

Research Article

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Derivation of septic B-spline function in n -dimensional to solve n -dimensional partial differential equations

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Abstract: In this study, a new structure for the septic B-spline collocation algorithm in n -dimensional is presented as a continuation of generating B-spline functions in n -dimensional to solve mathematical models in n -dimensional. The septic B-spline collocation algorithm is displayed in three forms: one dimensional, two dimensional, and three dimensional. In various domains, these constructs are essential for solving mathematical models. The effectiveness and correctness of the suggested method are demonstrated using a few two- and three-dimensional test problems. The proposed new structure provides better results than other methods because it deals with a larger number of points than the field. To create comparisons, we use different numerical approaches accessible in the literature.

Keywords: collocation method, n -dimensional B-spline functions, mathematical models

1 Introduction

Many researchers have solved some mathematical models in different dimensions using some analytical and numerical methods such as the (3+1)-dimensional Date–Jimbo–Kashiwara–Miwa equation [1], (3+1)-dimensional Boiti–Leon–Manna–Pempinelli equation [2], (2+1)-dimensional Schrödinger’s hyperbolic equation [3], and non-Newtonian fluid models [4,5]. Most mathematical models in several

fields, such as fluid mechanics and physics, are challenging to deal with analytically, prompting some researchers to consider numerical solutions. The finite differences approach, as seen in previous studies [6,7], is one of the strategies used in solving n -dimensional models. Various academics have also attempted to adapt some approaches for solving mathematical models in one dimension to solving models in n -dimensional, such as spectral methods [8,9]. However, most nonlinear models were challenging to solve using spectral approaches. Gardner and Gardner studied a two-dimensional bi-cubic B-spline finite element for solving two-dimensional problems [10]. To solve several diverse mathematical models, some researchers used the bi-cubic B-spline finite element method [11–13] and bi-quintic B-spline collocation method [14–16]. Raslan and Ali began to consider generalizing all types of B-spline functions. They talked about n -dimensional quadratic B-splines [17], new structure formulations for the cubic B-spline collocation method in three and four dimensional [18], construction of extended cubic B-splines in n -dimensional for solving n -dimensional partial differential equations [19], and a new structure for n -dimensional trigonometric cubic B-spline functions for solving n -dimensional partial differential equations [20]. The B-spline collocation method and other methods have been used in many articles to solve many mathematical models such as quintic B-spline method [21], cubic B-spline method [22,23], and novel collocation techniques [24–26].

The idea of solving mathematical models in different dimensional remains an idea that haunts most researchers in various fields. Although in previous articles we have presented solutions to these problems by generalizing B-spline collocation functions, we are continuing with these generalizations to deal with models that contain ranks higher than the fifth degree. In this article, we present a generalization of the septic B-spline function, where this method can deal with equations of the seventh order and below.

The structure of this article is as follows: In Section 2, n -dimensional septic B-spline formulas are presented. In Section 3, the numerical outcomes are presented. Section 4 introduces numerical examples. Finally, the conclusion of this work is presented.

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2 n -dimensional septic B-spline functions

In this section, we present the n -dimensional septic B-splines.

2.1 One-dimensional septic B-spline [27,28]

Let $l \leq x \leq m$ and $\mathcal{L}_i(x)$ are those septic B-spline with knots at the points x_q . Then, the set of septic B-splines $\mathcal{L}_{-3}(x)$, $\mathcal{L}_{-2}(x)$, $\mathcal{L}_0(x)$, ..., $\mathcal{L}_{N-1}(x)$, $\mathcal{L}_N(x)$, $\mathcal{L}_{N+2}(x)$, $\mathcal{L}_{N+3}$, serves as a basis for functions specified over a range of values. The $H^N(x)$ approximation to $H(x)$ is given by:

$$H^N(x) = \sum_{q=-3}^{N+3} \mathfrak{T}_q \mathcal{L}_q(x), \quad (1)$$

where $\mathfrak{T}_q$ is the unknown term and $\mathcal{L}_q(x)$ is a function given by:

$$\mathcal{L}_q(x) = \frac{1}{h^7} \begin{cases} a_1 = (x - x_{q-4})^7, & x_{q-4} \leq x < x_{q-3} \\ a_2 = a_1 - 8(x - x_{q-3})^7, & x_{q-3} \leq x < x_{q-2} \\ a_3 = a_2 + 28(x - x_{q-2})^7, & x_{q-2} \leq x < x_{q-1} \\ a_4 = a_3 - 56(x - x_{q-1})^7, & x_{q-1} \leq x < x_q \\ b_4 = b_3 - 56(x_{q-1} - x)^7, & x_q \leq x < x_{q+1} \\ b_3 = b_2 + 28(x_{q-2} - x)^7, & x_{q+1} \leq x < x_{q+2} \\ b_2 = b_1 - 8(x_{q-3} - x)^7, & x_{q+2} \leq x < x_{q+3} \\ b_1 = (x_{q-3} - x)^7, & x_{q+3} \leq x < x_{q+4} \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

We use (1) and (2) with substitution by collection points to find H_q , $\frac{dH_q}{dx}$ and $\frac{d^2H_q}{dx^2}$ as follows:

$$\begin{aligned} H_q &= \mathfrak{T}_{q-3} + 120\mathfrak{T}_{q-2} + 1,191\mathfrak{T}_{q-1} + 2,416\mathfrak{T}_q \\ &\quad + 1,191\mathfrak{T}_{q+1} + 120\mathfrak{T}_{q+2} + \mathfrak{T}_{q+3}, \\ \frac{dH_q}{dx} &= -\frac{7(\mathfrak{T}_{q-3} + 56\mathfrak{T}_{q-2} + 245\mathfrak{T}_{q-1} - 245\mathfrak{T}_{q+1} - 56\mathfrak{T}_{q+2} - \mathfrak{T}_{q+3})}{h}, \\ \frac{d^2H_q}{dx^2} &= \frac{42(\mathfrak{T}_{q-3} + 24\mathfrak{T}_{q-2} + 15\mathfrak{T}_{q-1} - 80\mathfrak{T}_q + 15\mathfrak{T}_{q+1} + 24\mathfrak{T}_{q+2} + \mathfrak{T}_{q+3})}{h^2}. \end{aligned} \quad (3)$$

The aforementioned analysis yields the following theorem.

Theorem 1. From (1) the approximation formulas to H_q , $\frac{dH_q}{dx}$, and $\frac{d^2H_q}{dx^2}$ are given in terms of $\mathfrak{T}_q$ at (3).

2.2 Two-dimensional septic B-spline

This subsection shows the formula for a two-dimensional septic B-spline on a rectangular grid divided into regular rectangular finite elements on both sides. $h = \Delta x$, $k = \Delta y$ by the knots (x_q, y_ζ) , where $q = 0, 1, \dots, N$, $\zeta = 0, 1, \dots, M$. The approximation $H^N(x, y)$ to $H(x, y)$ is given by:

$$H^N(x, y) = \sum_{q=-3}^{N+3} \sum_{\zeta=-3}^{M+3} \mathfrak{T}_q \zeta \Theta_{q,\zeta}(x, y), \quad (4)$$

where $\mathfrak{T}_{q,\zeta}$ are the amplitudes of the septic B-splines. $\Theta_{q,\zeta}(x, y)$ are given by:

$$\Theta_{q,\zeta}(x, y) = \mathcal{L}_q(x) \mathcal{L}_\zeta(y).$$

Moreover, $\mathcal{L}_\zeta(x)$, $\mathcal{L}_n(y)$ have the same shape as septic B-splines in one dimension. Then, the formulations of $H_{q,\zeta}$,

$\frac{\partial H_{q,\zeta}}{\partial x}$, $\frac{\partial H_{q,\zeta}}{\partial y}$, $\frac{\partial^2 H_{q,\zeta}}{\partial x^2}$, $\frac{\partial^2 H_{q,\zeta}}{\partial y^2}$, $\frac{\partial^2 H_{q,\zeta}}{\partial x \partial y}$, ... are given by:

$$\begin{aligned} H_{q,\zeta} &= \mathfrak{T}_{q-3,\zeta-3} + 120\mathfrak{T}_{q-3,\zeta-2} + 1,191\mathfrak{T}_{q-3,\zeta-1} + 2,416\mathfrak{T}_{q-3,\zeta} + 1,191\mathfrak{T}_{q-3,\zeta+1} + 120\mathfrak{T}_{q-3,\zeta+2} + \mathfrak{T}_{q-3,\zeta+3} + 120\mathfrak{T}_{q-2,\zeta-3} \\ &\quad + 14,400\mathfrak{T}_{q-2,\zeta-2} + 142,920\mathfrak{T}_{q-2,\zeta-1} + 289,920\mathfrak{T}_{q-2,\zeta} + 142,920\mathfrak{T}_{q-2,\zeta+1} + 14,400\mathfrak{T}_{q-2,\zeta+2} + 120\mathfrak{T}_{q-2,\zeta+3} \\ &\quad + 1,191\mathfrak{T}_{q-1,\zeta-3} + 142,920\mathfrak{T}_{q-1,\zeta-2} + 1,418,481\mathfrak{T}_{q-1,\zeta-1} + 2,877,456\mathfrak{T}_{q-1,\zeta} + 1,418,481\mathfrak{T}_{q-1,\zeta+1} + 142,920\mathfrak{T}_{q-1,\zeta+2} \\ &\quad + 1,191\mathfrak{T}_{q-1,\zeta+3} + 2,416\mathfrak{T}_{q,\zeta-3} + 289,920\mathfrak{T}_{q,\zeta-2} + 2,877,456\mathfrak{T}_{q,\zeta-1} + 5,837,056\mathfrak{T}_{q,\zeta} + 2,877,456\mathfrak{T}_{q,\zeta+1} \\ &\quad + 289,920\mathfrak{T}_{q,\zeta+2} + 2,416\mathfrak{T}_{q,\zeta+3} + 1,191\mathfrak{T}_{q+1,\zeta-3} + 142,920\mathfrak{T}_{q+1,\zeta-2} + 1,418,481\mathfrak{T}_{q+1,\zeta-1} + 2,877,456\mathfrak{T}_{q+1,\zeta} \\ &\quad + 1,418,481\mathfrak{T}_{q+1,\zeta+1} + 142,920\mathfrak{T}_{q+1,\zeta+2} + 1,191\mathfrak{T}_{q+1,\zeta+3} + 120\mathfrak{T}_{q+2,\zeta-3} + 14,400\mathfrak{T}_{q+2,\zeta-2} + 142,920\mathfrak{T}_{q+2,\zeta-1} \\ &\quad + 289,920\mathfrak{T}_{q+2,\zeta} + 142,920\mathfrak{T}_{q+2,\zeta+1} + 14,400\mathfrak{T}_{q+2,\zeta+2} + 120\mathfrak{T}_{q+2,\zeta+3} + \mathfrak{T}_{q+3,\zeta-3} + 120\mathfrak{T}_{q+3,\zeta-2} + 1,191\mathfrak{T}_{q+3,\zeta-1} \\ &\quad + 2,416\mathfrak{T}_{q+3,\zeta} + 1,191\mathfrak{T}_{q+3,\zeta+1} + 120\mathfrak{T}_{q+3,\zeta+2} + \mathfrak{T}_{q+3,\zeta+3}. \end{aligned} \quad (5)$$

$$\begin{aligned}
\frac{\partial H_{q,\zeta}}{\partial x} = & -\frac{7}{h}(\mathfrak{T}_{q-3,\zeta-3} + 120\mathfrak{T}_{q-3,\zeta-2} + 1,191\mathfrak{T}_{q-3,\zeta-1} + 2,416\mathfrak{T}_{q-3,\zeta} + 1,191\mathfrak{T}_{q-3,\zeta+1} + 120\mathfrak{T}_{q-3,\zeta+2} + \mathfrak{T}_{q-3,\zeta+3} + 56\mathfrak{T}_{q-2,\zeta-3} \\
& + 6,720\mathfrak{T}_{q-2,\zeta-2} + 66,696\mathfrak{T}_{q-2,\zeta-1} + 135,296\mathfrak{T}_{q-2,\zeta} + 66,696\mathfrak{T}_{q-2,\zeta+1} + 6,720\mathfrak{T}_{q-2,\zeta+2} + 56\mathfrak{T}_{q-2,\zeta+3} + 245\mathfrak{T}_{q-1,\zeta-3} \\
& + 29,400\mathfrak{T}_{q-1,\zeta-2} + 291,795\mathfrak{T}_{q-1,\zeta-1} + 591,920\mathfrak{T}_{q-1,\zeta} + 291,795\mathfrak{T}_{q-1,\zeta+1} + 29,400\mathfrak{T}_{q-1,\zeta+2} \\
& + 245\mathfrak{T}_{q-1,\zeta+3} - 245\mathfrak{T}_{q+1,\zeta-3} - 29,400\mathfrak{T}_{q+1,\zeta-2} - 291,795\mathfrak{T}_{q+1,\zeta-1} - 591,920\mathfrak{T}_{q+1,\zeta} - 291,795\mathfrak{T}_{q+1,\zeta+1} \\
& - 29,400\mathfrak{T}_{q+1,\zeta+2} - 245\mathfrak{T}_{q+1,\zeta+3} - 56\mathfrak{T}_{q+2,\zeta-3} - 6,720\mathfrak{T}_{q+2,\zeta-2} - 66,696\mathfrak{T}_{q+2,\zeta-1} - 135,296\mathfrak{T}_{q+2,\zeta} - 66,696\mathfrak{T}_{q+2,\zeta+1} \\
& - 6,720\mathfrak{T}_{q+2,\zeta+2} - 56\mathfrak{T}_{q+2,\zeta+3} - \mathfrak{T}_{q+3,\zeta-3} - 120\mathfrak{T}_{q+3,\zeta-2} - 1,191\mathfrak{T}_{q+3,\zeta-1} - 2,416\mathfrak{T}_{q+3,\zeta} - 1,191\mathfrak{T}_{q+3,\zeta+1} - 120\mathfrak{T}_{q+3,\zeta+2} \\
& - \mathfrak{T}_{q+3,\zeta+3}), \\
\frac{\partial H_{q,\zeta}}{\partial y} = & -\frac{7}{k}(\mathfrak{T}_{q-3,\zeta-3} + 56\mathfrak{T}_{q-3,\zeta-2} + 245\mathfrak{T}_{q-3,\zeta-1} - 245\mathfrak{T}_{q-3,\zeta+1} - 56\mathfrak{T}_{q-3,\zeta+2} - \mathfrak{T}_{q-3,\zeta+3} + 120\mathfrak{T}_{q-2,\zeta-3} + 6,720\mathfrak{T}_{q-2,\zeta-2} \\
& + 29,400\mathfrak{T}_{q-2,\zeta-1} - 29,400\mathfrak{T}_{q-2,\zeta+1} - 6,720\mathfrak{T}_{q-2,\zeta+2} - 120\mathfrak{T}_{q-2,\zeta+3} + 1,191\mathfrak{T}_{q-1,\zeta-3} + 66,696\mathfrak{T}_{q-1,\zeta-2} \\
& + 291,795\mathfrak{T}_{q-1,\zeta-1} - 291,795\mathfrak{T}_{q-1,\zeta+1} - 66,696\mathfrak{T}_{q-1,\zeta+2} - 1,191\mathfrak{T}_{q-1,\zeta+3} + 2,416\mathfrak{T}_{q,\zeta-3} + 135,296\mathfrak{T}_{q,\zeta-2} \\
& + 591,920\mathfrak{T}_{q,\zeta-1} - 591,920\mathfrak{T}_{q,\zeta+1} - 135,296\mathfrak{T}_{q,\zeta+2} - 2,416\mathfrak{T}_{q,\zeta+3} + 1,191\mathfrak{T}_{q+1,\zeta-3} + 66,696\mathfrak{T}_{q+1,\zeta-2} \\
& + 291,795\mathfrak{T}_{q+1,\zeta-1} - 291,795\mathfrak{T}_{q+1,\zeta+1} - 66,696\mathfrak{T}_{q+1,\zeta+2} - 1,191\mathfrak{T}_{q+1,\zeta+3} + 120\mathfrak{T}_{q+2,\zeta-3} + 6,720\mathfrak{T}_{q+2,\zeta-2} \\
& + 29,400\mathfrak{T}_{q+2,\zeta-1} - 29,400\mathfrak{T}_{q+2,\zeta+1} - 6,720\mathfrak{T}_{q+2,\zeta+2} - 120\mathfrak{T}_{q+2,\zeta+3} \\
& + \mathfrak{T}_{q+3,\tau-3} + 7(8\mathfrak{T}_{q+3,\tau-2} + 35\mathfrak{T}_{q+3,\tau-1} - 35\mathfrak{T}_{q+3,\tau+1} - 8\mathfrak{T}_{q+3,\tau+2}) - \mathfrak{T}_{q+3,\zeta+3}).
\end{aligned} \tag{6}$$

