{s})^{\gamma-1})\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}\right|,\\
&\le \frac{\mathcal{M}_{\mathcal{L}}}{\Gamma(\gamma)}\int_0^{\vartheta_1}((\vartheta_2-\mathfrak{s})^{\gamma-1} - (\vartheta_1-\mathfrak{s})^{\gamma-1})\nabla\mathfrak{s}\\
&\quad + \frac{\mathcal{M}_{\mathcal{L}}}{\Gamma(\gamma)}\int_0^{\vartheta_1}((\vartheta_2-\mathfrak{s})^{\gamma-1}\nabla\mathfrak{s}.
\end{aligned}$$

As $(\varphi - \mathfrak{s})^{\gamma-1}$ is continuous, then $|(\Omega_2\mathfrak{h})(\zeta_2) - (\Omega_2\mathfrak{h})(\zeta_1)| \to 0$ when $\vartheta_1 \to \vartheta_2$. The proof is same for $\zeta \le \varphi < T$. Then, the operator Ω_2 is equicontinuous. From the followed steps and by Arzela–Ascoli theorem, we find $\Omega_2(\mathfrak{B})$ is compact. And from the above steps, we find the Krasnoselskii's fixed point theorem holds and Eq. (1.1) contains atleast one solution in $\mathfrak{B}$. □

Theorem 3.4. *Consider* (*A*1)–(*A*5) *and inequality* (2.1) *hold. Eq.* (1.1) *is UH stable.*

Proof. Let $\mathfrak{h}$ be a unique solution of Eq. (1.1) and $\mathfrak{i}$ be the solution of the inequality

$$|\mathcal{D}^\gamma\mathfrak{i}(\zeta) - \mathcal{L}(\zeta, \mathfrak{i}(\zeta)\omega(\zeta))| \le \xi, \quad \zeta \in \mathscr{T},$$

then by Theorem (1.1), we have

$$\begin{aligned}
(\mathfrak{h})(\zeta) &= \frac{1}{\Gamma(\gamma)}\int_0^\zeta(\zeta-\mathfrak{s})^{\gamma-1}\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}\\
&\quad - \frac{\vartheta}{(\upsilon+\vartheta)\Gamma(\gamma)}\int_0^T(T-\mathfrak{s})^{\gamma-1}\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}\\
&\quad + \frac{1}{|\upsilon+\vartheta|\Gamma(\gamma)}\int_0^T(T-\mathfrak{s})^{\gamma-1}g(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}.
\end{aligned}$$

Then

$$\begin{aligned}
|(\mathfrak{i})(\zeta) - (\mathfrak{h})(\zeta)| &= \Bigg|\mathfrak{i}(\zeta) - \frac{1}{\Gamma(\gamma)}\int_0^\zeta(\zeta-\mathfrak{s})^{\gamma-1}\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}\\
&\quad - \frac{\vartheta}{(\upsilon+\vartheta)\Gamma(\gamma)}\int_0^T(T-\mathfrak{s})^{\gamma-1}\mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\omega(\mathfrak{s})\nabla\mathfrak{s}\\
&\quad + \frac{1}{|\upsilon+\vartheta|\Gamma(\gamma)}\int_0^T(T-\mathfrak{s})^{\gamma-1}g(\mathfrak{s},\mathfrak{h}(\mathfrak{s}))\nabla\mathfrak{s}\Bigg|,\\
&\le \xi\mathcal{M} + \Bigg|\frac{1}{\Gamma(\gamma)}\int_0^\zeta(\zeta-\mathfrak{s})^{\gamma-1}(\mathcal{L}(\mathfrak{s},\mathfrak{i}(\mathfrak{s}))\omega(\mathfrak{s})\\
&\quad - \mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s})))\omega(\mathfrak{s})\nabla\mathfrak{s}\Bigg|\\
&\quad + \Bigg|\frac{\vartheta}{(\upsilon+\vartheta)\Gamma(\gamma)}\int_0^T(T-\mathfrak{s})^{\gamma-1}(\mathcal{L}(\mathfrak{s},\mathfrak{i}(\mathfrak{s}))\omega(\mathfrak{s})\\
&\quad - \mathcal{L}(\mathfrak{s},\mathfrak{h}(\mathfrak{s})))\omega(\mathfrak{s})\nabla\mathfrak{s}\Bigg|\\
&\quad + \frac{1}{|\upsilon+\vartheta|\Gamma(\gamma)}\int_0^T(T-\mathfrak{s})^{\gamma-1}(g(\mathfrak{s},\mathfrak{i}(\mathfrak{s}))\\
&\quad - g(\mathfrak{s},\mathfrak{h}(\mathfrak{s})))\nabla\mathfrak{s}\Bigg|.
\end{aligned}$$

