

Research Article

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New kink-periodic and convex-concave-periodic solutions to the modified regularized long wave equation by means of modified rational trigonometric-hyperbolic functions

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Abstract: The significance of different types of periodic solutions in nonlinear equations is vital across various practical applications. Our objective in this study was to uncover novel forms of periodic solutions for the modified regularized long wave equation. This particular model holds great importance in the realm of physics as it characterizes the propagation of weak nonlinearity and space-time dispersion waves, encompassing phenomena like nonlinear transverse waves in shallow water, ion-acoustic waves in plasma, and phonon waves in nonlinear crystals. By employing the methodology of modified rational sine-cosine and sinh-cosh functions, we successfully derived new kink-periodic and convex-concave-periodic solutions. To showcase the superiority of our proposed approach, we conducted a comparative analysis with the alternative Kudryashov-expansion technique. Furthermore, we visually depicted the diverse recovery solutions through 2D and 3D plots to enhance the understanding of our findings.

Keywords: nonlinear equations, periodic solutions, time-space dispersion waves, trial function methods

1 Introduction

Nonlinear partial differential equations (NLPDEs) are commonly utilized as fundamental mathematical equations to model various physical phenomena in numerous fields of engineering, science, and physics. Explicit solutions to NLPDEs are highly beneficial for visualizing the dynamics of a wide range of applications and phenomena. Obtaining solutions with diverse physical structures for a particular model can lead to a better comprehension of the underlying mechanisms and processes of its dynamic system and help in its development and maintenance.

NLPDEs have solutions known as propagation wave-solutions, which encompass a diverse range of types such as soliton, kink, cusp, periodic, breather, lump, rogue, and more. No single method generates all these types of solutions simultaneously, and each approach has its own specific construction to generate a few types. In recent decades, many approaches have been developed to extract various types of solitary wave solutions, enriching the field of solitons. Some of the updated approaches and recent techniques include the Hirota bilinear method with Cole-Hopf transformations, which has offered new types of solitons such as multi-solitons, rogue, lump, and breather waves that have significant applications in water waves and optical pulses [1–4]. Other suggested scheme-solutions involve exponential, trigonometric, or hyperbolic functions, such as the simplified bilinear method [5,6], polynomial function method [7,8], modified (G'/G) -expansion [9], modified Kudryashov-expansion [10], Lie-symmetry [11], generalized exponential rational function method [12], and numerous other methods [13–18].

The main goal of this work is to explore periodic solutions, never been reported earlier, for the modified regularized long wave (MRLW) equation, which reads

$$\Omega_t + \Omega_x + \alpha \Omega^2 \Omega_x + \beta \Omega_{xxx} = 0, \quad \Omega = \Omega(x, t). \quad (1)$$

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The presence of different types of periodic solutions to nonlinear equations plays a crucial role in various real-life applications. For instance, in the field control systems and electrical engineering, periodic solutions provide valuable insights into the stability of a system. By studying the stability properties of periodic solutions, engineers and scientists can assess the behavior of a system over time and make predictions about its long-term stability. In nonlinear dynamics, different applications in nature and engineering are represented by nonlinear systems. Understanding the different types of periodic solutions helps in studying and predicting complex behaviors in these systems. Nonlinear dynamics provide insights into phenomena such as chaos, bifurcations, and attractors, which have applications in fields like weather prediction, population dynamics, and neural networks. Moreover, periodic solutions are also valuable in energy harvesting applications. Many natural and engineered systems exhibit periodic behavior that can be harnessed to generate electricity. Overall, the presence of different types of periodic solutions in nonlinear equations enhances our understanding of complex systems and provides practical benefits in various fields, ranging from engineering and physics to biology and environmental sciences.

In Eq. (1), the parameters α, β refer to the nonlinearity and time-space dispersion, respectively. The MRLW equation is very important application in the media of physics since it describes the propagation of weak nonlinearity and space-time dispersion waves, including nonlinear transverse waves that arise in shallow water, ion-acoustic waves in plasma, and phonon waves in nonlinear crystals.