$$\begin{aligned}
\frac{\partial^2 H_{q,\zeta}}{\partial x^2} = & \frac{42}{h^2}(\mathfrak{T}_{q-3,\zeta-3} + 120\mathfrak{T}_{q-3,\zeta-2} + 1,191\mathfrak{T}_{q-3,\zeta-1} + 2,416\mathfrak{T}_{q-3,\zeta} + 1,191\mathfrak{T}_{q-3,\zeta+1} + 120\mathfrak{T}_{q-3,\zeta+2} + \mathfrak{T}_{q-3,\zeta+3} + 24\mathfrak{T}_{q-2,\zeta-3} \\
& + 2,880\mathfrak{T}_{q-2,\zeta-2} + 28,584\mathfrak{T}_{q-2,\zeta-1} + 57,984\mathfrak{T}_{q-2,\zeta} + 28,584\mathfrak{T}_{q-2,\zeta+1} + 2,880\mathfrak{T}_{q-2,\zeta+2} + 24\mathfrak{T}_{q-2,\zeta+3} + 15\mathfrak{T}_{q-1,\zeta-3} \\
& + 1,800\mathfrak{T}_{q-1,\zeta-2} + 17,865\mathfrak{T}_{q-1,\zeta-1} + 36,240\mathfrak{T}_{q-1,\zeta} + 17,865\mathfrak{T}_{q-1,\zeta+1} + 1,800\mathfrak{T}_{q-1,\zeta+2} + 15\mathfrak{T}_{q-1,\zeta+3} - 80\mathfrak{T}_{q,\zeta-3} \\
& - 9,600\mathfrak{T}_{q,\zeta-2} - 95,280\mathfrak{T}_{q,\zeta-1} - 193,280\mathfrak{T}_{q,\zeta} - 95,280\mathfrak{T}_{q,\zeta+1} - 9,600\mathfrak{T}_{q,\zeta+2} - 80\mathfrak{T}_{q,\zeta+3} + 15\mathfrak{T}_{q+1,\zeta-3} \\
& + 1,800\mathfrak{T}_{q+1,\zeta-2} + 17,865\mathfrak{T}_{q+1,\zeta-1} + 36,240\mathfrak{T}_{q+1,\zeta} + 17,865\mathfrak{T}_{q+1,\zeta+1} + 1,800\mathfrak{T}_{q+1,\zeta+2} + 15\mathfrak{T}_{q+1,\zeta+3} + 24\mathfrak{T}_{q+2,\zeta-3} \\
& + 2,880\mathfrak{T}_{q+2,\zeta-2} + 28,584\mathfrak{T}_{q+2,\zeta-1} + 57,984\mathfrak{T}_{q+2,\zeta} + 28,584\mathfrak{T}_{q+2,\zeta+1} + 2,880\mathfrak{T}_{q+2,\zeta+2} + 24\mathfrak{T}_{q+2,\zeta+3} + \mathfrak{T}_{q+3,\zeta-3} \\
& + 120\mathfrak{T}_{q+3,\zeta-2} + 1,191\mathfrak{T}_{q+3,\zeta-1} + 2,416\mathfrak{T}_{q+3,\zeta} + 1,191\mathfrak{T}_{q+3,\zeta+1} + 120\mathfrak{T}_{q+3,\zeta+2} + \mathfrak{T}_{q+3,\zeta+3}).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 H_{q,\zeta}}{\partial y^2} = & \frac{42}{k^2}(\mathfrak{T}_{q-3,\zeta-3} + 24\mathfrak{T}_{q-3,\zeta-2} + 15\mathfrak{T}_{q-3,\zeta-1} - 80\mathfrak{T}_{q-3,\zeta} + 15\mathfrak{T}_{q-3,\zeta+1} + 24\mathfrak{T}_{q-3,\zeta+2} + \mathfrak{T}_{q-3,\zeta+3} + 120\mathfrak{T}_{q-2,\zeta-3} \\
& + 2,880\mathfrak{T}_{q-2,\zeta-2} + 1,800\mathfrak{T}_{q-2,\zeta-1} - 9,600\mathfrak{T}_{q-2,\zeta} + 1,800\mathfrak{T}_{q-2,\zeta+1} + 2,880\mathfrak{T}_{q-2,\zeta+2} + 120\mathfrak{T}_{q-2,\zeta+3} \\
& + 1,191\mathfrak{T}_{q-1,\zeta-3} + 28,584\mathfrak{T}_{q-1,\zeta-2} + 17,865\mathfrak{T}_{q-1,\zeta-1} - 95,280\mathfrak{T}_{q-1,\zeta} + 17,865\mathfrak{T}_{q-1,\zeta+1} + 28,584\mathfrak{T}_{q-1,\zeta+2} \\
& + 1,191\mathfrak{T}_{q-1,\zeta+3} + 2,416\mathfrak{T}_{q,\zeta-3} + 57,984\mathfrak{T}_{q,\zeta-2} + 36,240\mathfrak{T}_{q,\zeta-1} - 193,280\mathfrak{T}_{q,\zeta} + 36,240\mathfrak{T}_{q,\zeta+1} + 57,984\mathfrak{T}_{q,\zeta+2} \\
& + 2,416\mathfrak{T}_{q,\zeta+3} + 1,191\mathfrak{T}_{q+1,\zeta-3} + 28,584\mathfrak{T}_{q+1,\zeta-2} + 17,865\mathfrak{T}_{q+1,\zeta-1} - 95,280\mathfrak{T}_{q+1,\zeta} + 17,865\mathfrak{T}_{q+1,\zeta+1} \\
& + 28,584\mathfrak{T}_{q+1,\zeta+2} + 1,191\mathfrak{T}_{q+1,\zeta+3} + 120\mathfrak{T}_{q+2,\zeta-3} + 2,880\mathfrak{T}_{q+2,\zeta-2} + 1,800\mathfrak{T}_{q+2,\zeta-1} - 9,600\mathfrak{T}_{q+2,\zeta} \\
& + 1,800\mathfrak{T}_{q+2,\zeta+1} + 2,880\mathfrak{T}_{q+2,\zeta+2} + 120\mathfrak{T}_{q+2,\zeta+3} + \mathfrak{T}_{q+3,\zeta-3} + 24\mathfrak{T}_{q+3,\zeta-2} + 15\mathfrak{T}_{q+3,\zeta-1} - 80\mathfrak{T}_{q+3,\zeta} \\
& + 15\mathfrak{T}_{q+3,\zeta+1} + 24\mathfrak{T}_{q+3,\zeta+2} + \mathfrak{T}_{q+3,\zeta+3}), \\
& \vdots
\end{aligned} \tag{7}$$

The aforementioned analysis yields the following theorem.

Theorem 2. From (4), the approximation formulas to $H_{q,\zeta}$, $\frac{\partial H_{q,\zeta}}{\partial x}$, $\frac{\partial H_{q,\zeta}}{\partial y}$, $\frac{\partial^2 H_{q,\zeta}}{\partial x^2}$, $\frac{\partial^2 H_{q,\zeta}}{\partial y^2}$, $\frac{\partial^2 H_{q,\zeta}}{\partial x \partial y}$, ... are given in terms of $\mathfrak{T}_{q,\zeta}$ at (5)–(7).

2.3 Three-dimensional septic B-spline

Now, we obtain the septic B-spline in three measurements approximates on a framework divided up into limited components of sides $h = \Delta x$, $k = \Delta y$, and $q = \Delta z$ by the knots (x_q, y_ζ, z_s) , where $q = 0, 1, \dots, N$, $\zeta = 0, 1, \dots, M$, $s = 0, 1, \dots, R$ can be interpolated in terms of piece-wise septic B-splines.

If $H(x, y, z)$ is a function of x, y , and z , it can be shown there exists a unique approximation $H^N(x, y, z)$ as

$$H^N(x, y, z) = \sum_{q=-3}^{N+3} \sum_{\zeta=-3}^{M+3} \sum_{s=-3}^{R+3} \mathfrak{T}_{q,\zeta,s} B_{q,\zeta,s}(x, y, z), \quad (8)$$

where $\mathfrak{T}_{q,\zeta,s}$ are the septic B-spline amplitudes $B_{q,\zeta,s}(x, y, z)$ given by:

$$B_{q,\zeta,s}(x, y, z) = \mathcal{L}_q(x) \mathcal{L}_\zeta(y) \mathcal{L}_s(z).$$

Also, $\mathcal{L}_q(x)$, $\mathcal{L}_\zeta(y)$, and $\mathcal{L}_s(z)$ have the same shape as septic B-splines in one dimension. The compositions of $H_{q,\zeta,s}$, $\frac{\partial H_{q,\zeta,s}}{\partial x}$,

$\frac{\partial H_{q,\zeta,s}}{\partial y}$, $\frac{\partial H_{q,\zeta,s}}{\partial z}$, $\frac{\partial^2 H_{q,\zeta,s}}{\partial x^2}$, $\frac{\partial^2 H_{q,\zeta,s}}{\partial y^2}$, $\frac{\partial^2 H_{q,\zeta,s}}{\partial z^2}$, $\frac{\partial^2 H_{q,\zeta,s}}{\partial x \partial y}$, $\frac{\partial^2 H_{q,\zeta,s}}{\partial x \partial z}$, ... , are given in terms of $\mathfrak{T}_{q,\zeta,s}$ by:

$$\begin{aligned} H_{q,\zeta,s} = & \mathfrak{T}_{q-3,\zeta-3,s-3} + 120\mathfrak{T}_{q-3,\zeta-3,s-2} + 1,191\mathfrak{T}_{q-3,\zeta-3,s-1} + 2,416\mathfrak{T}_{q-3,\zeta-3,s} + 1,191\mathfrak{T}_{q-3,\zeta-3,s+1} + 120\mathfrak{T}_{q-3,\zeta-3,s+2} \\ & + \mathfrak{T}_{q-3,\zeta-3,s+3} + 120\mathfrak{T}_{q-3,\zeta-2,s-3} + 14,400\mathfrak{T}_{q-3,\zeta-2,s-2} + 142,920\mathfrak{T}_{q-3,\zeta-2,s-1} + 289,920\mathfrak{T}_{q-3,\zeta-2,s} + 142,920\mathfrak{T}_{q-3,\zeta-2,s+1} \\ & + 14,400\mathfrak{T}_{q-3,\zeta-2,s+2} + 120\mathfrak{T}_{q-3,\zeta-2,s+3} + 1,191\mathfrak{T}_{q-3,\zeta-1,s-3} + 142,920\mathfrak{T}_{q-3,\zeta-1,s-2} + 1,418,481\mathfrak{T}_{q-3,\zeta-1,s-1} \\ & + 2,877,456\mathfrak{T}_{q-3,\zeta-1,s} + 1,418,481\mathfrak{T}_{q-3,\zeta-1,s+1} + 142,920\mathfrak{T}_{q-3,\zeta-1,s+2} + 1,191\mathfrak{T}_{q-3,\zeta-1,s+3} + 2,416\mathfrak{T}_{q-3,\zeta-1,s+4} \\ & + 289,920\mathfrak{T}_{q-3,\zeta-1,s+5} + 2,877,456\mathfrak{T}_{q-3,\zeta-1,s+6} + 5,837,056\mathfrak{T}_{q-3,\zeta-1,s+7} + 2,877,456\mathfrak{T}_{q-3,\zeta-1,s+8} + 289,920\mathfrak{T}_{q-3,\zeta-1,s+9} \\ & + 2,416\mathfrak{T}_{q-3,\zeta-1,s+10} + 1,191\mathfrak{T}_{q-3,\zeta-1,s+11} + 142,920\mathfrak{T}_{q-3,\zeta-1,s+12} + 1,418,481\mathfrak{T}_{q-3,\zeta-1,s+13} + 2,877,456\mathfrak{T}_{q-3,\zeta-1,s+14} \\ & + 1,418,481\mathfrak{T}_{q-3,\zeta-1,s+15} + 142,920\mathfrak{T}_{q-3,\zeta-1,s+16} + 1,191\mathfrak{T}_{q-3,\zeta-1,s+17} + 120\mathfrak{T}_{q-3,\zeta-1,s+18} + 14,400\mathfrak{T}_{q-3,\zeta-1,s+19} \\ & + 142,920\mathfrak{T}_{q-3,\zeta-1,s+20} + 289,920\mathfrak{T}_{q-3,\zeta-1,s+21} + 142,920\mathfrak{T}_{q-3,\zeta-1,s+22} + 14,400\mathfrak{T}_{q-3,\zeta-1,s+23} + 120\mathfrak{T}_{q-3,\zeta-1,s+24} \\ & + \mathfrak{T}_{q-3,\zeta-1,s+25} + 120\mathfrak{T}_{q-3,\zeta-1,s+26} + 1,191\mathfrak{T}_{q-3,\zeta-1,s+27} + 2,416\mathfrak{T}_{q-3,\zeta-1,s+28} + 1,191\mathfrak{T}_{q-3,\zeta-1,s+29} + 120\mathfrak{T}_{q-3,\zeta-1,s+30} \\ & + \mathfrak{T}_{q-3,\zeta-1,s+31} + 120\mathfrak{T}_{q-2,\zeta-3,s-3} + 14,400\mathfrak{T}_{q-2,\zeta-3,s-2} + 142,920\mathfrak{T}_{q-2,\zeta-3,s-1} + 289,920\mathfrak{T}_{q-2,\zeta-3,s} + 142,920\mathfrak{T}_{q-2,\zeta-3,s+1} \\ & + 14,400\mathfrak{T}_{q-2,\zeta-3,s+2} + 120\mathfrak{T}_{q-2,\zeta-3,s+3} + 14,400\mathfrak{T}_{q-2,\zeta-2,s-3} + 1,728,000\mathfrak{T}_{q-2,\zeta-2,s-2} + 17,150,400\mathfrak{T}_{q-2,\zeta-2,s-1} \\ & + 34,790,400\mathfrak{T}_{q-2,\zeta-2,s} + 17,150,400\mathfrak{T}_{q-2,\zeta-2,s+1} + 1,728,000\mathfrak{T}_{q-2,\zeta-2,s+2} + 14,400\mathfrak{T}_{q-2,\zeta-2,s+3} + 142,920\mathfrak{T}_{q-2,\zeta-2,s+4} \\ & + 17,150,400\mathfrak{T}_{q-2,\zeta-2,s+5} + 170,217,720\mathfrak{T}_{q-2,\zeta-2,s+6} + 345,294,720\mathfrak{T}_{q-2,\zeta-2,s+7} + 170,217,720\mathfrak{T}_{q-2,\zeta-2,s+8} \\ & + 17,150,400\mathfrak{T}_{q-2,\zeta-2,s+9} + 142,920\mathfrak{T}_{q-2,\zeta-2,s+10} + 289,920\mathfrak{T}_{q-2,\zeta-2,s+11} + 34,790,400\mathfrak{T}_{q-2,\zeta-2,s+12} + 345,294,720\mathfrak{T}_{q-2,\zeta-2,s+13} \\ & + 700,446,720\mathfrak{T}_{q-2,\zeta-2,s+14} + 345,294,720\mathfrak{T}_{q-2,\zeta-2,s+15} + 34,790,400\mathfrak{T}_{q-2,\zeta-2,s+16} + 289,920\mathfrak{T}_{q-2,\zeta-2,s+17} + 142,920\mathfrak{T}_{q-2,\zeta-2,s+18} \\ & + 17,150,400\mathfrak{T}_{q-2,\zeta-2,s+19} + 170,217,720\mathfrak{T}_{q-2,\zeta-2,s+20} + 345,294,720\mathfrak{T}_{q-2,\zeta-2,s+21} + 170,217,720\mathfrak{T}_{q-2,\zeta-2,s+22} \\ & + 17,150,400\mathfrak{T}_{q-2,\zeta-2,s+23} + 142,920\mathfrak{T}_{q-2,\zeta-2,s+24} + 14,400\mathfrak{T}_{q-2,\zeta-2,s+25} + 1,728,000\mathfrak{T}_{q-2,\zeta-2,s+26} + 17,150,400\mathfrak{T}_{q-2,\zeta-2,s+27} \\ & + 34,790,400\mathfrak{T}_{q-2,\zeta-2,s+28} + 17,150,400\mathfrak{T}_{q-2,\zeta-2,s+29} + 1,728,000\mathfrak{T}_{q-2,\zeta-2,s+30} + 14,400\mathfrak{T}_{q-2,\zeta-2,s+31} + 120\mathfrak{T}_{q-2,\zeta-2,s+32} \\ & + 14,400\mathfrak{T}_{q-2,\zeta-2,s+33} + 142,920\mathfrak{T}_{q-2,\zeta-2,s+34} + 289,920\mathfrak{T}_{q-2,\zeta-2,s+35} + 142,920\mathfrak{T}_{q-2,\zeta-2,s+36} + 14,400\mathfrak{T}_{q-2,\zeta-2,s+37} \\ & + 120\mathfrak{T}_{q-2,\zeta-2,s+38} + 1,191\mathfrak{T}_{q-2,\zeta-2,s+39} + 142,920\mathfrak{T}_{q-2,\zeta-2,s+40} + 1,418,481\mathfrak{T}_{q-2,\zeta-2,s+41} + 2,877,456\mathfrak{T}_{q-2,\zeta-2,s+42} \\ & + 1,418,481\mathfrak{T}_{q-2,\zeta-2,s+43} + 142,920\mathfrak{T}_{q-2,\zeta-2,s+44} + 1,191\mathfrak{T}_{q-2,\zeta-2,s+45} + 142,920\mathfrak{T}_{q-2,\zeta-2,s+46} + 17,150,400\mathfrak{T}_{q-2,\zeta-2,s+47} \\ & + 170,217,720\mathfrak{T}_{q-2,\zeta-2,s+48} + 345,294,720\mathfrak{T}_{q-2,\zeta-2,s+49} + 170,217,720\mathfrak{T}_{q-2,\zeta-2,s+50} + 17,150,400\mathfrak{T}_{q-2,\zeta-2,s+51} \\ & + 142,920\mathfrak{T}_{q-2,\zeta-2,s+52} + 1,418,481\mathfrak{T}_{q-2,\zeta-2,s+53} + 170,217,720\mathfrak{T}_{q-2,\zeta-2,s+54} + 1,689,410,871\mathfrak{T}_{q-2,\zeta-2,s+55} \\ & + 3,427,050,096\mathfrak{T}_{q-2,\zeta-2,s+56} + 1,689,410,871\mathfrak{T}_{q-2,\zeta-2,s+57} + 170,217,720\mathfrak{T}_{q-2,\zeta-2,s+58} + 1,418,481\mathfrak{T}_{q-2,\zeta-2,s+59} \\ & + 2,877,456\mathfrak{T}_{q-2,\zeta-2,s+60} + 345,294,720\mathfrak{T}_{q-2,\zeta-2,s+61} + 3,427,050,096\mathfrak{T}_{q-2,\zeta-2,s+62} + 6,951,933,696\mathfrak{T}_{q-2,\zeta-2,s+63} \\ & + 3,427,050,096\mathfrak{T}_{q-2,\zeta-2,s+64} + 345,294,720\mathfrak{T}_{q-2,\zeta-2,s+65} + 2,877,456\mathfrak{T}_{q-2,\zeta-2,s+66} + 1,418,481\mathfrak{T}_{q-2,\zeta-2,s+67} \\ & + 170,217,720\mathfrak{T}_{q-2,\zeta-2,s+68} + 1,689,410,871\mathfrak{T}_{q-2,\zeta-2,s+69} \end{aligned}$$