Hence,

$$||\mathfrak{i} - \mathfrak{h}||_c \le \xi\mathcal{M} + \frac{T^\gamma}{\Gamma(\gamma+1)}\left[\mathcal{E}_{\mathcal{L}} + \frac{\mathcal{E}_{\mathcal{L}}|\vartheta|}{|\upsilon+\vartheta|} + \frac{\mathcal{L}_g}{|\upsilon+\vartheta|}\right]||\mathfrak{i} - \mathfrak{h}||_c,$$

$$||\mathfrak{i} - \mathfrak{h}||_c \le \frac{\xi\mathcal{M}}{1-\xi_{\mathcal{F}}}.$$

Thus,

$$||\mathfrak{i} - \mathfrak{h}||_c \le \mathbb{H}_{(\mathcal{E}_{\mathcal{L}},\mathcal{E}_g,\gamma)}\xi,$$

where

$$\mathbb{H}_{(\mathcal{E}_{\mathcal{L}},\mathcal{E}_g,\gamma)} = \frac{\mathcal{M}}{1-\xi_{\mathcal{F}}}.$$

Therefore, Eq. (1.1) is UH stable.

Setting

$$\mathbb{H}_{(\mathcal{E}_{\mathcal{L}},\mathcal{E}_g,\gamma)}(\xi) = \mathbb{H}_{(\mathcal{E}_{\mathcal{L}},\mathcal{E}_g,\gamma)}\xi,$$
$$\mathbb{H}_{(\mathcal{E}_{\mathcal{L}},\mathcal{E}_g,\gamma)}(0) = 0.$$

Therefore, Eq. (1.1) is generalized UH stable. Based on assumptions (A1)–(A5) and the validity of inequality (2.1), the unique solution $\mathfrak{h}$ of Eq. (1.1) satisfies the UH stability criteria. Specifically, for any solution $\mathfrak{i}$ that approximately satisfies Eq. (1.1) within a bounded error ξ, the difference between $\mathfrak{i}$ and $\mathfrak{h}$ is bounded by a linear function of ξ. Moreover, the solution $\mathfrak{h}$ exhibits generalized UH stability, where the bound on $||\mathfrak{i} - \mathfrak{h}||$ vanishes as $\xi \to 0$. Thus, Eq. (1.1) is both UH stable and generalized UH stable under the given conditions. □

4 Nonlinear fractional dynamic equations (NFDE) with nonlocal condition

The research of nonlocal problems is driven by physical issues. As an illustration, it is employed to ascertain certain unidentified physical factors involved in inverse heat conduction problems. Byszewski first formulated and demonstrated the outcome regarding the presence and singularity of solutions to abstract Cauchy problems with nonlocal initial conditions. Many studies have discussed the subject of existence and uniqueness results in different kinds of nonlinear DEs. Initially, Byzewski formulated the nonlocal condition, where he mentioned [48], theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. Chang and Li [49] discussed the existence results for impulsive dynamic equations on time scales with nonlocal initial conditions. Subashini et al. [50] discussed the new results on nonlocal functional integro-DEs via Hilfer fractional derivative. Gogoi et al. [51] discussed the impulsive fractional dynamic equation with nonlocal initial condition on time scales. Inspired by the above work, we discuss the results for NFDE with nonlocal condition.

$$\mathfrak{h}(0) + g(\mathfrak{h}) = \mathfrak{h}_o, \tag{4.1}$$

where $g : \mathscr{T} \times \mathbb{T} \to \mathbb{T}$, which satisfies the following assumption.