The MRLW has been investigated in a few occasions, for example, the B-spline collocation numerical scheme is used to find approximate solutions for specific values of the nonlinearity and dispersion coefficients [19]. Also, by using the Fourier spectral method, bell-shaped solution is obtained [20]. By implementing of the cosine-function algorithm, the secant solution is obtained for the case of $\alpha = 1$ and $\beta = -1$ [21]. Moreover, via the conservation laws, the sech-solution is reported for the MRLW [22]. Finally, numerical solutions of the MRLW equation are obtained by means of quintic B-splines, quartic B-splines, septic B-spline collocation, and Petrov Galerkin finite element method [23–25].

To the best of our knowledge, the contributions to the MRLW equation are limited. It is manifested from the literature point of view that there are some scopes for further investigations on the MRLW equation to explore new periodic solutions via the modified rational sine-cosine/sinh-cosh function method and the Kudryashov-expansion method as well as to draw their physical clarifications.

2 Explicit solutions to the MRLW model

To recognize travelling wave solutions to a NLPDE, we use linear transformation to reduce it into simplified ordinary differential equation. In particular, we consider the new independent variable $z = x - ct$ to reduce (1) to the following differential equation:

$$(1 - c)U(z) + \frac{\alpha}{3}U^3(z) - \beta cU''(z) = 0, \quad (2)$$

where $U(z) = \Omega(x, t)$. Next, we solve (2) by implementing two recent effective approaches: the Kudryashov's method [26,27] and the modified rational trigonometric and hyperbolic functions schemes [28–30].

2.1 Approach I: Kudryashov-expansion

The Kudryashov solution of (2) is of the following form:

$$U(z) = A + BY, \quad (3)$$

where $Y = Y(z) = \frac{1}{1 + \delta e^{\mu z}}$ satisfies $Y' = Y'(z) = \mu Y(Y - 1)$. By plugging (3) into (2) and collecting the coefficients of Y^i , we obtain the nonlinear system:

$$\begin{aligned} 0 &= \alpha A^3 - 3Ac + 3A, \\ 0 &= 3\alpha A^2 B - 3\beta Bc\mu^2 - 3Bc + 3B, \\ 0 &= 3\alpha AB^2 + 9\beta Bc\mu^2, \\ 0 &= \alpha B^3 - 6\beta Bc\mu^2. \end{aligned} \quad (4)$$

By solving the aforementioned four equations, we reach at the following findings:

$$\begin{aligned} A &= \mp \frac{\sqrt{3}\sqrt{\beta}\mu}{\sqrt{-\alpha(\beta\mu^2 - 2)}}, \\ B &= \pm \frac{2\sqrt{3}\sqrt{\beta}\mu}{\sqrt{-\alpha(\beta\mu^2 - 2)}}, \\ c &= -\frac{2}{\beta\mu^2 - 2}. \end{aligned} \quad (5)$$

Accordingly, the Kudryashov solution of MRLW is

$$\Omega(x, t) = \mp \frac{\sqrt{3}\sqrt{\beta}\mu \left(\delta e^{\mu \left(\frac{-2t}{\beta\mu^2 - 2} + x \right)} - 1 \right)}{\sqrt{-\alpha(\beta\mu^2 - 2)} \left(\delta e^{\mu \left(\frac{-2t}{\beta\mu^2 - 2} + x \right)} + 1 \right)}. \quad (6)$$

The parameter δ that appear in (6) is known as the Kudryashov index, which determines whether the wave is singular ($\delta < 0$) or nonsingular ($\delta > 0$), and Figure 1