$$\begin{aligned}
& + 3,427,050,096 \mathfrak{T}_{q-1,\zeta+1,s} + 1,689,410,871 \mathfrak{T}_{q-1,\zeta+1,s+1} + 170,217,720 \mathfrak{T}_{q-1,\zeta+1,s+2} + 1,418,481 \mathfrak{T}_{q-1,\zeta+1,s+3} \\
& + 142,920 \mathfrak{T}_{q-1,\zeta+2,s-3} + 17,150,400 \mathfrak{T}_{q-1,\zeta+2,s-2} + 170,217,720 \mathfrak{T}_{q-1,\zeta+2,s-1} + 345,294,720 \mathfrak{T}_{q-1,\zeta+2,s} \\
& + 170,217,720 \mathfrak{T}_{q-1,\zeta+2,s+1} + 17,150,400 \mathfrak{T}_{q-1,\zeta+2,s+2} + 142,920 \mathfrak{T}_{q-1,\zeta+2,s+3} + 1,191 \mathfrak{T}_{q-1,\zeta+3,s-3} + 142,920 \mathfrak{T}_{q-1,\zeta+3,s-2} \\
& + 1,418,481 \mathfrak{T}_{q-1,\zeta+3,s-1} + 2,877,456 \mathfrak{T}_{q-1,\zeta+3,s} + 1,418,481 \mathfrak{T}_{q-1,\zeta+3,s+1} + 142,920 \mathfrak{T}_{q-1,\zeta+3,s+2} + 1,191 \mathfrak{T}_{q-1,\zeta+3,s+3} \\
& + 2,416 \mathfrak{T}_{q,\zeta-3,s-3} + 289,920 \mathfrak{T}_{q,\zeta-3,s-2} + 2,877,456 \mathfrak{T}_{q,\zeta-3,s-1} + 5,837,056 \mathfrak{T}_{q,\zeta-3,s} + 2,877,456 \mathfrak{T}_{q,\zeta-3,s+1} \\
& + 289,920 \mathfrak{T}_{q,\zeta-3,s+2} + 2,416 \mathfrak{T}_{q,\zeta-3,s+3} + 289,920 \mathfrak{T}_{q,\zeta-2,s-3} + 34,790,400 \mathfrak{T}_{q,\zeta-2,s-2} + 345,294,720 \mathfrak{T}_{q,\zeta-2,s-1} \\
& + 700,446,720 \mathfrak{T}_{q,\zeta-2,s} + 345,294,720 \mathfrak{T}_{q,\zeta-2,s+1} + 34,790,400 \mathfrak{T}_{q,\zeta-2,s+2} + 289,920 \mathfrak{T}_{q,\zeta-2,s+3} + 2,877,456 \mathfrak{T}_{q,\zeta-1,s-3} \\
& + 345,294,720 \mathfrak{T}_{q,\zeta-1,s-2} + 3,427,050,096 \mathfrak{T}_{q,\zeta-1,s-1} + 6,951,933,696 \mathfrak{T}_{q,\zeta-1,s} + 3,427,050,096 \mathfrak{T}_{q,\zeta-1,s+1} \\
& + 345,294,720 \mathfrak{T}_{q,\zeta-1,s+2} + 2,877,456 \mathfrak{T}_{q,\zeta-1,s+3} + 5,837,056 \mathfrak{T}_{q,\zeta,s-3} + 700,446,720 \mathfrak{T}_{q,\zeta,s-2} \\
& + 6,951,933,696 \mathfrak{T}_{q,\zeta,s-1} + 14,102,327,296 \mathfrak{T}_{q,\zeta,s} + 6,951,933,696 \mathfrak{T}_{q,\zeta,s+1} + 700,446,720 \mathfrak{T}_{q,\zeta,s+2} + 5,837,056 \mathfrak{T}_{q,\zeta,s+3} \\
& + 2,877,456 \mathfrak{T}_{q,\zeta+1,s-3} + 345,294,720 \mathfrak{T}_{q,\zeta+1,s-2} + 3,427,050,096 \mathfrak{T}_{q,\zeta+1,s-1} + 6,951,933,696 \mathfrak{T}_{q,\zeta+1,s} \\
& + 3,427,050,096 \mathfrak{T}_{q,\zeta+1,s+1} + 345,294,720 \mathfrak{T}_{q,\zeta+1,s+2} + 2,877,456 \mathfrak{T}_{q,\zeta+1,s+3} + 289,920 \mathfrak{T}_{q,\zeta+2,s-3} + 34,790,400 \mathfrak{T}_{q,\zeta+2,s-2} \\
& + 345,294,720 \mathfrak{T}_{q,\zeta+2,s-1} + 700,446,720 \mathfrak{T}_{q,\zeta+2,s} + 345,294,720 \mathfrak{T}_{q,\zeta+2,s+1} + 34,790,400 \mathfrak{T}_{q,\zeta+2,s+2} + 289,920 \mathfrak{T}_{q,\zeta+2,s+3} \\
& + 2,416 \mathfrak{T}_{q,\zeta+3,s-3} + 289,920 \mathfrak{T}_{q,\zeta+3,s-2} + 2,877,456 \mathfrak{T}_{q,\zeta+3,s-1} + 5,837,056 \mathfrak{T}_{q,\zeta+3,s} + 2,877,456 \mathfrak{T}_{q,\zeta+3,s+1} \\
& + 289,920 \mathfrak{T}_{q,\zeta+3,s+2} + 2,416 \mathfrak{T}_{q,\zeta+3,s+3} + 1,191 \mathfrak{T}_{q+1,\zeta-3,s-3} + 142,920 \mathfrak{T}_{q+1,\zeta-3,s-2} + 1,418,481 \mathfrak{T}_{q+1,\zeta-3,s-1} \\
& + 2,877,456 \mathfrak{T}_{q+1,\zeta-3,s} + 1,418,481 \mathfrak{T}_{q+1,\zeta-3,s+1} + 142,920 \mathfrak{T}_{q+1,\zeta-3,s+2} + 1,191 \mathfrak{T}_{q+1,\zeta-3,s+3} + 142,920 \mathfrak{T}_{q+1,\zeta-2,s-3} \\
& + 17,150,400 \mathfrak{T}_{q+1,\zeta-2,s-2} + 170,217,720 \mathfrak{T}_{q+1,\zeta-2,s-1} + 345,294,720 \mathfrak{T}_{q+1,\zeta-2,s} + 170,217,720 \mathfrak{T}_{q+1,\zeta-2,s+1} \\
& + 17,150,400 \mathfrak{T}_{q+1,\zeta-2,s+2} + 142,920 \mathfrak{T}_{q+1,\zeta-2,s+3} + 1,418,481 \mathfrak{T}_{q+1,\zeta-1,s-3} + 170,217,720 \mathfrak{T}_{q+1,\zeta-1,s-2} \\
& + 1,689,410,871 \mathfrak{T}_{q+1,\zeta-1,s-1} + 3,427,050,096 \mathfrak{T}_{q+1,\zeta-1,s} + 1,689,410,871 \mathfrak{T}_{q+1,\zeta-1,s+1} + 170,217,720 \mathfrak{T}_{q+1,\zeta-1,s+2} \\
& + 1,418,481 \mathfrak{T}_{q+1,\zeta-1,s+3} + 2,877,456 \mathfrak{T}_{q+1,\zeta,s-3} + 345,294,720 \mathfrak{T}_{q+1,\zeta,s-2} + 3,427,050,096 \mathfrak{T}_{q+1,\zeta,s-1} \\
& + 6,951,933,696 \mathfrak{T}_{q+1,\zeta,s} + 3,427,050,096 \mathfrak{T}_{q+1,\zeta,s+1} + 345,294,720 \mathfrak{T}_{q+1,\zeta,s+2} + 2,877,456 \mathfrak{T}_{q+1,\zeta,s+3} \\
& + 1,418,481 \mathfrak{T}_{q+1,\zeta+1,s-3} + 170,217,720 \mathfrak{T}_{q+1,\zeta+1,s-2} + 1,689,410,871 \mathfrak{T}_{q+1,\zeta+1,s-1} + 3,427,050,096 \mathfrak{T}_{q+1,\zeta+1,s} \\
& + 1,689,410,871 \mathfrak{T}_{q+1,\zeta+1,s+1} + 170,217,720 \mathfrak{T}_{q+1,\zeta+1,s+2} + 1,418,481 \mathfrak{T}_{q+1,\zeta+1,s+3} + 142,920 \mathfrak{T}_{q+1,\zeta+2,s-3} \\
& + 17,150,400 \mathfrak{T}_{q+1,\zeta+2,s-2} + 170,217,720 \mathfrak{T}_{q+1,\zeta+2,s-1} + 345,294,720 \mathfrak{T}_{q+1,\zeta+2,s} + 170,217,720 \mathfrak{T}_{q+1,\zeta+2,s+1} \\
& + 17,150,400 \mathfrak{T}_{q+1,\zeta+2,s+2} + 142,920 \mathfrak{T}_{q+1,\zeta+2,s+3} + 1,191 \mathfrak{T}_{q+1,\zeta+3,s-3} + 142,920 \mathfrak{T}_{q+1,\zeta+3,s-2} + 1,418,481 \mathfrak{T}_{q+1,\zeta+3,s-1} \\
& + 2,877,456 \mathfrak{T}_{q+1,\zeta+3,s} + 1,418,481 \mathfrak{T}_{q+1,\zeta+3,s+1} + 142,920 \mathfrak{T}_{q+1,\zeta+3,s+2} + 1,191 \mathfrak{T}_{q+1,\zeta+3,s+3} + 120 \mathfrak{T}_{q+2,\zeta-3,s-3} \\
& + 14,400 \mathfrak{T}_{q+2,\zeta-3,s-2} + 142,920 \mathfrak{T}_{q+2,\zeta-3,s-1} + 289,920 \mathfrak{T}_{q+2,\zeta-3,s} + 142,920 \mathfrak{T}_{q+2,\zeta-3,s+1} + 14,400 \mathfrak{T}_{q+2,\zeta-3,s+2} \\
& + 120 \mathfrak{T}_{q+2,\zeta-3,s+3} + 14,400 \mathfrak{T}_{q+2,\zeta-2,s-3} + 1,728,000 \mathfrak{T}_{q+2,\zeta-2,s-2} + 17,150,400 \mathfrak{T}_{q+2,\zeta-2,s-1} + 34,790,400 \mathfrak{T}_{q+2,\zeta-2,s} \\
& + 17,150,400 \mathfrak{T}_{q+2,\zeta-2,s+1} + 1,728,000 \mathfrak{T}_{q+2,\zeta-2,s+2} + 14,400 \mathfrak{T}_{q+2,\zeta-2,s+3} + 142,920 \mathfrak{T}_{q+2,\zeta-1,s-3} + 17,150,400 \mathfrak{T}_{q+2,\zeta-1,s-2} \\
& + 170,217,720 \mathfrak{T}_{q+2,\zeta-1,s-1} + 345,294,720 \mathfrak{T}_{q+2,\zeta-1,s} + 170,217,720 \mathfrak{T}_{q+2,\zeta-1,s+1} + 17,150,400 \mathfrak{T}_{q+2,\zeta-1,s+2} \\
& + 142,920 \mathfrak{T}_{q+2,\zeta-1,s+3} + 289,920 \mathfrak{T}_{q+2,\zeta,s-3} + 34,790,400 \mathfrak{T}_{q+2,\zeta,s-2} + 345,294,720 \mathfrak{T}_{q+2,\zeta,s-1} + 700,446,720 \mathfrak{T}_{q+2,\zeta,s} \\
& + 345,294,720 \mathfrak{T}_{q+2,\zeta,s+1} + 34,790,400 \mathfrak{T}_{q+2,\zeta,s+2} + 289,920 \mathfrak{T}_{q+2,\zeta,s+3} + 142,920 \mathfrak{T}_{q+2,\zeta+1,s-3} + 17,150,400 \mathfrak{T}_{q+2,\zeta+1,s-2} \\
& + 170,217,720 \mathfrak{T}_{q+2,\zeta+1,s-1} + 345,294,720 \mathfrak{T}_{q+2,\zeta+1,s} + 170,217,720 \mathfrak{T}_{q+2,\zeta+1,s+1} + 17,150,400 \mathfrak{T}_{q+2,\zeta+1,s+2} \\
& + 142,920 \mathfrak{T}_{q+2,\zeta+1,s+3} + 14,400 \mathfrak{T}_{q+2,\zeta+2,s-3} + 1,728,000 \mathfrak{T}_{q+2,\zeta+2,s-2} + 17,150,400 \mathfrak{T}_{q+2,\zeta+2,s-1} + 34,790,400 \mathfrak{T}_{q+2,\zeta+2,s} \\
& + 17,150,400 \mathfrak{T}_{q+2,\zeta+2,s+1} + 1,728,000 \mathfrak{T}_{q+2,\zeta+2,s+2} + 14,400 \mathfrak{T}_{q+2,\zeta+2,s+3} + 120 \mathfrak{T}_{q+2,\zeta+3,s-3} + 14,400 \mathfrak{T}_{q+2,\zeta+3,s-2} \\
& + 142,920 \mathfrak{T}_{q+2,\zeta+3,s-1} + 289,920 \mathfrak{T}_{q+2,\zeta+3,s} + 142,920 \mathfrak{T}_{q+2,\zeta+3,s+1} + 14,400 \mathfrak{T}_{q+2,\zeta+3,s+2} + 120 \mathfrak{T}_{q+2,\zeta+3,s+3} + \mathfrak{T}_{q+3,\zeta-3,s-3} \\
& + 120 \mathfrak{T}_{q+3,\zeta-3,s-2} + 1,191 \mathfrak{T}_{q+3,\zeta-3,s-1} + 2,416 \mathfrak{T}_{q+3,\zeta-3,s} + 1,191 \mathfrak{T}_{q+3,\zeta-3,s+1} + 120 \mathfrak{T}_{q+3,\zeta-3,s+2} \\
& + \mathfrak{T}_{q+3,\zeta-3,s+3} + 120 \mathfrak{T}_{q+3,\zeta-2,s-3} + 14,400 \mathfrak{T}_{q+3,\zeta-2,s-2} + 142,920 \mathfrak{T}_{q+3,\zeta-2,s-1}
\end{aligned}
\tag{9}$$