(A6) There appears a constant $\mathcal{P} > 0$ such that

$$||g(\mathfrak{h}) - g(\mathfrak{i})|| \le \mathcal{P}||\mathfrak{h} - \mathfrak{i}||, \quad \forall \mathfrak{h}, \mathfrak{i} \in ([0, T], \mathfrak{R}).$$

(A7) There exists a constant $\mathcal{K} > 0$ such that for all $\mathfrak{h}, \mathfrak{i} \in ([0, T], \mathfrak{R})$.

$$||\mathfrak{g}(\mathfrak{h})|| \le \mathcal{K},$$

ensuring that the nonlocal term is bounded and does not introduce instability into the solution.

Using the nonlocal condition in physics yields a more advantageous outcome compared to the classical initial condition $(\mathfrak{h}) = \mathfrak{h}_o$. For example, $g(\mathfrak{h})$ can be written as

$$g(\mathfrak{h}) = \sum_{\mathfrak{n}=1}^{\mathfrak{m}} \mathfrak{c}_\mathfrak{n}\mathfrak{h}(\mathfrak{t}_\mathfrak{n}),$$

where $\mathfrak{c}_\mathfrak{n}(\mathfrak{n} = 1, 2, \ldots \mathfrak{n})$ are mentioned constants and $0 < \mathfrak{t}_1 < \ldots < \mathfrak{t}_\mathfrak{n} \le T$. More than the initial condition, the nonlocal condition can be more useful.

Theorem 4.1. *Assume* $0 < \gamma \le 1$, *for* $\mathfrak{h} \in \mathfrak{X} \cap \mathcal{L}_\nabla(\mathscr{T}, \mathfrak{R})$ *as a solution of Eq.* (1.1), $\mathfrak{h}$ *is also the solution of the integral equation, and applying the nonlocal condition where the assumption* (A1)–(A7) *holds*

$$\mathfrak{h}(\zeta) = [\mathfrak{h}_o - g(\mathfrak{h})] + \int_{\zeta_o}^{\zeta} \mathfrak{h}_{\gamma-1}(\zeta, \rho(\varphi))\mathcal{L}(\varphi, \mathfrak{h}(\varphi, \mathcal{D}^\gamma)\mathfrak{h}(\varphi))\nabla\varphi.$$

Proof. Assume $\mathcal{D}^\gamma\mathfrak{h}(\zeta) = g(\zeta)$, then using Eq. (1.1) in the above equation, we obtain

$$\mathfrak{h}(\zeta) = [\mathfrak{h}_o - g(\mathfrak{h})] + \int_{\zeta_o}^{\zeta} \mathfrak{h}_{\gamma-1}(\zeta, \rho(\varphi))g(\varphi)\nabla\varphi.$$

By using the technique used in theorem 3.2, one can show that Ω has a fixed point. Then, one can show that fractional dynamic Eq. (1.1) has fixed point by employing the technique applied in Theorems 3.2 and 3.3. The proof is same as Theorems 3.2 and 3.3 and hence is ommitted. □

5 Example

The following example serves as an illustration for our theoretical findings. To obtain the nonlinear problem (1.1) numerical solution, we will also provide the path of the proposed numerical scheme.

Example 5.1. Assume a fractional dynamic equation having the nonlinear integral boundary condition on time scale $T = [0, 1]_\mathbb{T}$ such that

$$\begin{cases} \mathcal{D}^{\mathfrak{h}(\zeta)} = \dfrac{\mathfrak{e}^{-3\zeta}}{10} + \dfrac{\mathfrak{h}(\zeta) + \mathcal{D}^{\mathfrak{h}(\zeta)}}{6(1 + \mathfrak{h}(\zeta) + \mathcal{D}^{\mathfrak{h}(\zeta)})}, \\ \mathfrak{h}(0) + \mathfrak{h}(T) = \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_0^1 \dfrac{3(1 - s)^{\gamma-1}\mathfrak{e}^{-2\mathfrak{s}}|\mathfrak{h}|}{2 + \mathfrak{e}^{\mathfrak{s}}(1 + |\mathfrak{h}|)}\nabla\mathfrak{s}, \end{cases} \tag{5.1}$$