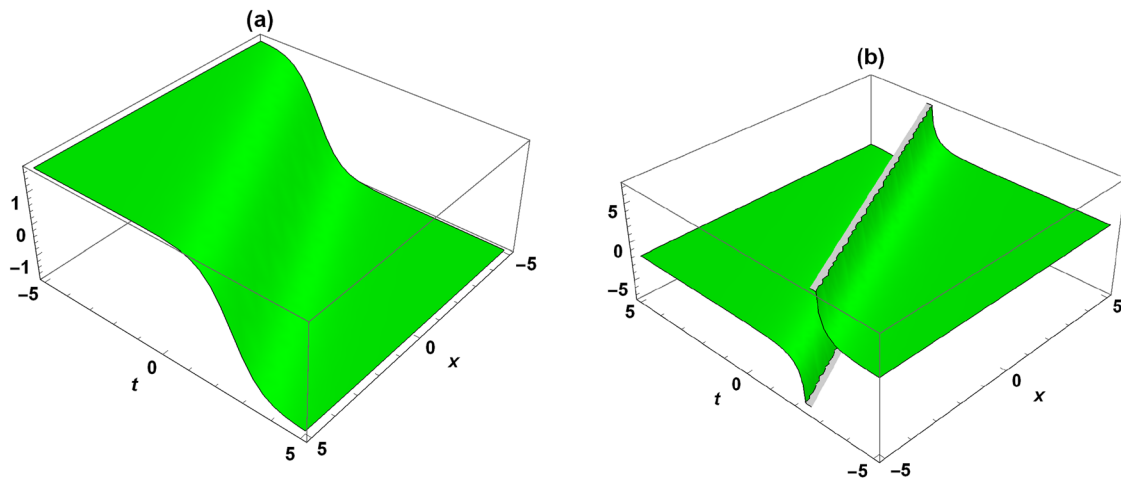


Figure 1: Kink and singular-kink of MRLW as depicted in (6): (a) $\delta > 0$ and (b) $\delta < 0$.

shows the physical structure of (6), kink-wave $\delta > 0$, and singular-kink-wave $\delta < 0$.

2.2 Approach II: modified rational trigonometric/hyperbolic functions

We aim here to investigate travelling periodic-wave solutions to the MRLW by considering different types of rational functions in terms of trigonometric and hyperbolic functions. Four suggestions will be offered in this section.

2.2.1 Rational sine-cosine

The suggested solution is

$$U(z) = \frac{1 + a_1 \sin(\mu z)}{a_2 + a_3 \cos(\mu z)}. \quad (7)$$

Then, we insert (7) in (2) to obtain

$$\frac{P(1, \sin(\mu z), \cos(\mu z), \sin(2\mu z), \sin^2(\mu z), \sin^3(\mu z))}{(a_2 + a_3 \cos(\mu z))^3} = 0. \quad (8)$$

From (8), we null the coefficients of $1, \sin(\mu z), \dots, \sin^3(\mu z)$ to obtain

$$\begin{aligned} 0 &= \alpha - 3a_2^2(c - 1) - 3a_3^2(\beta c \mu^2 + c - 1), \\ 0 &= -3a_2 a_3(c(\beta \mu^2 + 2) - 2), \\ 0 &= 3a_1(\alpha + a_2^2(c(\beta \mu^2 - 1) + 1) - a_3^2(2\beta c \mu^2 + c - 1)), \\ 0 &= \frac{1}{2}(-3)a_1 a_2 a_3(c(\beta \mu^2 + 2) - 2), \\ 0 &= -3(a_3^2(c(\beta \mu^2 - 1) + 1) - a a_1^2), \\ 0 &= a_1(a a_1^2 + 3a_3^2(c - 1)). \end{aligned} \quad (9)$$

By solving the aforementioned system, we deduce the following three cases:

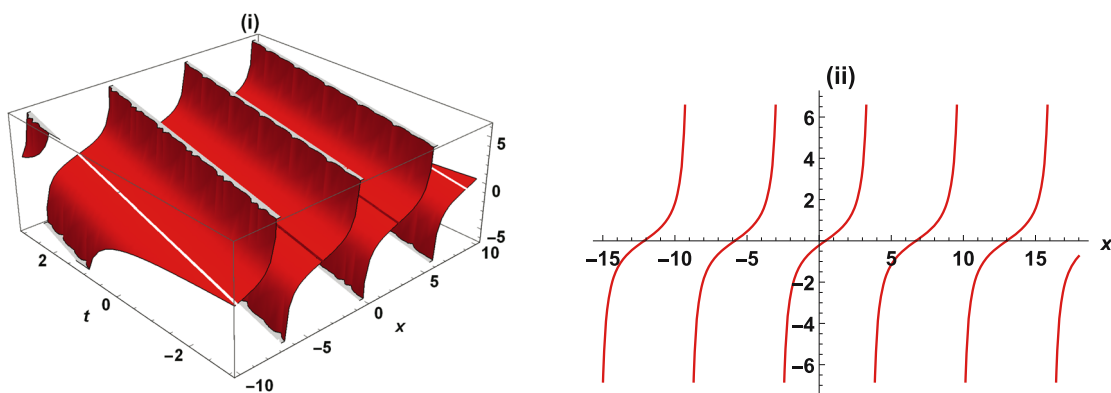


Figure 2: Kink-periodic waves of MRLW as depicted in Ω_1 .