$$\begin{aligned}
& + 289,920 \mathfrak{T}_{q+3,\zeta-2,s} + 142,920 \mathfrak{T}_{q+3,\zeta-2,s+1} + 14,400 \mathfrak{T}_{q+3,\zeta-2,s+2} + 120 \mathfrak{T}_{q+3,\zeta-2,s+3} + 1,191 \mathfrak{T}_{q+3,\zeta-1,s-3} + 142,920 \mathfrak{T}_{q+3,\zeta-1,s-2} \\
& + 1,418,481 \mathfrak{T}_{q+3,\zeta-1,s-1} + 2,877,456 \mathfrak{T}_{q+3,\zeta-1,s} + 1,418,481 \mathfrak{T}_{q+3,\zeta-1,s+1} + 142,920 \mathfrak{T}_{q+3,\zeta-1,s+2} + 1,191 \mathfrak{T}_{q+3,\zeta-1,s+3} \\
& + 2,416 \mathfrak{T}_{q+3,\zeta,s-3} + 289,920 \mathfrak{T}_{q+3,\zeta,s-2} + 2,877,456 \mathfrak{T}_{q+3,\zeta,s-1} + 5,837,056 \mathfrak{T}_{q+3,\zeta,s} + 2,877,456 \mathfrak{T}_{q+3,\zeta,s+1} + 289,920 \mathfrak{T}_{q+3,\zeta,s+2} \\
& + 2,416 \mathfrak{T}_{q+3,\zeta,s+3} + 1,191 \mathfrak{T}_{q+3,\zeta+1,s-3} + 142,920 \mathfrak{T}_{q+3,\zeta+1,s-2} + 1,418,481 \mathfrak{T}_{q+3,\zeta+1,s-1} + 2,877,456 \mathfrak{T}_{q+3,\zeta+1,s} \\
& + 1,418,481 \mathfrak{T}_{q+3,\zeta+1,s+1} + 142,920 \mathfrak{T}_{q+3,\zeta+1,s+2} + 1,191 \mathfrak{T}_{q+3,\zeta+1,s+3} + 120 \mathfrak{T}_{q+3,\zeta+2,s-3} + 14,400 \mathfrak{T}_{q+3,\zeta+2,s-2} \\
& + 142,920 \mathfrak{T}_{q+3,\zeta+2,s-1} + 289,920 \mathfrak{T}_{q+3,\zeta+2,s} + 142,920 \mathfrak{T}_{q+3,\zeta+2,s+1} + 14,400 \mathfrak{T}_{q+3,\zeta+2,s+2} + 120 \mathfrak{T}_{q+3,\zeta+2,s+3} + \mathfrak{T}_{q+3,\zeta+3,s-3} \\
& + 120 \mathfrak{T}_{q+3,\zeta+3,s-2} + 1,191 \mathfrak{T}_{q+3,\zeta+3,s-1} + 2,416 \mathfrak{T}_{q+3,\zeta+3,s} + 1,191 \mathfrak{T}_{q+3,\zeta+3,s+1} + 120 \mathfrak{T}_{q+3,\zeta+3,s+2} + \mathfrak{T}_{q+3,\zeta+3,s+3}.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial H_{q,\zeta,s}}{\partial x} = & -\frac{7}{h} (\mathfrak{T}_{q-3,\zeta-3,s-3} + 120 \mathfrak{T}_{q-3,\zeta-3,s-2} + 1,191 \mathfrak{T}_{q-3,\zeta-3,s-1} + 2,416 \mathfrak{T}_{q-3,\zeta-3,s} + 1,191 \mathfrak{T}_{q-3,\zeta-3,s+1} \\
& + 120 \mathfrak{T}_{q-3,\zeta-3,s+2} + \mathfrak{T}_{q-3,\zeta-3,s+3} + 120 \mathfrak{T}_{q-3,\zeta-2,s-3} + 14,400 \mathfrak{T}_{q-3,\zeta-2,s-2} + 142,920 \mathfrak{T}_{q-3,\zeta-2,s-1} + 289,920 \mathfrak{T}_{q-3,\zeta-2,s} \\
& + 142,920 \mathfrak{T}_{q-3,\zeta-2,s+1} + 14,400 \mathfrak{T}_{q-3,\zeta-2,s+2} + 120 \mathfrak{T}_{q-3,\zeta-2,s+3} + 1,191 \mathfrak{T}_{q-3,\zeta-1,s-3} + 142,920 \mathfrak{T}_{q-3,\zeta-1,s-2} \\
& + 1,418,481 \mathfrak{T}_{q-3,\zeta-1,s-1} + 2,877,456 \mathfrak{T}_{q-3,\zeta-1,s} + 1,418,481 \mathfrak{T}_{q-3,\zeta-1,s+1} + 142,920 \mathfrak{T}_{q-3,\zeta-1,s+2} + 1,191 \mathfrak{T}_{q-3,\zeta-1,s+3} \\
& + 2,416 \mathfrak{T}_{q-3,\zeta,s-3} + 289,920 \mathfrak{T}_{q-3,\zeta,s-2} + 2,877,456 \mathfrak{T}_{q-3,\zeta,s-1} + 5,837,056 \mathfrak{T}_{q-3,\zeta,s} + 2,877,456 \mathfrak{T}_{q-3,\zeta,s+1} \\
& + 289,920 \mathfrak{T}_{q-3,\zeta,s+2} + 2,416 \mathfrak{T}_{q-3,\zeta,s+3} + 1,191 \mathfrak{T}_{q-3,\zeta+1,s-3} + 142,920 \mathfrak{T}_{q-3,\zeta+1,s-2} + 1,418,481 \mathfrak{T}_{q-3,\zeta+1,s-1} \\
& + 2,877,456 \mathfrak{T}_{q-3,\zeta+1,s} + 1,418,481 \mathfrak{T}_{q-3,\zeta+1,s+1} + 142,920 \mathfrak{T}_{q-3,\zeta+1,s+2} + 1,191 \mathfrak{T}_{q-3,\zeta+1,s+3} + 120 \mathfrak{T}_{q-3,\zeta+2,s-3} \\
& + 14,400 \mathfrak{T}_{q-3,\zeta+2,s-2} + 142,920 \mathfrak{T}_{q-3,\zeta+2,s-1} + 289,920 \mathfrak{T}_{q-3,\zeta+2,s} + 142,920 \mathfrak{T}_{q-3,\zeta+2,s+1} \\
& + 14,400 \mathfrak{T}_{q-3,\zeta+2,s+2} + 120 \mathfrak{T}_{q-3,\zeta+2,s+3} + \mathfrak{T}_{q-3,\zeta+3,s-3} + 120 \mathfrak{T}_{q-3,\zeta+3,s-2} + 1,191 \mathfrak{T}_{q-3,\zeta+3,s-1} + 2,416 \mathfrak{T}_{q-3,\zeta+3,s} \\
& + 1,191 \mathfrak{T}_{q-3,\zeta+3,s+1} + 120 \mathfrak{T}_{q-3,\zeta+3,s+2} + \mathfrak{T}_{q-3,\zeta+3,s+3} + 56 \mathfrak{T}_{q-2,\zeta-3,s-3} + 6,720 \mathfrak{T}_{q-2,\zeta-3,s-2} + 66,696 \mathfrak{T}_{q-2,\zeta-3,s-1} \\
& + 135,296 \mathfrak{T}_{q-2,\zeta-3,s} + 66,696 \mathfrak{T}_{q-2,\zeta-3,s+1} + 6,720 \mathfrak{T}_{q-2,\zeta-3,s+2} + 56 \mathfrak{T}_{q-2,\zeta-3,s+3} + 6,720 \mathfrak{T}_{q-2,\zeta-2,s-3} \\
& + 806,400 \mathfrak{T}_{q-2,\zeta-2,s-2} + 8,003,520 \mathfrak{T}_{q-2,\zeta-2,s-1} + 16,235,520 \mathfrak{T}_{q-2,\zeta-2,s} + 8,003,520 \mathfrak{T}_{q-2,\zeta-2,s+1} + 806,400 \mathfrak{T}_{q-2,\zeta-2,s+2} \\
& + 6,720 \mathfrak{T}_{q-2,\zeta-2,s+3} + 66,696 \mathfrak{T}_{q-2,\zeta-1,s-3} + 8,003,520 \mathfrak{T}_{q-2,\zeta-1,s-2} + 79,434,936 \mathfrak{T}_{q-2,\zeta-1,s-1} + 161,137,536 \mathfrak{T}_{q-2,\zeta-1,s} \\
& + 79,434,936 \mathfrak{T}_{q-2,\zeta-1,s+1} + 8,003,520 \mathfrak{T}_{q-2,\zeta-1,s+2} + 66,696 \mathfrak{T}_{q-2,\zeta-1,s+3} + 135,296 \mathfrak{T}_{q-2,\zeta,s-3} + 16,235,520 \mathfrak{T}_{q-2,\zeta,s-2} \\
& + 161,137,536 \mathfrak{T}_{q-2,\zeta,s-1} + 326,875,136 \mathfrak{T}_{q-2,\zeta,s} + 161,137,536 \mathfrak{T}_{q-2,\zeta,s+1} + 16,235,520 \mathfrak{T}_{q-2,\zeta,s+2} + 135,296 \mathfrak{T}_{q-2,\zeta,s+3} \\
& + 66,696 \mathfrak{T}_{q-2,\zeta+1,s-3} + 8,003,520 \mathfrak{T}_{q-2,\zeta+1,s-2} + 79,434,936 \mathfrak{T}_{q-2,\zeta+1,s-1} + 161,137,536 \mathfrak{T}_{q-2,\zeta+1,s} + 79,434,936 \mathfrak{T}_{q-2,\zeta+1,s+1} \\
& + 8,003,520 \mathfrak{T}_{q-2,\zeta+1,s+2} + 66,696 \mathfrak{T}_{q-2,\zeta+1,s+3} + 6,720 \mathfrak{T}_{q-2,\zeta+2,s-3} + 806,400 \mathfrak{T}_{q-2,\zeta+2,s-2} + 8,003,520 \mathfrak{T}_{q-2,\zeta+2,s-1} \\
& + 16,235,520 \mathfrak{T}_{q-2,\zeta+2,s} + 8,003,520 \mathfrak{T}_{q-2,\zeta+2,s+1} + 806,400 \mathfrak{T}_{q-2,\zeta+2,s+2} + 6,720 \mathfrak{T}_{q-2,\zeta+2,s+3} + 56 \mathfrak{T}_{q-2,\zeta+3,s-3} \\
& + 6,720 \mathfrak{T}_{q-2,\zeta+3,s-2} + 66,696 \mathfrak{T}_{q-2,\zeta+3,s-1} + 135,296 \mathfrak{T}_{q-2,\zeta+3,s} + 66,696 \mathfrak{T}_{q-2,\zeta+3,s+1} \\
& + 6,720 \mathfrak{T}_{q-2,\zeta+3,s+2} + 56 \mathfrak{T}_{q-2,\zeta+3,s+3} + 245 \mathfrak{T}_{q-1,\zeta-3,s-3} + 29,400 \mathfrak{T}_{q-1,\zeta-3,s-2} + 291,795 \mathfrak{T}_{q-1,\zeta-3,s-1} + 591,920 \mathfrak{T}_{q-1,\zeta-3,s} \\
& + 291,795 \mathfrak{T}_{q-1,\zeta-3,s+1} + 29,400 \mathfrak{T}_{q-1,\zeta-3,s+2} + 245 \mathfrak{T}_{q-1,\zeta-3,s+3} + 29,400 \mathfrak{T}_{q-1,\zeta-2,s-3} + 3,528,000 \mathfrak{T}_{q-1,\zeta-2,s-2} \\
& + 35,015,400 \mathfrak{T}_{q-1,\zeta-2,s-1} + 71,030,400 \mathfrak{T}_{q-1,\zeta-2,s} + 35,015,400 \mathfrak{T}_{q-1,\zeta-2,s+1} + 3,528,000 \mathfrak{T}_{q-1,\zeta-2,s+2} + 29,400 \mathfrak{T}_{q-1,\zeta-2,s+3} \\
& + 291,795 \mathfrak{T}_{q-1,\zeta-1,s-3} + 35,015,400 \mathfrak{T}_{q-1,\zeta-1,s-2} + 347,527,845 \mathfrak{T}_{q-1,\zeta-1,s-1} + 704,976,720 \mathfrak{T}_{q-1,\zeta-1,s} \\
& + 347,527,845 \mathfrak{T}_{q-1,\zeta-1,s+1} + 35,015,400 \mathfrak{T}_{q-1,\zeta-1,s+2} + 291,795 \mathfrak{T}_{q-1,\zeta-1,s+3} + 591,920 \mathfrak{T}_{q-1,\zeta,s-3} \\
& + 71,030,400 \mathfrak{T}_{q-1,\zeta,s-2} + 704,976,720 \mathfrak{T}_{q-1,\zeta,s-1} + 1,430,078,720 \mathfrak{T}_{q-1,\zeta,s} + 704,976,720 \mathfrak{T}_{q-1,\zeta,s+1} \\
& + 71,030,400 \mathfrak{T}_{q-1,\zeta,s+2} + 591,920 \mathfrak{T}_{q-1,\zeta,s+3} + 291,795 \mathfrak{T}_{q-1,\zeta+1,s-3} + 35,015,400 \mathfrak{T}_{q-1,\zeta+1,s-2} + 347,527,845 \mathfrak{T}_{q-1,\zeta+1,s-1}
\end{aligned}$$