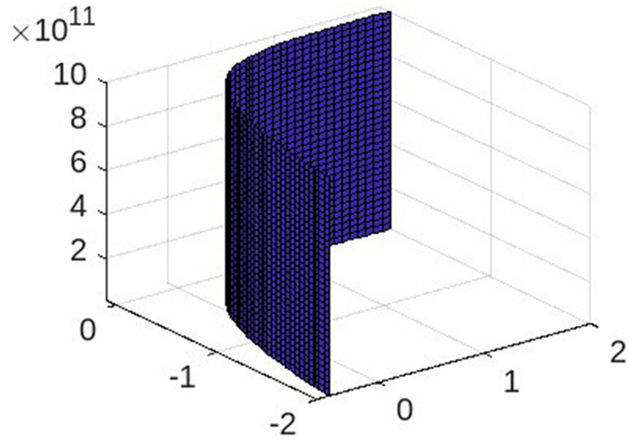

Figure 1: Graph of the approximate solution of $\mathfrak{h}(\zeta)$.

where $\zeta \in [0,1] \cap T_{\mathfrak{k}}$. Now for $\varphi_1, \varphi_2 \in \Re$, we set

$$\mathcal{L}(\zeta, \varphi_1, \varphi_2) = \frac{\mathfrak{e}^{-3\zeta}}{10} + \frac{\varphi_1 + \varphi_2}{6(1 + \varphi_1 + \varphi_2)},$$

satisfying the condition $|\mathcal{L}(\zeta, \mathfrak{h})| \le \mathcal{M}_{\mathcal{L}}(1 + |\mathfrak{h}|)$.

$$\therefore \mathcal{L}(\zeta, \varphi_1, \varphi_2) < 1.$$

Therefore, Eq. (5.1) has a solution at the interval $[0,1] \cap T_{\mathfrak{k}}$. Figure 1 reveals a commendable correspondence between the numerical solution and the exact solution across entire interval.

6 Conclusion

This study explores nonlinear fractional dynamic systems governed by RL nabla FDEs with integral boundary conditions. We established the existence, uniqueness, and stability of solutions using Krasnoselskii's fixed point theorem and the Banach contraction principle, enhancing the applicability of our results to real-world systems. Our MATLAB-supported example highlights the practical implications of these findings.

These contributions deepen the understanding of fractional dynamic systems, offering insights for mathematicians and engineers in complex modeling. Future research will focus on advanced numerical methods, innovative control strategies, and applications in emerging technologies, laying a strong foundation for further advancements in fractional calculus.

Acknowledgments: The authors extend their appreciation to Prince Sattam bin Abdulaziz University, Saudi Arabia for funding this research work through the project number (PSAU/2025/R/1446).

Funding information: Authors state no funding involved.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: Authors state no conflict of interest.

Data availability statement: All data generated or analysed during this study are included in this published article.

References

[1] Hioual A, Ouannas A, Grassi G, Oussaeif TE. Nonlinear nabla variable-order fractional discrete systems: Asymptotic stability and application to neural networks. J Comput Appl Math. 2023;423:114939.

[2] Jothimani K, Ravichandran C, Kumar V, Djemai M, Nisar KS. Interpretation of trajectory control and optimization for the nondense fractional system. Int J Appl Comput Math. 2022;8(6):273.

[3] Mohammed PO, Baleanu D, Abdeljawad T, Al-Sarairah E, Hamed YS. Monotonicity and extremality analysis of difference operators in Riemann–Liouville family. AIMS Math. 2023;8;5303–17.

[4] Ravichandran C, Sowbakiya V, Nisar KS. Study on existence and data dependence results for fractional order differential equations. Chaos Solitons Fractals. 2022;160:112232.

[5] Paul SK, Mishra LN, Mishra VN, Baleanu D. An effective method for solving nonlinear integral equations involving the Riemann–Liouville fractional operator. AIMS Math. 2023;8:17448–69.

[6] Nisar KS, Anusha C, Ravichandran C. A non-linear fractional neutral dynamic equations: existence and stability results on time scales. AIMS Math. 2024;9(1):1911–25.

[7] Kaliraj K. An investigation of fractional mixed functional integro-differential equations with impulsive conditions. Discontinuity Nonlinearity Complexity. 2024;13(01):189–202.