Case 1:

$$\begin{aligned} a_1 &= \mp \frac{\sqrt{3}a_3\sqrt{\beta\mu}}{\sqrt{\alpha(\beta\mu^2+2)}}, \\ a_2 &= \mp \frac{\sqrt{3a_3^2\beta\mu^2-\alpha(\beta\mu^2+2)}}{\sqrt{\beta\mu^2+2}\sqrt{3-\frac{6}{\beta\mu^2+2}}}, \\ c &= \frac{2}{\beta\mu^2+2}. \end{aligned} \quad (10)$$

Case 2:

$$\begin{aligned} a_1 &= \mp 1, \quad a_2 = 0, \quad \mu = \mp \frac{\sqrt{2}\sqrt{\alpha}}{\sqrt{\beta(3a_3^2-\alpha)}}, \\ c &= 1 - \frac{\alpha}{3a_3^2}. \end{aligned} \quad (11)$$

Case 3:

$$\begin{aligned} a_1 &= 0, \quad a_2 = 0, \quad \mu = \mp \frac{\sqrt{\alpha}}{\sqrt{\beta(\alpha+6a_3^2)}}, \\ c &= \frac{\alpha}{6a_3^2} + 1. \end{aligned} \quad (12)$$

As a result, the first three periodic solutions to MRLW labeled as $\Omega_1, \Omega_2, \Omega_3$ are as follows:

$$\begin{aligned} \Omega_1(x, t) &= \frac{1 \mp \frac{\sqrt{3}a_3\sqrt{\beta\mu} \sin\left(\mu\left(x - \frac{2t}{\beta\mu^2+2}\right)\right)}{\sqrt{\alpha(\beta\mu^2+2)}}}{a_3 \cos\left(\mu\left(x - \frac{2t}{\beta\mu^2+2}\right)\right) \mp \frac{\sqrt{3a_3^2\beta\mu^2-\alpha(\beta\mu^2+2)}}{\sqrt{\beta\mu^2+2}\sqrt{3-\frac{6}{\beta\mu^2+2}}}}, \\ \Omega_2(x, t) &= \frac{1}{a_3} \tan\left[\frac{\sqrt{2}\sqrt{\alpha}\left(t\left(\frac{\alpha}{3a_3^2}-1\right)+x\right)}{\sqrt{\beta(3a_3^2-\alpha)}}\right] \\ &\quad \mp \frac{1}{a_3} \sec\left[\frac{\sqrt{2}\sqrt{\alpha}\left(t\left(\frac{\alpha}{3a_3^2}-1\right)+x\right)}{\sqrt{\beta(3a_3^2-\alpha)}}\right], \\ \Omega_3(x, t) &= \frac{1}{a_3} \sec\left[\frac{\sqrt{\alpha}\left(t\left(-\frac{\alpha}{6a_3^2}-1\right)+x\right)}{\sqrt{\beta(\alpha+6a_3^2)}}\right]. \end{aligned} \quad (13)$$

By selecting $a_3 = \mu = \alpha = \beta = 1$, Figure 2 shows the propagation of kink-periodic waves as depicted in Ω_1 , and Figure 3 shows the motion of convex-concave-periodic waves as depicted in Ω_3 . For the same assigned values, Ω_2 has the same physical shape as for Ω_3 .

2.2.2 Rational cosine-sine

The suggested solution is

$$U(z) = \frac{1 + a_1 \cos(\mu z)}{a_2 + a_3 \sin(\mu z)}. \quad (14)$$

By substituting (14) into (2), we obtain the same system as in (9). Thus, the same findings as in (10)–(12). Accordingly, three new more periodic solutions will be attained to MRLW labeled as $\Omega_4, \Omega_5, \Omega_6$ and given by