$$\begin{aligned}
& + 704,976,720 \mathfrak{T}_{q-1,\zeta+1,s} + 347,527,845 \mathfrak{T}_{q-1,\zeta+1,s+1} + 35,015,400 \mathfrak{T}_{q-1,\zeta+1,s+2} + 291,795 \mathfrak{T}_{q-1,\zeta+1,s+3} + 29,400 \mathfrak{T}_{q-1,\zeta+2,s-3} \\
& + 3,528,000 \mathfrak{T}_{q-1,\zeta+2,s-2} + 35,015,400 \mathfrak{T}_{q-1,\zeta+2,s-1} + 71,030,400 \mathfrak{T}_{q-1,\zeta+2,s} + 35,015,400 \mathfrak{T}_{q-1,\zeta+2,s+1} \\
& + 3,528,000 \mathfrak{T}_{q-1,\zeta+2,s+2} + 29,400 \mathfrak{T}_{q-1,\zeta+2,s+3} + 245 \mathfrak{T}_{q-1,\zeta+3,s-3} + 29,400 \mathfrak{T}_{q-1,\zeta+3,s-2} + 291,795 \mathfrak{T}_{q-1,\zeta+3,s-1} \\
& + 591,920 \mathfrak{T}_{q-1,\zeta+3,s} + 291,795 \mathfrak{T}_{q-1,\zeta+3,s+1} + 29,400 \mathfrak{T}_{q-1,\zeta+3,s+2} + 245 \mathfrak{T}_{q-1,\zeta+3,s+3} - 245 \mathfrak{T}_{q+1,\zeta-3,s-3} \\
& - 29,400 \mathfrak{T}_{q+1,\zeta-3,s-2} - 291,795 \mathfrak{T}_{q+1,\zeta-3,s-1} - 591,920 \mathfrak{T}_{q+1,\zeta-3,s} - 291,795 \mathfrak{T}_{q+1,\zeta-3,s+1} - 29,400 \mathfrak{T}_{q+1,\zeta-3,s+2} \\
& - 245 \mathfrak{T}_{q+1,\zeta-3,s+3} - 29,400 \mathfrak{T}_{q+1,\zeta-2,s-3} - 3,528,000 \mathfrak{T}_{q+1,\zeta-2,s-2} - 35,015,400 \mathfrak{T}_{q+1,\zeta-2,s-1} - 71,030,400 \mathfrak{T}_{q+1,\zeta-2,s} \\
& - 35,015,400 \mathfrak{T}_{q+1,\zeta-2,s+1} - 3,528,000 \mathfrak{T}_{q+1,\zeta-2,s+2} - 29,400 \mathfrak{T}_{q+1,\zeta-2,s+3} - 291,795 \mathfrak{T}_{q+1,\zeta-1,s-3} - 35,015,400 \mathfrak{T}_{q+1,\zeta-1,s-2} \\
& - 347,527,845 \mathfrak{T}_{q+1,\zeta-1,s-1} - 704,976,720 \mathfrak{T}_{q+1,\zeta-1,s} - 347,527,845 \mathfrak{T}_{q+1,\zeta-1,s+1} - 35,015,400 \mathfrak{T}_{q+1,\zeta-1,s+2} \\
& - 291,795 \mathfrak{T}_{q+1,\zeta-1,s+3} - 591,920 \mathfrak{T}_{q+1,\zeta,s-3} - 71,030,400 \mathfrak{T}_{q+1,\zeta,s-2} - 704,976,720 \mathfrak{T}_{q+1,\zeta,s-1} - 1,430,078,720 \mathfrak{T}_{q+1,\zeta,s} \\
& - 704,976,720 \mathfrak{T}_{q+1,\zeta,s+1} - 71,030,400 \mathfrak{T}_{q+1,\zeta,s+2} - 591,920 \mathfrak{T}_{q+1,\zeta,s+3} - 291,795 \mathfrak{T}_{q+1,\zeta+1,s-3} - 35,015,400 \mathfrak{T}_{q+1,\zeta+1,s-2} \\
& - 347,527,845 \mathfrak{T}_{q+1,\zeta+1,s-1} - 704,976,720 \mathfrak{T}_{q+1,\zeta+1,s} - 347,527,845 \mathfrak{T}_{q+1,\zeta+1,s+1} - 35,015,400 \mathfrak{T}_{q+1,\zeta+1,s+2} \\
& - 291,795 \mathfrak{T}_{q+1,\zeta+1,s+3} - 29,400 \mathfrak{T}_{q+1,\zeta+2,s-3} - 3,528,000 \mathfrak{T}_{q+1,\zeta+2,s-2} - 35,015,400 \mathfrak{T}_{q+1,\zeta+2,s-1} - 71,030,400 \mathfrak{T}_{q+1,\zeta+2,s} \\
& - 35,015,400 \mathfrak{T}_{q+1,\zeta+2,s+1} - 3,528,000 \mathfrak{T}_{q+1,\zeta+2,s+2} - 29,400 \mathfrak{T}_{q+1,\zeta+2,s+3} - 245 \mathfrak{T}_{q+1,\zeta+3,s-3} - 29,400 \mathfrak{T}_{q+1,\zeta+3,s-2} \\
& - 291,795 \mathfrak{T}_{q+1,\zeta+3,s-1} - 591,920 \mathfrak{T}_{q+1,\zeta+3,s} - 291,795 \mathfrak{T}_{q+1,\zeta+3,s+1} - 29,400 \mathfrak{T}_{q+1,\zeta+3,s+2} \\
& - 245 \mathfrak{T}_{q+1,\zeta+3,s+3} - 56 \mathfrak{T}_{q+2,\zeta-3,s-3} - 6,720 \mathfrak{T}_{q+2,\zeta-3,s-2} - 66,696 \mathfrak{T}_{q+2,\zeta-3,s-1} - 135,296 \mathfrak{T}_{q+2,\zeta-3,s} - 66,696 \mathfrak{T}_{q+2,\zeta-3,s+1} \\
& - 6,720 \mathfrak{T}_{q+2,\zeta-3,s+2} - 56 \mathfrak{T}_{q+2,\zeta-3,s+3} - 6,720 \mathfrak{T}_{q+2,\zeta-2,s-3} - 806,400 \mathfrak{T}_{q+2,\zeta-2,s-2} - 8,003,520 \mathfrak{T}_{q+2,\zeta-2,s-1} \\
& - 16,235,520 \mathfrak{T}_{q+2,\zeta-2,s} - 8,003,520 \mathfrak{T}_{q+2,\zeta-2,s+1} - 806,400 \mathfrak{T}_{q+2,\zeta-2,s+2} - 6,720 \mathfrak{T}_{q+2,\zeta-2,s+3} - 66,696 \mathfrak{T}_{q+2,\zeta-1,s-3} \\
& - 8,003,520 \mathfrak{T}_{q+2,\zeta-1,s-2} - 79,434,936 \mathfrak{T}_{q+2,\zeta-1,s-1} - 161,137,536 \mathfrak{T}_{q+2,\zeta-1,s} - 79,434,936 \mathfrak{T}_{q+2,\zeta-1,s+1} \\
& - 8,003,520 \mathfrak{T}_{q+2,\zeta-1,s+2} - 66,696 \mathfrak{T}_{q+2,\zeta-1,s+3} - 135,296 \mathfrak{T}_{q+2,\zeta,s-3} - 16,235,520 \mathfrak{T}_{q+2,\zeta,s-2} - 161,137,536 \mathfrak{T}_{q+2,\zeta,s-1} \\
& - 326,875,136 \mathfrak{T}_{q+2,\zeta,s} - 161,137,536 \mathfrak{T}_{q+2,\zeta,s+1} - 16,235,520 \mathfrak{T}_{q+2,\zeta,s+2} - 135,296 \mathfrak{T}_{q+2,\zeta,s+3} - 66,696 \mathfrak{T}_{q+2,\zeta+1,s-3} \\
& - 8,003,520 \mathfrak{T}_{q+2,\zeta+1,s-2} - 79,434,936 \mathfrak{T}_{q+2,\zeta+1,s-1} - 161,137,536 \mathfrak{T}_{q+2,\zeta+1,s} - 79,434,936 \mathfrak{T}_{q+2,\zeta+1,s+1} \\
& - 8,003,520 \mathfrak{T}_{q+2,\zeta+1,s+2} - 66,696 \mathfrak{T}_{q+2,\zeta+1,s+3} - 6,720 \mathfrak{T}_{q+2,\zeta+2,s-3} - 806,400 \mathfrak{T}_{q+2,\zeta+2,s-2} - 8,003,520 \mathfrak{T}_{q+2,\zeta+2,s-1} \\
& - 16,235,520 \mathfrak{T}_{q+2,\zeta+2,s} - 8,003,520 \mathfrak{T}_{q+2,\zeta+2,s+1} - 806,400 \mathfrak{T}_{q+2,\zeta+2,s+2} - 6,720 \mathfrak{T}_{q+2,\zeta+2,s+3} - 56 \mathfrak{T}_{q+2,\zeta+3,s-3} \\
& - 6,720 \mathfrak{T}_{q+2,\zeta+3,s-2} - 66,696 \mathfrak{T}_{q+2,\zeta+3,s-1} - 135,296 \mathfrak{T}_{q+2,\zeta+3,s} - 66,696 \mathfrak{T}_{q+2,\zeta+3,s+1} \\
& - 6,720 \mathfrak{T}_{q+2,\zeta+3,s+2} - 56 \mathfrak{T}_{q+2,\zeta+3,s+3} - \mathfrak{T}_{q+3,\zeta-3,s-3} - 120 \mathfrak{T}_{q+3,\zeta-3,s-2} - 1,191 \mathfrak{T}_{q+3,\zeta-3,s-1} - 2,416 \mathfrak{T}_{q+3,\zeta-3,s} \\
& - 1,191 \mathfrak{T}_{q+3,\zeta-3,s+1} - 120 \mathfrak{T}_{q+3,\zeta-3,s+2} - \mathfrak{T}_{q+3,\zeta-3,s+3} - 120 \mathfrak{T}_{q+3,\zeta-2,s-3} - 14,400 \mathfrak{T}_{q+3,\zeta-2,s-2} - 142,920 \mathfrak{T}_{q+3,\zeta-2,s-1} \\
& - 289,920 \mathfrak{T}_{q+3,\zeta-2,s} - 142,920 \mathfrak{T}_{q+3,\zeta-2,s+1} - 14,400 \mathfrak{T}_{q+3,\zeta-2,s+2} - 120 \mathfrak{T}_{q+3,\zeta-2,s+3} - 1,191 \mathfrak{T}_{q+3,\zeta-1,s-3} \\
& - 142,920 \mathfrak{T}_{q+3,\zeta-1,s-2} - 1,418,481 \mathfrak{T}_{q+3,\zeta-1,s-1} - 2,877,456 \mathfrak{T}_{q+3,\zeta-1,s} - 1,418,481 \mathfrak{T}_{q+3,\zeta-1,s+1} - 142,920 \mathfrak{T}_{q+3,\zeta-1,s+2} \\
& - 1,191 \mathfrak{T}_{q+3,\zeta-1,s+3} - 2,416 \mathfrak{T}_{q+3,\zeta,s-3} - 289,920 \mathfrak{T}_{q+3,\zeta,s-2} - 2,877,456 \mathfrak{T}_{q+3,\zeta,s-1} - 5,837,056 \mathfrak{T}_{q+3,\zeta,s} \\
& - 2,877,456 \mathfrak{T}_{q+3,\zeta,s+1} - 289,920 \mathfrak{T}_{q+3,\zeta,s+2} - 2,416 \mathfrak{T}_{q+3,\zeta,s+3} - 1,191 \mathfrak{T}_{q+3,\zeta+1,s-3} - 142,920 \mathfrak{T}_{q+3,\zeta+1,s-2} \\
& - 1,418,481 \mathfrak{T}_{q+3,\zeta+1,s-1} - 2,877,456 \mathfrak{T}_{q+3,\zeta+1,s} - 1,418,481 \mathfrak{T}_{q+3,\zeta+1,s+1} - 142,920 \mathfrak{T}_{q+3,\zeta+1,s+2} - 1,191 \mathfrak{T}_{q+3,\zeta+1,s+3} \\
& - 120 \mathfrak{T}_{q+3,\zeta+2,s-3} - 14,400 \mathfrak{T}_{q+3,\zeta+2,s-2} - 142,920 \mathfrak{T}_{q+3,\zeta+2,s-1} - 289,920 \mathfrak{T}_{q+3,\zeta+2,s} - 142,920 \mathfrak{T}_{q+3,\zeta+2,s+1} \\
& - 14,400 \mathfrak{T}_{q+3,\zeta+2,s+2} - 120 \mathfrak{T}_{q+3,\zeta+2,s+3} - \mathfrak{T}_{q+3,\zeta+3,\sigma-3} - 120 \mathfrak{T}_{q+3,\zeta+3,\sigma-2} - 1,191 \mathfrak{T}_{q+3,\zeta+3,\sigma-1} - 2,416 \mathfrak{T}_{q+3,\zeta+3,\sigma} \\
& - 1,191 \mathfrak{T}_{q+3,\zeta+3,\sigma+1} - 120 \mathfrak{T}_{q+3,\zeta+3,\sigma+2} - \mathfrak{T}_{q+3,\zeta+3,\sigma+3}),
\end{aligned} \tag{10}$$