[8] Munusamy K, Ravichandran C, Nisar KS, Munjam SR. Investigation on continuous dependence and regularity solutions of functional integrodifferential equations. Results Control Optim. 2024;14:100376.

[9] Zhu J, Wu L. Fractional Cauchy problem with Caputo nabla derivative on time scales. In: Abstract and applied analysis. Vol. 2015. No. 1. Egypt: Hindawi Publishing Corporation; 2015. p. 486054.

[10] Benchohra M, Ouaar F. Existence results for nonlinear fractional differential equations with integral boundary conditions. Bull Math Anal Appl. 2010;2(4):7–15.

[11] Mohiuddine SA, Das A, Alotaibi A. Existence of solutions for nonlinear integral equations in tempered sequence spaces via generalized Darbo-type theorem. J Funct Spaces. 2022;2022(1):4527439.

[12] Veeresha P, Prakasha DG, Ravichandran C, Akinyemi L, Nisar KS. Numerical approach to generalized coupled fractional Ramani equations. Int J Modern Phys B. 2022;36(05):2250047.

[13] Wei Y, Zhao X, Wei Y, Chen Y. Lyapunov stability analysis for incommensurate nabla fractional order systems. J Syst Sci Complexity. 2023;36(2):555–76.

[14] Kumar V, Djemai M, Defoort M, Malik M. Total controllability results for a class of time-varying switched dynamical systems with impulses on time scales. Asian J Control. 2022;24(1):474–82.
[15] Kumar V, Malik M. Controllability results of fractional integro-differential equation with non-instantaneous impulses on time scales. IMA J Math Control Inform. 2021;38(1):211–31.
[16] Kumar V, Malik M. Existence and stability results of nonlinear fractional differential equations with nonlinear integral boundary condition on time scales. Appl Appl Math Int J (AAM). 2020;15(3):10.
[17] Kumar V, Malik M. Existence stability and controllability results of fractional dynamic system on time scales with application to population dynamics. Int J Nonl Sci Numer Simulat. 2021;22(6):741–66.
[18] Agarwal RP, O'Regan D. Nonlinear boundary value problems on time scales. Nonl Anal Theory Methods Appl. 2001;44(4):527–35.
[19] Bendouma B, Cherif AB, Hammoudi A. Existence of solutions for nonlocal nabla conformable fractional thermistor problem on time scales. Mem Differ Equ Math Phys. 2023;88:73–87.
[20] Benkhettou N, Hammoudi A, Torres DF. Existence and uniqueness of solution for a fractional Riemann–Liouville initial value problem on time scales. J King Saud Univ-Sci. 2016;28(1):87–92.
[21] Kaliraj K, Priya PL, Ravichandran C. An explication of finite-time stability for fractional delay model with neutral impulsive conditions. Qualitative Theory Dynam Syst. 2022;21(4):161.
[22] Agarwal R, Bohner M, O'Regan D, Peterson A. Dynamic equations on time scales: a survey. J Comput Appl Math. 2002;141(1–2):1–26.
[23] Mahdi NK, Khudair AR. Stability of nonlinear q-fractional dynamical systems on time scale. Partial Differ Equ Appl Math. 2023;7:100496.
[24] Bohner M, Peterson A. Dynamic equations on time scales: An introduction with applications. Boston: Springer Science & Business Media; 2001.
[25] Bohner M, Peterson AC. Advances in dynamic equations on time scales. Birkäuser Boston. Inc ., Boston, MA. 2003.
[26] Sarkar N, Sen M, Saha D, Hazarika B. A qualitative study on fractional logistic integro-differential equations in an arbitrary time scale. Kragujevac J Math. 2026;50(3):403–14.
[27] Tian Z. Analysis and research on chaotic dynamics behaviour of wind power time series at different time scales. J Ambient Intel Human Comput. 2023;14(2):897–921.
[28] Ilhan E, Veeresha P, Baskonus HM. Fractional approach for a mathematical model of atmospheric dynamics of CO2 gas with an efficient method. Chaos Solitons Fractals. 2021;152:111347.
[29] Veeresha P. The efficient fractional order based approach to analyze chemical reaction associated with pattern formation. Chaos Solitons Fractals. 2022;165:112862.