$$\begin{aligned} \Omega_4(x, t) &= \frac{1 \mp \frac{\sqrt{3}a_3\sqrt{\beta\mu} \cos\left(\mu\left(x - \frac{2t}{\beta\mu^2+2}\right)\right)}{\sqrt{\alpha(\beta\mu^2+2)}}}{a_3 \sin\left(\mu\left(x - \frac{2t}{\beta\mu^2+2}\right)\right) \mp \frac{\sqrt{3a_3^2\beta\mu^2-\alpha(\beta\mu^2+2)}}{\sqrt{\beta\mu^2+2}\sqrt{3-\frac{6}{\beta\mu^2+2}}}}, \\ \Omega_5(x, t) &= \frac{1}{a_3} \cot\left[\frac{\sqrt{2}\sqrt{\alpha}\left(t\left(\frac{\alpha}{3a_3^2}-1\right)+x\right)}{\sqrt{\beta(3a_3^2-\alpha)}}\right] \\ &\quad \mp \frac{1}{a_3} \csc\left[\frac{\sqrt{2}\sqrt{\alpha}\left(t\left(\frac{\alpha}{3a_3^2}-1\right)+x\right)}{\sqrt{\beta(3a_3^2-\alpha)}}\right], \\ \Omega_6(x, t) &= \frac{1}{a_3} \csc\left[\frac{\sqrt{\alpha}\left(t\left(-\frac{\alpha}{6a_3^2}-1\right)+x\right)}{\sqrt{\beta(\alpha+6a_3^2)}}\right]. \end{aligned} \quad (15)$$

We point here that the physical types of $\Omega_4, \Omega_5, \Omega_6$ are the same as reported in $\Omega_1, \Omega_2, \Omega_3$.

2.2.3 Rational sinh-cosh

The suggested solution is

$$U(z) = \frac{1 + a_1 \sinh(\mu z)}{a_2 + a_3 \cosh(\mu z)}. \quad (16)$$

Then, we insert (16) in (2) to obtain

$$\frac{Q(1, \sinh(\mu z), \cosh(\mu z), \sinh(2\mu z), \sinh^2(\mu z), \sinh^3(\mu z))}{(a_2 + a_3 \cosh(\mu z))^3} = 0. \quad (17)$$

From (8), we collect the coefficients of $1, \sinh(\mu z), \dots, \sinh^3(\mu z)$ and null them to zero to arrive a nonlinear algebraic system in the unknowns a_1, a_2, a_3, μ, c . By solving the resulting system, we obtain two cases:

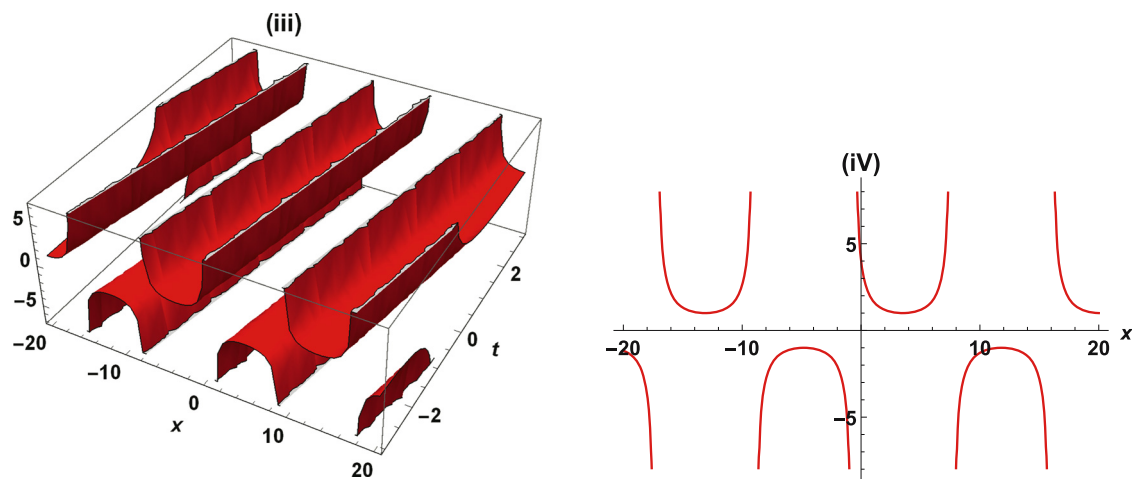


Figure 3: Convex-concave-periodic waves of MRLW as depicted in Ω_3 .