$$\begin{aligned}
\frac{\partial H_{0,c,s}}{\partial y} = & -\frac{7}{k}(\mathfrak{T}_{q-3,c-3,s-3} + 120\mathfrak{T}_{q-3,c-3,s-2} + 1,191\mathfrak{T}_{q-3,c-3,s-1} + 2,416\mathfrak{T}_{q-3,c-3,s} + 1,191\mathfrak{T}_{q-3,c-3,s+1} + 120\mathfrak{T}_{q-3,c-3,s+2} + \mathfrak{T}_{q-3,c-3,s+3} + 56\mathfrak{T}_{q-3,c-2,s-3} \\
& + 6,720\mathfrak{T}_{q-3,c-2,s-2} + 66,696\mathfrak{T}_{q-3,c-2,s-1} + 135,296\mathfrak{T}_{q-3,c-2,s} + 66,696\mathfrak{T}_{q-3,c-2,s+1} + 6,720\mathfrak{T}_{q-3,c-2,s+2} + 56\mathfrak{T}_{q-3,c-2,s+3} + 245\mathfrak{T}_{q-3,c-1,s-3} \\
& + 29,400\mathfrak{T}_{q-3,c-1,s-2} + 291,795\mathfrak{T}_{q-3,c-1,s-1} + 591,920\mathfrak{T}_{q-3,c-1,s} + 291,795\mathfrak{T}_{q-3,c-1,s+1} + 29,400\mathfrak{T}_{q-3,c-1,s+2} + 245\mathfrak{T}_{q-3,c-1,s+3} - 245\mathfrak{T}_{q-3,c+1,s-3} \\
& - 29,400\mathfrak{T}_{q-3,c+1,s-2} - 291,795\mathfrak{T}_{q-3,c+1,s-1} - 591,920\mathfrak{T}_{q-3,c+1,s} - 291,795\mathfrak{T}_{q-3,c+1,s+1} - 29,400\mathfrak{T}_{q-3,c+1,s+2} - 245\mathfrak{T}_{q-3,c+1,s+3} - 56\mathfrak{T}_{q-3,c+2,s-3} \\
& - 6,720\mathfrak{T}_{q-3,c+2,s-2} - 66,696\mathfrak{T}_{q-3,c+2,s-1} - 135,296\mathfrak{T}_{q-3,c+2,s} - 66,696\mathfrak{T}_{q-3,c+2,s+1} - 6,720\mathfrak{T}_{q-3,c+2,s+2} - 56\mathfrak{T}_{q-3,c+2,s+3} - \mathfrak{T}_{q-3,c+3,s-3} \\
& - 120\mathfrak{T}_{q-3,c+3,s-2} - 1,191\mathfrak{T}_{q-3,c+3,s-1} - 2,416\mathfrak{T}_{q-3,c+3,s} - 1,191\mathfrak{T}_{q-3,c+3,s+1} - 120\mathfrak{T}_{q-3,c+3,s+2} - \mathfrak{T}_{q-3,c+3,s+3} + 120\mathfrak{T}_{q-2,c-3,s-3} \\
& + 14,400\mathfrak{T}_{q-2,c-3,s-2} + 142,920\mathfrak{T}_{q-2,c-3,s-1} + 289,920\mathfrak{T}_{q-2,c-3,s} + 6,720\mathfrak{T}_{q-2,c-3,s+1} + 806,400\mathfrak{T}_{q-2,c-3,s+2} + 8,003,520\mathfrak{T}_{q-2,c-3,s+3} \\
& + 16,235,520\mathfrak{T}_{q-2,c-2,s} + 8,003,520\mathfrak{T}_{q-2,c-2,s+1} + 806,400\mathfrak{T}_{q-2,c-2,s+2} + 6,720\mathfrak{T}_{q-2,c-2,s+3} + 29,400\mathfrak{T}_{q-2,c-1,s} + 3,528,000\mathfrak{T}_{q-2,c-1,s+1} \\
& + 35,015,400\mathfrak{T}_{q-2,c-1,s+2} + 71,030,400\mathfrak{T}_{q-2,c-1,s+3} + 35,015,400\mathfrak{T}_{q-2,c-1,s+4} + 3,528,000\mathfrak{T}_{q-2,c-1,s+5} + 29,400\mathfrak{T}_{q-2,c-1,s+6} - 29,400\mathfrak{T}_{q-2,c+1,s-3} \\
& - 3,528,000\mathfrak{T}_{q-2,c+1,s-2} - 35,015,400\mathfrak{T}_{q-2,c+1,s-1} - 71,030,400\mathfrak{T}_{q-2,c+1,s} - 35,015,400\mathfrak{T}_{q-2,c+1,s+1} - 3,528,000\mathfrak{T}_{q-2,c+1,s+2} \\
& - 29,400\mathfrak{T}_{q-2,c+1,s+3} - 6,720\mathfrak{T}_{q-2,c+1,s+4} - 806,400\mathfrak{T}_{q-2,c+1,s+5} - 8,003,520\mathfrak{T}_{q-2,c+1,s+6} - 16,235,520\mathfrak{T}_{q-2,c+2,s-3} - 8,003,520\mathfrak{T}_{q-2,c+2,s-2} \\
& - 806,400\mathfrak{T}_{q-2,c+2,s-1} - 6,720\mathfrak{T}_{q-2,c+2,s} - 120\mathfrak{T}_{q-2,c+2,s+1} - 14,400\mathfrak{T}_{q-2,c+2,s+2} - 142,920\mathfrak{T}_{q-2,c+2,s+3} - 289,920\mathfrak{T}_{q-2,c+2,s+4} \\
& - 142,920\mathfrak{T}_{q-2,c+3,s-1} - 14,400\mathfrak{T}_{q-2,c+3,s} - 120\mathfrak{T}_{q-2,c+3,s+1} + 1,191\mathfrak{T}_{q-1,c-3,s-3} + 142,920\mathfrak{T}_{q-1,c-3,s-2} + 1,418,481\mathfrak{T}_{q-1,c-3,s-1} \\
& + 2,877,456\mathfrak{T}_{q-1,c-3,s} + 1,418,481\mathfrak{T}_{q-1,c-3,s+1} + 142,920\mathfrak{T}_{q-1,c-3,s+2} + 1,191\mathfrak{T}_{q-1,c-3,s+3} + 66,696\mathfrak{T}_{q-1,c-2,s-3} + 8,003,520\mathfrak{T}_{q-1,c-2,s-2} \\
& + 79,434,936\mathfrak{T}_{q-1,c-2,s-1} + 161,137,536\mathfrak{T}_{q-1,c-2,s} + 79,434,936\mathfrak{T}_{q-1,c-2,s+1} + 8,003,520\mathfrak{T}_{q-1,c-2,s+2} + 66,696\mathfrak{T}_{q-1,c-2,s+3} + 291,795\mathfrak{T}_{q-1,c-1,s-3} \\
& + 35,015,400\mathfrak{T}_{q-1,c-1,s-2} + 347,527,845\mathfrak{T}_{q-1,c-1,s-1} + 704,976,720\mathfrak{T}_{q-1,c-1,s} + 347,527,845\mathfrak{T}_{q-1,c-1,s+1} + 35,015,400\mathfrak{T}_{q-1,c-1,s+2} \\
& + 291,795\mathfrak{T}_{q-1,c-1,s+3} - 291,795\mathfrak{T}_{q-1,c+1,s-3} - 35,015,400\mathfrak{T}_{q-1,c+1,s-2} - 347,527,845\mathfrak{T}_{q-1,c+1,s-1} - 704,976,720\mathfrak{T}_{q-1,c+1,s} \\
& - 347,527,845\mathfrak{T}_{q-1,c+1,s+1} - 35,015,400\mathfrak{T}_{q-1,c+1,s+2} - 291,795\mathfrak{T}_{q-1,c+1,s+3} - 66,696\mathfrak{T}_{q-1,c+2,s-3} - 8,003,520\mathfrak{T}_{q-1,c+2,s-2} - 79,434,936\mathfrak{T}_{q-1,c+2,s-1} \\
& - 161,137,536\mathfrak{T}_{q-1,c+2,s} - 79,434,936\mathfrak{T}_{q-1,c+2,s+1} - 8,003,520\mathfrak{T}_{q-1,c+2,s+2} - 66,696\mathfrak{T}_{q-1,c+2,s+3} - 1,191\mathfrak{T}_{q-1,c+3,s-3} \\
& - 142,920\mathfrak{T}_{q-1,c+3,s-2} - 1,418,481\mathfrak{T}_{q-1,c+3,s-1} - 2,877,456\mathfrak{T}_{q-1,c+3,s} - 1,418,481\mathfrak{T}_{q-1,c+3,s+1} - 142,920\mathfrak{T}_{q-1,c+3,s+2} - 1,191\mathfrak{T}_{q-1,c+3,s+3} \\
& + 2,416\mathfrak{T}_{q,c-3,s-3} + 289,920\mathfrak{T}_{q,c-3,s-2} + 2,877,456\mathfrak{T}_{q,c-3,s-1} + 5,837,056\mathfrak{T}_{q,c-3,s} + 2,877,456\mathfrak{T}_{q,c-3,s+1} + 289,920\mathfrak{T}_{q,c-3,s+2} + 2,416\mathfrak{T}_{q,c-3,s+3} \\
& + 135,296\mathfrak{T}_{q,c-2,s-3} + 16,235,520\mathfrak{T}_{q,c-2,s-2} + 161,137,536\mathfrak{T}_{q,c-2,s-1} + 326,875,136\mathfrak{T}_{q,c-2,s} + 161,137,536\mathfrak{T}_{q,c-2,s+1} + 16,235,520\mathfrak{T}_{q,c-2,s+2} \\
& + 135,296\mathfrak{T}_{q,c-2,s+3} + 591,920\mathfrak{T}_{q,c-1,s-3} + 71,030,400\mathfrak{T}_{q,c-1,s-2} + 704,976,720\mathfrak{T}_{q,c-1,s-1} + 1,430,078,720\mathfrak{T}_{q,c-1,s} \\
& + 704,976,720\mathfrak{T}_{q,c-1,s+1} + 71,030,400\mathfrak{T}_{q,c-1,s+2} + 591,920\mathfrak{T}_{q,c-1,s+3} - 591,920\mathfrak{T}_{q,c+1,s-3} - 71,030,400\mathfrak{T}_{q,c+1,s-2} - 704,976,720\mathfrak{T}_{q,c+1,s-1} \\
& - 1,430,078,720\mathfrak{T}_{q,c+1,s} - 704,976,720\mathfrak{T}_{q,c+1,s+1} - 71,030,400\mathfrak{T}_{q,c+1,s+2} - 591,920\mathfrak{T}_{q,c+1,s+3} - 135,296\mathfrak{T}_{q,c+2,s-3} - 16,235,520\mathfrak{T}_{q,c+2,s-2} \\
& - 161,137,536\mathfrak{T}_{q,c+2,s-1} - 326,875,136\mathfrak{T}_{q,c+2,s} - 161,137,536\mathfrak{T}_{q,c+2,s+1} - 16,235,520\mathfrak{T}_{q,c+2,s+2} - 135,296\mathfrak{T}_{q,c+2,s+3} - 2,416\mathfrak{T}_{q,c+3,s-3} \\
& - 289,920\mathfrak{T}_{q,c+3,s-2} - 2,877,456\mathfrak{T}_{q,c+3,s-1} - 5,837,056\mathfrak{T}_{q,c+3,s} - 2,877,456\mathfrak{T}_{q,c+3,s+1} - 289,920\mathfrak{T}_{q,c+3,s+2} - 2,416\mathfrak{T}_{q,c+3,s+3} + 1,191\mathfrak{T}_{q+1,c-3,s-3} \\
& + 142,920\mathfrak{T}_{q+1,c-3,s-2} + 1,418,481\mathfrak{T}_{q+1,c-3,s-1} + 2,877,456\mathfrak{T}_{q+1,c-3,s} + 1,418,481\mathfrak{T}_{q+1,c-3,s+1} + 142,920\mathfrak{T}_{q+1,c-3,s+2} + 1,191\mathfrak{T}_{q+1,c-3,s+3} \\
& + 66,696\mathfrak{T}_{q+1,c-2,s-3} + 8,003,520\mathfrak{T}_{q+1,c-2,s-2} + 79,434,936\mathfrak{T}_{q+1,c-2,s-1} + 161,137,536\mathfrak{T}_{q+1,c-2,s} + 79,434,936\mathfrak{T}_{q+1,c-2,s+1} + 8,003,520\mathfrak{T}_{q+1,c-2,s+2} \\
& + 66,696\mathfrak{T}_{q+1,c-2,s+3} + 291,795\mathfrak{T}_{q+1,c-1,s-3} + 35,015,400\mathfrak{T}_{q+1,c-1,s-2} + 347,527,845\mathfrak{T}_{q+1,c-1,s-1} + 704,976,720\mathfrak{T}_{q+1,c-1,s} \\
& + 347,527,845\mathfrak{T}_{q+1,c-1,s+1} + 35,015,400\mathfrak{T}_{q+1,c-1,s+2} + 291,795\mathfrak{T}_{q+1,c-1,s+3} - 291,795\mathfrak{T}_{q+1,c+1,s-3} - 35,015,400\mathfrak{T}_{q+1,c+1,s-2} \\
& - 347,527,845\mathfrak{T}_{q+1,c+1,s-1} - 704,976,720\mathfrak{T}_{q+1,c+1,s} - 347,527,845\mathfrak{T}_{q+1,c+1,s+1} - 35,015,400\mathfrak{T}_{q+1,c+1,s+2} - 291,795\mathfrak{T}_{q+1,c+1,s+3} \\
& - 66,696\mathfrak{T}_{q+1,c+2,s-3} - 8,003,520\mathfrak{T}_{q+1,c+2,s-2} - 79,434,936\mathfrak{T}_{q+1,c+2,s-1} - 161,137,536\mathfrak{T}_{q+1,c+2,s} - 79,434,936\mathfrak{T}_{q+1,c+2,s+1} - 8,003,520\mathfrak{T}_{q+1,c+2,s+2} \\
& - 66,696\mathfrak{T}_{q+1,c+2,s+3} - 1,191\mathfrak{T}_{q+1,c+3,s-3} - 142,920\mathfrak{T}_{q+1,c+3,s-2} - 1,418,481\mathfrak{T}_{q+1,c+3,s-1} - 2,877,456\mathfrak{T}_{q+1,c+3,s} \\
& - 1,418,481\mathfrak{T}_{q+1,c+3,s+1} - 142,920\mathfrak{T}_{q+1,c+3,s+2} - 1,191\mathfrak{T}_{q+1,c+3,s+3} + 120\mathfrak{T}_{q+2,c-3,s-3} + 14,400\mathfrak{T}_{q+2,c-3,s-2} + 142,920\mathfrak{T}_{q+2,c-3,s-1} \\
& + 289,920\mathfrak{T}_{q+2,c-3,s} + 142,920\mathfrak{T}_{q+2,c-3,s+1} + 14,400\mathfrak{T}_{q+2,c-3,s+2} + 120\mathfrak{T}_{q+2,c-3,s+3} + 6,720\mathfrak{T}_{q+2,c-2,s-3} + 806,400\mathfrak{T}_{q+2,c-2,s-2} \\
& + 8,003,520\mathfrak{T}_{q+2,c-2,s-1} + 16,235,520\mathfrak{T}_{q+2,c-2,s} + 8,003,520\mathfrak{T}_{q+2,c-2,s+1} + 806,400\mathfrak{T}_{q+2,c-2,s+2} + 6,720\mathfrak{T}_{q+2,c-2,s+3} + 29,400\mathfrak{T}_{q+2,c-1,s-3} \\
& + 3,528,000\mathfrak{T}_{q+2,c-1,s-2} + 35,015,400\mathfrak{T}_{q+2,c-1,s-1} + 71,030,400\mathfrak{T}_{q+2,c-1,s} + 35,015,400\mathfrak{T}_{q+2,c-1,s+1} + 3,528,000\mathfrak{T}_{q+2,c-1,s+2} + 29,400\mathfrak{T}_{q+2,c-1,s+3} \\
& - 29,400\mathfrak{T}_{q+2,c+1,s-3} - 3,528,000\mathfrak{T}_{q+2,c+1,s-2} - 35,015,400\mathfrak{T}_{q+2,c+1,s-1} - 71,030,400\mathfrak{T}_{q+2,c+1,s} - 35,015,400\mathfrak{T}_{q+2,c+1,s+1} - 3,528,000\mathfrak{T}_{q+2,c+1,s+2} \\
& - 29,400\mathfrak{T}_{q+2,c+1,s+3} - 6,720\mathfrak{T}_{q+2,c+2,s-3} - 806,400\mathfrak{T}_{q+2,c+2,s-2} - 8,003,520\mathfrak{T}_{q+2,c+2,s-1} - 16,235,520\mathfrak{T}_{q+2,c+2,s} - 8,003,520\mathfrak{T}_{q+2,c+2,s+1} \\
& - 806,400\mathfrak{T}_{q+2,c+2,s+2} - 6,720\mathfrak{T}_{q+2,c+2,s+3} - 120\mathfrak{T}_{q+2,c+3,s-3} - 14,400\mathfrak{T}_{q+2,c+3,s-2} - 142,920\mathfrak{T}_{q+2,c+3,s-1} - 289,920\mathfrak{T}_{q+2,c+3,s} \\
& - 142,920\mathfrak{T}_{q+2,c+3,s+1} - 14,400\mathfrak{T}_{q+2,c+3,s+2} - 120\mathfrak{T}_{q+2,c+3,s+3} + \mathfrak{T}_{q+3,c-3,s-3} + 120\mathfrak{T}_{q+3,c-3,s-2} + 1,191\mathfrak{T}_{q+3,c-3,s-1} + 2,416\mathfrak{T}_{q+3,c-3,s} \\
& + 1,191\mathfrak{T}_{q+3,c-3,s+1} + 120\mathfrak{T}_{q+3,c-3,s+2} + \mathfrak{T}_{q+3,c-3,s+3} + 56\mathfrak{T}_{q+3,c-2,s-3} + 6,720\mathfrak{T}_{q+3,c-2,s-2} + 66,696\mathfrak{T}_{q+3,c-2,s-1} + 135,296\mathfrak{T}_{q+3,c-2,s} \\
& + 66,696\mathfrak{T}_{q+3,c-2,s+1} + 6,720\mathfrak{T}_{q+3,c-2,s+2} + 56\mathfrak{T}_{q+3,c-2,s+3} + 245\mathfrak{T}_{q+3,c-1,s-3} + 29,400\mathfrak{T}_{q+3,c-1,s-2} + 291,795\mathfrak{T}_{q+3,c-1,s-1} + 591,920\mathfrak{T}_{q+3,c-1,s} \\
& + 291,795\mathfrak{T}_{q+3,c-1,s+1} + 29,400\mathfrak{T}_{q+3,c-1,s+2} + 245\mathfrak{T}_{q+3,c-1,s+3} - 245\mathfrak{T}_{q+3,c+1,s-3} - 29,400\mathfrak{T}_{q+3,c+1,s-2} - 291,795\mathfrak{T}_{q+3,c+1,s-1} - 591,920\mathfrak{T}_{q+3,c+1,s} \\
& - 291,795\mathfrak{T}_{q+3,c+1,s+1} - 29,400\mathfrak{T}_{q+3,c+1,s+2} - 245\mathfrak{T}_{q+3,c+1,s+3} - 56\mathfrak{T}_{q+3,c+2,s-3} - 6,720\mathfrak{T}_{q+3,c+2,s-2} - 66,696\mathfrak{T}_{q+3,c+2,s-1} - 135,296\mathfrak{T}_{q+3,c+2,s} \\
& - 66,696\mathfrak{T}_{q+3,c+2,s+1} - 6,720\mathfrak{T}_{q+3,c+2,s+2} - 56\mathfrak{T}_{q+3,c+2,s+3} - \mathfrak{T}_{q+3,c+3,s-3} - 120\mathfrak{T}_{q+3,c+3,s-2} - 2,416\mathfrak{T}_{q+3,c+3,s-1} - 1,191\mathfrak{T}_{q+3,c+3,s} - 120\mathfrak{T}_{q+3,c+3,s+1} \\
& - \mathfrak{T}_{q+3,c+3,s+2} - \mathfrak{T}_{q+3,c+3,s+3}).
\end{aligned}
\tag{11}$$