[30] Sherly K, Veeresha P. Mathematical model for effective CO_2 emission control with forest biomass using fractional operator. Model Earth Syst Environ. 2024;10(4):5469–88.
[31] Baishya C, Achar SJ, Veeresha P, Kumar D. Dynamical analysis of fractional yellow fever virus model with efficient numerical approach. J Comput Anal Appl. 2023;31(1):140–57.
[32] Boutiara A, Abdo MS, Benbachir M. Existence results for ψ-Caputo fractional neutral functional integro-differential equations with finite delay. Turkish J Math. 2020;44(6):2380–401.
[33] Wahash HA, Abdo M, Panchal SK. Existence and stability of a nonlinear fractional differential equation involving a ψ-Caputo operator. Adv Theory Nonl Anal Appl. 2020;4(4):266–78.
[34] Derbazi C, Baitiche Z, Abdo MS, Abdeljawad T. Qualitative analysis of fractional relaxation equation and coupled system with ψ-Caputo fractional derivative in Banach spaces. AIMS Math. 2021;6(3):2486–509.
[35] Goufo EF, Ravichandran C, Birajdar GA. Self-similarity techniques for chaotic attractors with many scrolls using step series switching. Math Model Anal. 2021;26(4):591–611.
[36] Nisar KS, Jagatheeshwari R, Ravichandran C, Veeresha P. High performance computational method for fractional model of solid tumour invasion. Ain Shams Eng J. 2023;14(12):102226.
[37] Ma YK, Dineshkumar C, Vijayakumar V, Udhayakumar R, Shukla A, Nisar KS. Approximate controllability of Atangana-Baleanu fractional neutral delay integrodifferential stochastic systems with nonlocal conditions. Ain Shams Eng J. 2023;14(3):101882.
[38] Ma YK, Dineshkumar C, Vijayakumar V, Udhayakumar R, Shukla A, Nisar KS. Hilfer fractional neutral stochastic Sobolev-type evolution hemivariational inequality: Existence and controllability. Ain Shams Eng J. 2023;14(9):102126.
[39] Gogoi B, Saha UK, Hazarika B. Existence of solution of a nonlinear fractional dynamic equation with initial and boundary conditions on time scales. J Anal. 2024;32(1):85–102.
[40] Gogoi B, Saha UK, Hazarika B, Torres DF, Ahmad H. Nabla fractional derivative and fractional integral on time scales. Axioms. 2021;10(4):317.
[41] Guseinov GS. Integration on time scales. J Math Anal Appl. 2003;285(1):107–27.
[42] Anastassiou GA. Foundations of nabla fractional calculus on time scales and inequalities. Comput Math Appl. 2010;59(12):3750–62.
[43] Tikare S, Tisdell CC. Nonlinear dynamic equations on time scales with impulses and nonlocal conditions. J Class Anal. 2020;16(2):125–40.
[44] Feng M, Zhang X, Li X, Ge W. Necessary and sufficient conditions for the existence of positive solution for singular boundary value problems on time scales. Adv Differ Equ. 2009;2009:1–4.
[45] Sahir MJ. Coordination of classical and dynamic inequalities complying on time scales. Eur J Math Anal. 2023;3:12.
[46] Morsy A, Nisar KS, Ravichandran C, Anusha C. Sequential fractional order neutral functional Integro differential equations on time scales with Caputo fractional operator over Banach spaces. AIMS Math. 2023;8(3):5934–49.
[47] Wang J, Fec M, Zhou Y. Ulam's type stability of impulsive ordinary differential equations. J Math Anal Appl. 2012;395(1):258–64.
[48] Byszewski L. Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. J Math Anal Appl. 1991;162(2):494–505.
[49] Chang YK, Li WT. Existence results for impulsive dynamic equations on time scales with nonlocal initial conditions. Math Comput Model. 2006;43(3–4):377–84.
[50] Subashini R, Jothimani K, Nisar KS, Ravichandran C. New results on nonlocal functional integro-differential equations via Hilfer fractional derivative. Alexandr Eng J. 2020;59(5):2891–9.
[51] Gogoi B, Hazarika B, Saha UK. Impulsive fractional dynamic equation with non-local initial condition on time scales. 2022 Jun 29. arXiv: http://arXiv.org/abs/arXiv:2207.01517.