Case A:

$$\begin{aligned} a_1 &= \mp \frac{\sqrt{3}a_3\sqrt{\beta}\mu}{\sqrt{-\alpha(\beta\mu^2 - 2)}}, \\ a_2 &= \mp \frac{\sqrt{\alpha(2 - \beta\mu^2) + 3a_3^2\beta\mu^2}}{\sqrt{\beta\mu^2 - 2}\sqrt{\frac{6}{\beta\mu^2 - 2} + 3}}, \\ c &= -\frac{2}{\beta\mu^2 - 2}. \end{aligned} \quad (18)$$

Case B:

$$\begin{aligned} a_1 &= \mp i, \quad a_2 = 0, \\ \mu &= \mp \frac{i\sqrt{2}\sqrt{\alpha}}{\sqrt{\beta(\alpha - 3a_3^2)}}, \quad c = 1 - \frac{\alpha}{3a_3^2}. \end{aligned} \quad (19)$$

From case A, we obtain the following new kink-soliton solution to MRLW labeled as Ω_7 and given by

$$\Omega_7(x, t) = \frac{1 \mp \frac{\sqrt{3}a_3\sqrt{\beta}\mu \sinh\left(\mu\left(\frac{2t}{\beta\mu^2 - 2} + x\right)\right)}{\sqrt{-\alpha(\beta\mu^2 - 2)}}}{a_3 \cosh\left(\mu\left(\frac{2t}{\beta\mu^2 - 2} + x\right)\right) \mp \frac{\sqrt{\alpha(2 - \beta\mu^2) + 3a_3^2\beta\mu^2}}{\sqrt{\beta\mu^2 - 2}\sqrt{\frac{6}{\beta\mu^2 - 2} + 3}}}. \quad (20)$$

From case B, we obtain the same solution as depicted in Ω_2 .

2.2.4 Rational cosh-sinh

The suggested solution is

$$U(z) = \frac{1 + a_1 \cosh(\mu z)}{a_2 + a_3 \sinh(\mu z)}. \quad (21)$$

By using the same steps applied to the aforementioned suggested solutions, we insert (21) in (2) and collect the coefficients of $1, \cosh(\mu z), \dots, \cosh^3(\mu z)$ and null them to zero to arrive a nonlinear algebraic system in the unknowns a_1, a_2, a_3, μ, c . By solving the resulting system, we obtain four cases:

Case i:

$$\begin{aligned} a_1 &= \mp \frac{\sqrt{3}a_3\sqrt{\beta}\mu}{\sqrt{-\alpha(\beta\mu^2 - 2)}}, \\ a_2 &= \mp \frac{\sqrt{\alpha(\beta\mu^2 - 2) + 3a_3^2\beta\mu^2}}{\sqrt{\beta\mu^2 - 2}\sqrt{\frac{6}{\beta\mu^2 - 2} - 3}}, \\ c &= -\frac{2}{\beta\mu^2 - 2}. \end{aligned} \quad (22)$$

Case ii:

$$\begin{aligned} a_1 &= a_2 = 0, \quad \mu = \mp \frac{\sqrt{\alpha}}{\sqrt{\beta(6a_3^2 - \alpha)}}, \\ c &= 1 - \frac{\alpha}{6a_3^2}. \end{aligned} \quad (23)$$

Case iii:

$$\begin{aligned} a_1 &= -1, \quad a_2 = 0, \quad \mu = \mp \frac{\sqrt{2}\sqrt{\alpha}}{\sqrt{\beta(\alpha + 3a_3^2)}}, \\ c &= \frac{\alpha}{3a_3^2} + 1. \end{aligned} \quad (24)$$

Case iv:

$$\begin{aligned} a_1 &= 1, \quad a_2 = 0, \quad \mu = \mp \frac{\sqrt{2}\sqrt{\alpha}}{\sqrt{\beta(\alpha + 3a_3^2)}}, \\ c &= \frac{\alpha}{3a_3^2} + 1. \end{aligned} \quad (25)$$

Now, by combining the aforementioned four cases with (21), more new solutions of the MRLW will be attained, labeled as Ω_8 , Ω_9 , Ω_{10} , and Ω_{11} and given by:

$$\Omega_8(x, t) = \frac{1 \mp \frac{\sqrt{3}a_3\sqrt{\beta}\mu \cosh\left(\mu\left(\frac{2t}{\beta\mu^2-2} + x\right)\right)}{\sqrt{-a(\beta\mu^2-2)}}}{a_3 \sinh\left(\mu\left(\frac{2t}{\beta\mu^2-2} + x\right)\right) \mp \frac{\sqrt{a(\beta\mu^2-2)+3a_3^2\beta\mu^2}}{\sqrt{\beta\mu^2-2}\sqrt{-\frac{6}{\beta\mu^2-2}-3}}}. \quad (26)$$