$$\begin{aligned} \frac{\partial H_{0,C,S}}{\partial Z} = & -\frac{7}{q}(\mathfrak{T}_{0,-3,-3,-3} + 56\mathfrak{T}_{0,-3,-3,-2} + 245\mathfrak{T}_{0,-3,-3,-1} - 245\mathfrak{T}_{0,-3,-3,+1} - 56\mathfrak{T}_{0,-3,-3,+2} - \mathfrak{T}_{0,-3,-3,+3} + 120\mathfrak{T}_{0,-3,-2,-3} + 6,720\mathfrak{T}_{0,-3,-2,-2} \\ & + 29,400\mathfrak{T}_{0,-3,-2,-1} - 29,400\mathfrak{T}_{0,-3,-2,+1} - 6,720\mathfrak{T}_{0,-3,-2,+2} - 120\mathfrak{T}_{0,-3,-2,+3} + 1,191\mathfrak{T}_{0,-3,-1,-3} + 66,696\mathfrak{T}_{0,-3,-1,-2} + 291,795\mathfrak{T}_{0,-3,-1,-1} \\ & - 291,795\mathfrak{T}_{0,-3,-1,+1} - 66,696\mathfrak{T}_{0,-3,-1,+2} - 1,191\mathfrak{T}_{0,-3,-1,+3} + 2,416\mathfrak{T}_{0,-3,-3,-3} + 135,296\mathfrak{T}_{0,-3,-3,-2} + 591,920\mathfrak{T}_{0,-3,-3,-1} - 591,920\mathfrak{T}_{0,-3,-3,+1} \\ & - 135,296\mathfrak{T}_{0,-3,-3,+2} - 2,416\mathfrak{T}_{0,-3,-3,+3} + 1,191\mathfrak{T}_{0,-3,-1,-3} + 66,696\mathfrak{T}_{0,-3,-1,-2} + 291,795\mathfrak{T}_{0,-3,-1,-1} - 291,795\mathfrak{T}_{0,-3,-1,+1} - 66,696\mathfrak{T}_{0,-3,-1,+2} \\ & - 1,191\mathfrak{T}_{0,-3,-1,+3} + 120\mathfrak{T}_{0,-3,-2,-3} + 6,720\mathfrak{T}_{0,-3,-2,-2} + 29,400\mathfrak{T}_{0,-3,-2,-1} - 29,400\mathfrak{T}_{0,-3,-2,+1} - 6,720\mathfrak{T}_{0,-3,-2,+2} \\ & - 120\mathfrak{T}_{0,-3,-2,+3} + \mathfrak{T}_{0,-3,-3,-3} + 56\mathfrak{T}_{0,-3,-3,-2} + 245\mathfrak{T}_{0,-3,-3,-1} - 245\mathfrak{T}_{0,-3,-3,+1} - 56\mathfrak{T}_{0,-3,-3,+2} - \mathfrak{T}_{0,-3,-3,+3} + 120\mathfrak{T}_{0,-2,-3,-3} \\ & + 6,720\mathfrak{T}_{0,-2,-3,-2} + 29,400\mathfrak{T}_{0,-2,-3,-1} - 29,400\mathfrak{T}_{0,-2,-3,+1} - 6,720\mathfrak{T}_{0,-2,-3,+2} - 120\mathfrak{T}_{0,-2,-3,+3} + 14,400\mathfrak{T}_{0,-2,-2,-3} \\ & + 806,400\mathfrak{T}_{0,-2,-2,-2} + 3,528,000\mathfrak{T}_{0,-2,-2,-1} - 3,528,000\mathfrak{T}_{0,-2,-2,+1} - 806,400\mathfrak{T}_{0,-2,-2,+2} - 14,400\mathfrak{T}_{0,-2,-2,+3} + 142,920\mathfrak{T}_{0,-2,-1,-3} \\ & + 8,003,520\mathfrak{T}_{0,-2,-1,-2} + 35,015,400\mathfrak{T}_{0,-2,-1,-1} - 35,015,400\mathfrak{T}_{0,-2,-1,+1} - 8,003,520\mathfrak{T}_{0,-2,-1,+2} - 142,920\mathfrak{T}_{0,-2,-1,+3} + 289,920\mathfrak{T}_{0,-2,-3,-3} \\ & + 16,235,520\mathfrak{T}_{0,-2,-3,-2} + 71,030,400\mathfrak{T}_{0,-2,-3,-1} - 71,030,400\mathfrak{T}_{0,-2,-3,+1} - 16,235,520\mathfrak{T}_{0,-2,-3,+2} - 289,920\mathfrak{T}_{0,-2,-3,+3} + 142,920\mathfrak{T}_{0,-2,-1,-3} \\ & + 8,003,520\mathfrak{T}_{0,-2,-1,-2} + 35,015,400\mathfrak{T}_{0,-2,-1,-1} - 35,015,400\mathfrak{T}_{0,-2,-1,+1} - 8,003,520\mathfrak{T}_{0,-2,-1,+2} - 142,920\mathfrak{T}_{0,-2,-1,+3} + 14,400\mathfrak{T}_{0,-2,-3,-3} \\ & + 806,400\mathfrak{T}_{0,-2,-3,-2} + 3,528,000\mathfrak{T}_{0,-2,-3,-1} - 3,528,000\mathfrak{T}_{0,-2,-3,+1} - 806,400\mathfrak{T}_{0,-2,-3,+2} - 14,400\mathfrak{T}_{0,-2,-3,+3} + 120\mathfrak{T}_{0,-2,-1,-3} \\ & + 6,720\mathfrak{T}_{0,-2,-1,-2} + 29,400\mathfrak{T}_{0,-2,-1,-1} - 29,400\mathfrak{T}_{0,-2,-1,+1} - 6,720\mathfrak{T}_{0,-2,-1,+2} - 120\mathfrak{T}_{0,-2,-1,+3} + 1,191\mathfrak{T}_{0,-1,-3,-3} + 66,696\mathfrak{T}_{0,-1,-3,-2} \\ & + 291,795\mathfrak{T}_{0,-1,-3,-1} - 291,795\mathfrak{T}_{0,-1,-3,+1} - 66,696\mathfrak{T}_{0,-1,-3,+2} - 1,191\mathfrak{T}_{0,-1,-3,+3} + 142,920\mathfrak{T}_{0,-1,-2,-3} + 8,003,520\mathfrak{T}_{0,-1,-2,-2} \\ & + 35,015,400\mathfrak{T}_{0,-1,-2,-1} - 35,015,400\mathfrak{T}_{0,-1,-2,+1} - 8,003,520\mathfrak{T}_{0,-1,-2,+2} - 142,920\mathfrak{T}_{0,-1,-2,+3} + 1,418,481\mathfrak{T}_{0,-1,-1,-3} \\ & + 79,434,936\mathfrak{T}_{0,-1,-1,-2} + 347,527,845\mathfrak{T}_{0,-1,-1,-1} - 347,527,845\mathfrak{T}_{0,-1,-1,+1} - 79,434,936\mathfrak{T}_{0,-1,-1,+2} - 1,418,481\mathfrak{T}_{0,-1,-1,+3} \\ & + 2,877,456\mathfrak{T}_{0,-1,-3,-3} + 161,137,536\mathfrak{T}_{0,-1,-3,-2} + 704,976,720\mathfrak{T}_{0,-1,-3,-1} - 704,976,720\mathfrak{T}_{0,-1,-3,+1} - 161,137,536\mathfrak{T}_{0,-1,-3,+2} - 2,877,456\mathfrak{T}_{0,-1,-3,+3} \\ & + 1,418,481\mathfrak{T}_{0,-1,-1,-3} + 79,434,936\mathfrak{T}_{0,-1,-1,-2} + 347,527,845\mathfrak{T}_{0,-1,-1,-1} - 347,527,845\mathfrak{T}_{0,-1,-1,+1} - 79,434,936\mathfrak{T}_{0,-1,-1,+2} \\ & - 1,418,481\mathfrak{T}_{0,-1,-1,+3} + 142,920\mathfrak{T}_{0,-1,-2,-3} + 8,003,520\mathfrak{T}_{0,-1,-2,-2} + 35,015,400\mathfrak{T}_{0,-1,-2,-1} - 35,015,400\mathfrak{T}_{0,-1,-2,+1} \\ & - 8,003,520\mathfrak{T}_{0,-1,-2,+2} - 142,920\mathfrak{T}_{0,-1,-2,+3} + 1,191\mathfrak{T}_{0,-1,-3,-3} + 66,696\mathfrak{T}_{0,-1,-3,-2} + 291,795\mathfrak{T}_{0,-1,-3,-1} - 291,795\mathfrak{T}_{0,-1,-3,+1} \\ & - 66,696\mathfrak{T}_{0,-1,-3,+2} - 1,191\mathfrak{T}_{0,-1,-3,+3} + 2,416\mathfrak{T}_{0,-3,-3,-3} + 135,296\mathfrak{T}_{0,-3,-3,-2} + 591,920\mathfrak{T}_{0,-3,-3,-1} - 591,920\mathfrak{T}_{0,-3,-3,+1} - 135,296\mathfrak{T}_{0,-3,-3,+2} \\ & - 2,416\mathfrak{T}_{0,-3,-3,+3} + 289,920\mathfrak{T}_{0,-3,-2,-3} + 16,235,520\mathfrak{T}_{0,-3,-2,-2} + 71,030,400\mathfrak{T}_{0,-3,-2,-1} - 71,030,400\mathfrak{T}_{0,-3,-2,+1} - 16,235,520\mathfrak{T}_{0,-3,-2,+2} \\ & - 289,920\mathfrak{T}_{0,-3,-2,+3} + 2,877,456\mathfrak{T}_{0,-3,-1,-3} + 161,137,536\mathfrak{T}_{0,-3,-1,-2} + 704,976,720\mathfrak{T}_{0,-3,-1,-1} - 704,976,720\mathfrak{T}_{0,-3,-1,+1} - 161,137,536\mathfrak{T}_{0,-3,-1,+2} \\ & - 2,877,456\mathfrak{T}_{0,-3,-1,+3} + 5,837,056\mathfrak{T}_{0,-3,-3,-3} + 326,875,136\mathfrak{T}_{0,-3,-3,-2} + 1,430,078,720\mathfrak{T}_{0,-3,-3,-1} - 1,430,078,720\mathfrak{T}_{0,-3,-3,+1} - 326,875,136\mathfrak{T}_{0,-3,-3,+2} \\ & - 5,837,056\mathfrak{T}_{0,-3,-3,+3} + 2,877,456\mathfrak{T}_{0,-3,-1,-3} + 161,137,536\mathfrak{T}_{0,-3,-1,-2} + 704,976,720\mathfrak{T}_{0,-3,-1,-1} - 704,976,720\mathfrak{T}_{0,-3,-1,+1} - 161,137,536\mathfrak{T}_{0,-3,-1,+2} \\ & - 2,877,456\mathfrak{T}_{0,-3,-1,+3} + 289,920\mathfrak{T}_{0,-3,-2,-3} + 16,235,520\mathfrak{T}_{0,-3,-2,-2} + 71,030,400\mathfrak{T}_{0,-3,-2,-1} - 71,030,400\mathfrak{T}_{0,-3,-2,+1} - 16,235,520\mathfrak{T}_{0,-3,-2,+2} \\ & - 289,920\mathfrak{T}_{0,-3,-2,+3} + 2,416\mathfrak{T}_{0,-3,-3,-3} + 135,296\mathfrak{T}_{0,-3,-3,-2} + 591,920\mathfrak{T}_{0,-3,-3,-1} - 591,920\mathfrak{T}_{0,-3,-3,+1} - 135,296\mathfrak{T}_{0,-3,-3,+2} - 2,416\mathfrak{T}_{0,-3,-3,+3} \\ & + 1,191\mathfrak{T}_{0,-1,-3,-3} + 66,696\mathfrak{T}_{0,-1,-3,-2} + 291,$$

The aforementioned analysis yields the following theorem:

Theorem 3. From (8) the approximation formulas to $H_{0,c,s}$,

$\frac{\partial H_{0,c,s}}{\partial x}, \frac{\partial H_{0,c,s}}{\partial y}, \frac{\partial H_{0,c,s}}{\partial z}, \frac{\partial^2 H_{0,c,s}}{\partial x^2}, \frac{\partial^2 H_{0,c,s}}{\partial y^2}, \frac{\partial^2 H_{0,c,s}}{\partial z^2}, \frac{\partial^2 H_{0,c,s}}{\partial x \partial y}, \frac{\partial^2 H_{0,c,s}}{\partial x \partial z}, \dots$, are given in terms of $\mathcal{T}_{0,c,s}$ at (9)–(12).

3 Numerical outcomes

Now, to determine whether or not this method, which was developed by presenting its constructions in n -dimensional space, is correct and effective. In this section, we provide several numerical examples in various dimensions to demonstrate the accuracy of this method. In addition to the comparison of our results with those previously obtained, we also exhibit some of the obtained figures. We should point out that all of the examples were created using the Mathematica 12.1 package and ran on a standard computer (Intel(R) core(TM) i7-351U, CPU@1.90 Hz 2.40 GHz).

The first test problem: [8,9,13,17,18,29]

We take the two-dimensional problem in the following form:

$$h_{xx}(x, y) + h_{yy}(x, y) - \sin(\pi x) \sin(\pi y) = 0, \quad (13)$$

$$x, y \in [l, m].$$

The following is the exact solution to that problem:

$$h(x, y) = -\frac{\sin(\pi x) \sin(\pi y)}{2\pi^2}. \quad (14)$$

We take the boundary conditions to the third problem in the following form:

Table 1: Numerical results for the third issue are available at $y = 0.4$ and $x, y \in [0, 1]$

x	Numerical results	Exact results	Absolute error
0.2	-0.028320	-0.028320	8.07856×10^{-11}
0.4	-0.045822	-0.045822	1.30714×10^{-10}
0.6	-0.045822	-0.045822	1.30714×10^{-10}
0.8	-0.028320	-0.028320	8.07856×10^{-11}

Table 2: Maximum absolute error based on the approach used to solve the problem

The proposed method	Cubic B-spline approach [18]	Quadratic B-spline approach [17]	MCBDQM approach [13]	Spline-based DQM approach [29]	Haar wavelet approach [8]	SCA based on Haarwavelets [9]
8.07×10^{-11}	1.67×10^{-4}	3.72×10^{-5}	2.11×10^{-5}	1.62×10^{-4}	3.08×10^{-4}	3.08×10^{-4}

$$h(l, y) = h(x, l) = \alpha, \quad h(m, y) = h(x, m) = \beta. \quad (15)$$

By substituting from (5)–(7) into (13) with (15), we obtain the numerical results as in Table 1.

The results of the two-dimensional septic B-spline approach at 15×15 are shown in Table 1. In terms of outcomes, we can presume that they are satisfactory. We compare the suggested method's results to those of several approaches [8,9,13,17,18,29] that are presented in Table 2 and use mesh 15×15 grid points.

Now, we show the numerical results and absolute errors at $y = 0.4$ (Figures 1 and 2). Also, Figure 3 depicts a three-dimensional graph of numerical results.

The second test problem:

We take the two-dimensional nonlinear problem in the following form:

$$h_{xx}(x, y) + \exp(x^2)h_{yy}(x, y) + \sin(h(x, y)) - f(x, y) = 0, \quad x, y \in [l, m], \quad (16)$$

where

$$f(x, y) = e^{x^2}(- (1 - e^{1-x})x^2(1 - e^{1-y})\sin(xy) - (1 - e^{1-x})e^{1-y}\sin(xy) + 2(1 - e^{1-x})xe^{1-y}\cos(xy)) - (1 - e^{1-x})(1 - e^{1-y})y^2\sin(xy) - e^{1-x}(1 - e^{1-y})\sin(xy) + \sin((1 - e^{1-x})(1 - e^{1-y})\sin(xy)) + 2e^{1-x}(1 - e^{1-y})y\cos(xy). \quad (17)$$

The following is the exact solution to that problem:

$$h(x, y) = \sin(yx)(1 - \exp(1 - x))(1 - \exp(1 - y)). \quad (18)$$

We take the boundary conditions to the third problem in the following form:

$$h(l, y) = h(x, l) = \alpha, \quad h(m, y) = h(x, m) = \beta. \quad (19)$$

By substituting from (5)–(7) into (16) with (19), we obtain the numerical results as in Table 3.

The results of the two-dimensional septic B-spline approach at 50×50 are shown in Table 3. From outcomes, we can presume that the method is satisfactory. Now, we show the numerical results and absolute errors at $y = 0.4$ in Figures 4 and 5. Also, Figure 6 depicts a three-dimensional graph of numerical results.