$$\Omega_9(x, t) = \mp \frac{1}{a_3} \operatorname{csch} \left(\frac{\sqrt{\alpha} \left(t \left(\frac{\alpha}{6a_3^2} - 1 \right) + x \right)}{\sqrt{\beta(6a_3^2 - \alpha)}} \right). \quad (27)$$

$$\Omega_{10}(x, t) = \mp \frac{1}{a_3} \tanh \left(\frac{\sqrt{\alpha} \left(t \left(-\frac{\alpha}{3a_3^2} - 1 \right) + x \right)}{\sqrt{2}\sqrt{\beta(\alpha + 3a_3^2)}} \right). \quad (28)$$

$$\Omega_{11}(x, t) = \mp \frac{1}{a_3} \coth \left(\frac{\sqrt{\alpha} \left(t \left(-\frac{\alpha}{3a_3^2} - 1 \right) + x \right)}{\sqrt{2}\sqrt{\beta(\alpha + 3a_3^2)}} \right). \quad (29)$$

For instance, if we assign the values $a_3 = \alpha = \beta = 1$ and $\mu = 0.5$, one can check that the types of Ω_8 , Ω_9 , Ω_{10} , and Ω_{11} are kink, singular-kink, kink, and singular-kink, respectively.

3 Discussions

Periodic solution refers to a solution that repeats itself after a certain period. In other words, the solution exhibits a recurring pattern over time. Kink solution is a type of soliton that describes a localized wave disturbance or discontinuity in a system. It is characterized by a sharp change or jump in the field or variable being described. The kink solution typically connects different stable equilibrium states. Singular-Kink solution: A singular-kink solution refers to a special type of kink solution where the wave disturbance or discontinuity becomes particularly sharp or intense. It is associated with a singularity or a point of non-analytic behavior in the solution. Kink periodic solution combines the concepts of a kink solution and a periodic solution, where the wave shape or kink repeats itself periodically. Convex-concave periodic solution refers to a periodic solution that alternates between convex and concave shapes. In other words, the solution exhibits regions where it is curved outward (convex) and regions where it is curved inward (concave) in a repeating manner.

Now, we demonstrate the advantages of employing the adapted rational sine-cosine/sinh-cosh functions for identifying periodic solutions to the MRLW equation. Our intention is to offer a concise summary of the research outcomes by presenting them as bullet points accompanied by simplified explanations.

- The tanh-solution Ω_{10} and coth-solution Ω_{11} can be derived directly by using the tanh-coth expansion method.
- Both Ω_2 and Ω_3 can be obtained by using the sec-tan expansion method.
- Both Ω_5 and Ω_6 can be extracted by using the csc-cot method.
- The solution Ω_9 can be attained by using the csch-expansion method.
- For $\mu = \mp \frac{\sqrt{2}\sqrt{\alpha}}{\sqrt{\beta(\alpha + 3a_3^2)}}$, for the case of $\delta = 1$, the Kudryashov solution (6) is the same as Ω_{10} . Also, for $\delta = -1$, the Kudryashov solution (6) is the same as Ω_{11} .
- The solutions Ω_1 , Ω_4 , Ω_7 , Ω_8 are presented in this work for the first time and they are of type kink-periodic and convex-concave-periodic.

In summary, we may say that the modified rational sine-cosine/sinh-cosh approach is a comprehensive hybrid scheme that generates wave solutions of different physical shapes that cannot be found using a single method. The proposed approach includes other well-known methods in terms of giving similar solutions as shown in the cases of tanh-coth expansion, csc-cot expansion, csch-expansion, and the Kudryashov-expansion.

4 Conclusion

This work introduces a novel investigation of the MRLW equation, focusing on the discovery of previously unrevealed periodic solutions. These solutions were derived using a contemporary approach known as modified rational sine-cosine and sinh-cosh functions. The effectiveness of this method is demonstrated through a comparative analysis with previous approaches, highlighting its capability to identify multiple solutions with diverse physical characteristics.

There are possibilities for exploring new avenues in future research that are directly relevant to the current work. For instance, one can broaden the scope of the MRLW model by incorporating time/space fractional derivatives and examining their influence on the propagation of its solitary waves. By employing various explicit and numerical techniques [31–38], both explicit fractional soliton solutions and numerical-analytical solutions can be obtained.

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