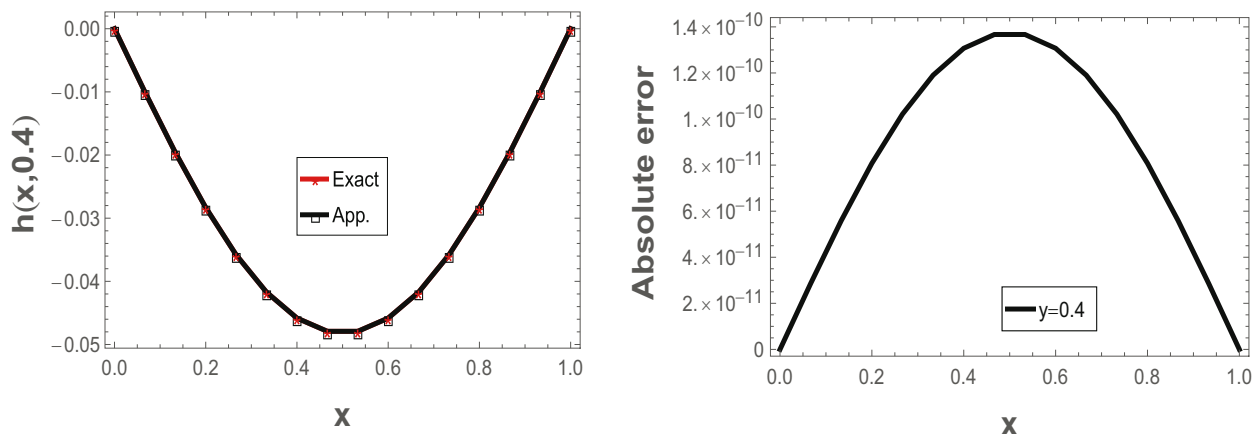


Figure 1: The graph of numerical results and absolute error at $y = 0.4$.

The third test problem: [17,18]

We take the second test problem in the three-dimensional in the following form:

$$h_{xx}(x, y, z) + h_{yy}(x, y, z) + h_{zz}(x, y, z) - f(x, y, z) = 0, \quad x, y, z \in [l, m], \quad (20)$$

where

$$f(x, y, z) = xyz(e^{x+y+z})(3yxz + yx + zx - 5x + zy - 5y - 5z + 9). \quad (21)$$

The exact solution to that problem is given as follows:

$$h(x, y, z) = (x - x^2)(y - y^2)(z - z^2)e^{x+y+z}. \quad (22)$$

We take the boundary conditions to the fourth problem in the following form:

$$\begin{aligned} h(l, y, z) &= h(x, l, z) = h(x, y, l) = \alpha, \\ h(m, y, z) &= h(x, m, z) = h(x, y, m) = \beta. \end{aligned} \quad (23)$$

By substituting from (9)–(12) into (20) with (23), we obtain the numerical results as in Table 4.

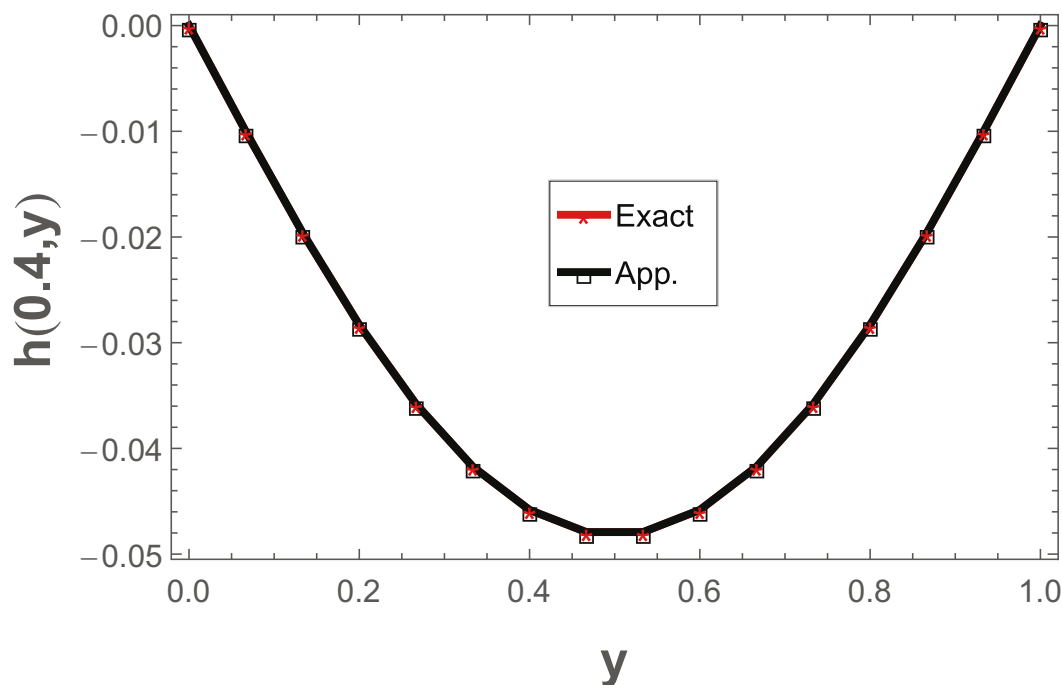


Figure 2: The graph of numerical results at $x = 0.4$.

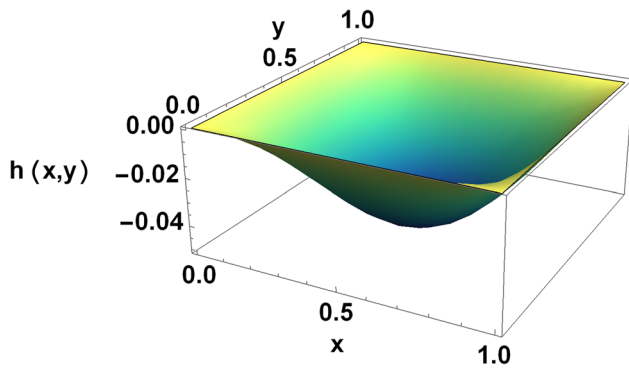


Figure 3: Three-dimensional graph for numerical results.

Table 3: Numerical results for the third issue are available at $y = 0.5$ and $x, y \in [0, 1]$

x	Numerical results	Exact results	Absolute error
0.2	0.079328	0.079371	4.28156×10^{-5}
0.4	0.105922	0.105956	3.39476×10^{-5}
0.6	0.094261	0.094287	2.61754×10^{-5}
0.8	0.055914	0.055931	1.71551×10^{-5}

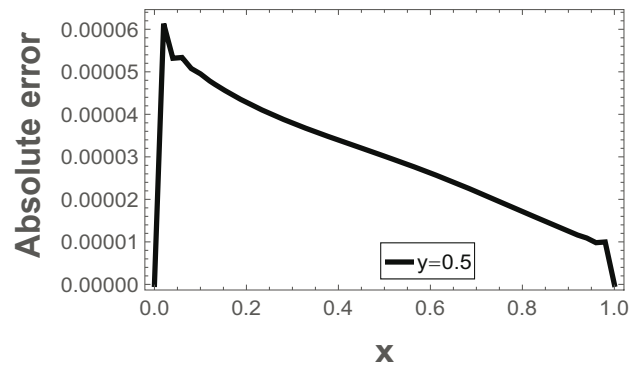
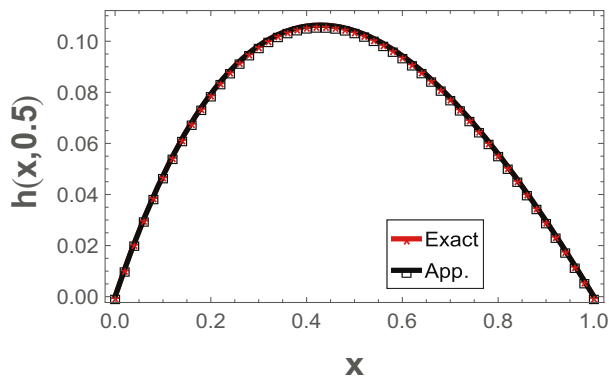


Figure 4: The graph of numerical results and absolute error at $y = 0.5$.

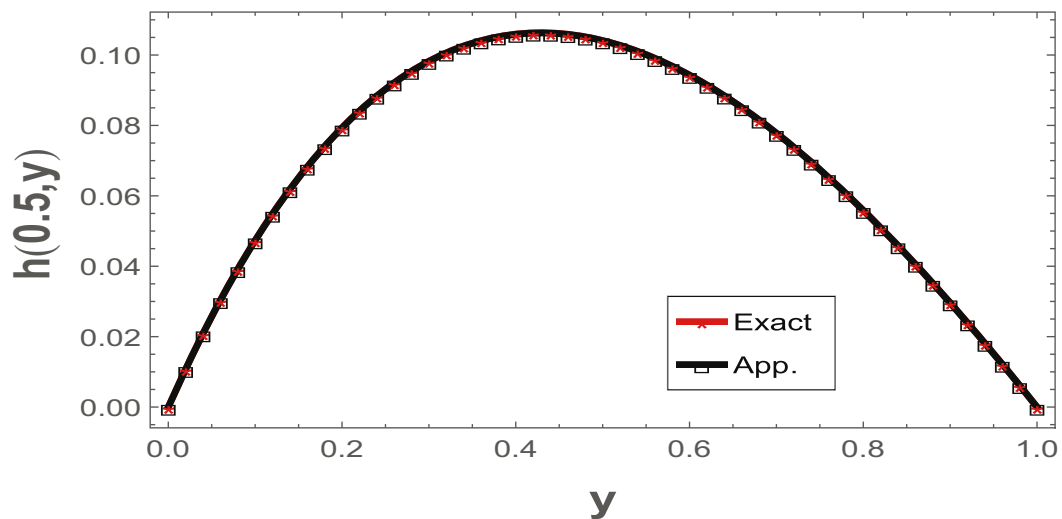


Figure 5: The graph of numerical results at $x = 0.5$.

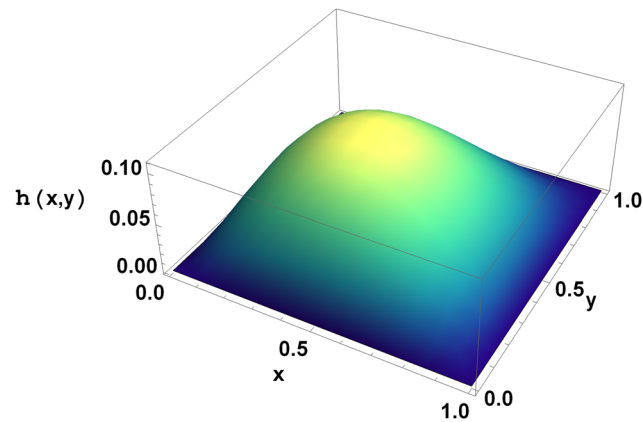


Figure 6: Three-dimensional graph for numerical results.

Table 4: Numerical results for test problem at $z = y = 0.5$ and $x, y, z \in [0, 1]$

x	Numerical solution	Exact solution	Absolute error	Quadratic B-spline method [17]	Cubic B-spline method [18]
0.1	0.016868	0.0168984	3.00566×10^{-5}	3.24947×10^{-5}	3.48009×10^{-5}
0.2	0.033142	0.0332012	5.91788×10^{-5}	6.49943×10^{-5}	7.05722×10^{-5}
0.3	0.048073	0.0481595	8.65621×10^{-5}	9.65554×10^{-5}	1.06205×10^{-4}
0.4	0.060716	0.0608280	1.11853×10^{-4}	1.27075×10^{-4}	1.41846×10^{-4}
0.5	0.069891	0.0700264	1.35497×10^{-4}	1.57835×10^{-4}	1.79618×10^{-4}
0.6	0.074136	0.0742955	1.59281×10^{-4}	1.92337×10^{-4}	2.24718×10^{-4}
0.7	0.071658	0.0718456	1.86992×10^{-4}	2.37433×10^{-4}	2.86918×10^{-4}
0.8	0.060271	0.0604965	2.24725×10^{-4}	3.04639×10^{-4}	3.82276×10^{-4}
0.9	0.050423	0.0376082	2.77977×10^{-4}	4.11161×10^{-4}	5.34402×10^{-4}

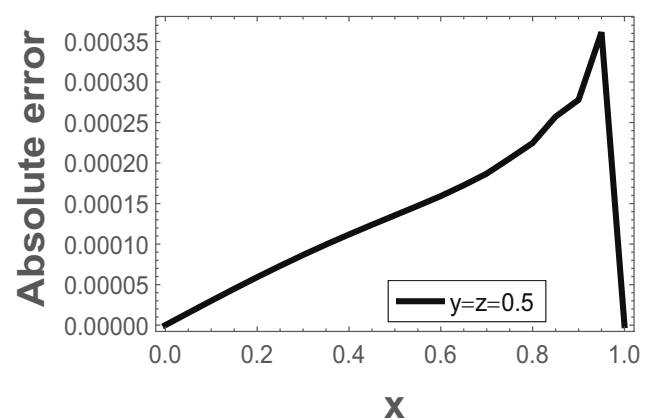
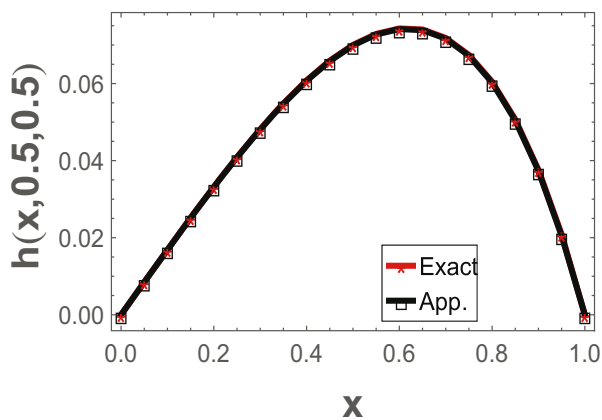


Figure 7: The graph of numerical results and absolute error at $y = z = 0.5$.

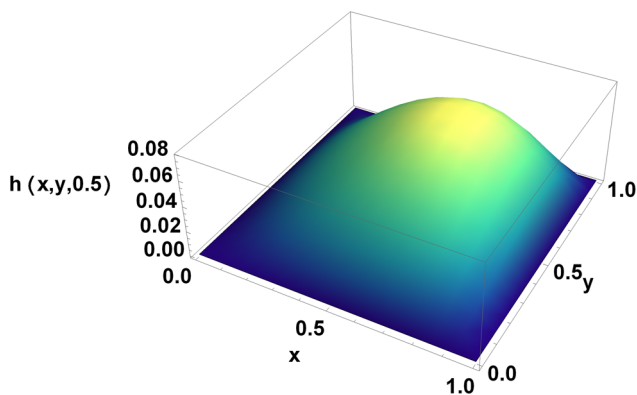


Figure 8: Three-dimensional graph for numerical results.

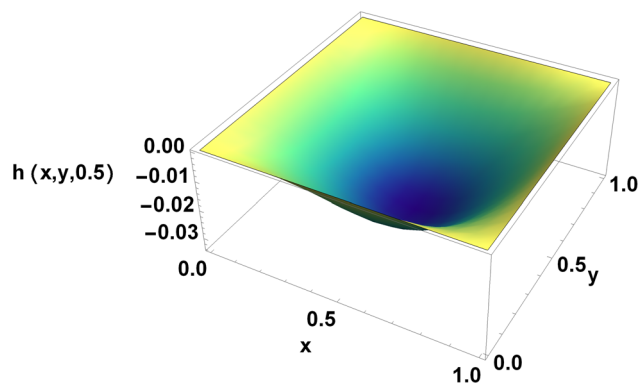


Figure 10: Three-dimensional graph for numerical results.

Table 4 shows a comparison of our results with those obtained using quadratic B-spline and cubic B-spline approaches with meshes of 20×20 . In terms of the results, we can see that they are acceptable based on our observations. At $y = z = 0.5$, Figure 7 displays the numerical results and absolute errors. Figure 8 depicts a three-dimensional graph of numerical results.

The fourth test problem: [9,20]

We take the test problem in the three-dimensional in the following form:

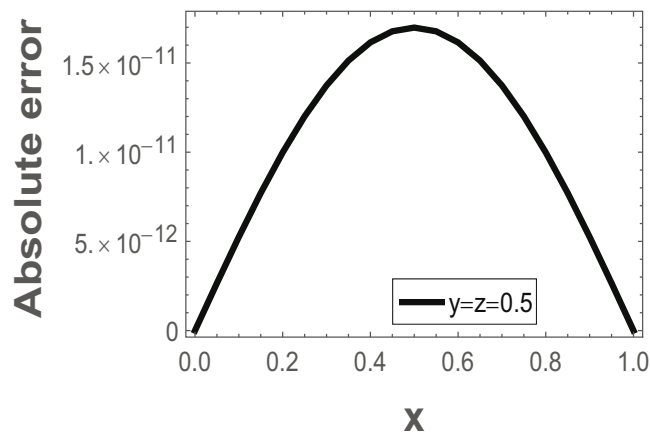
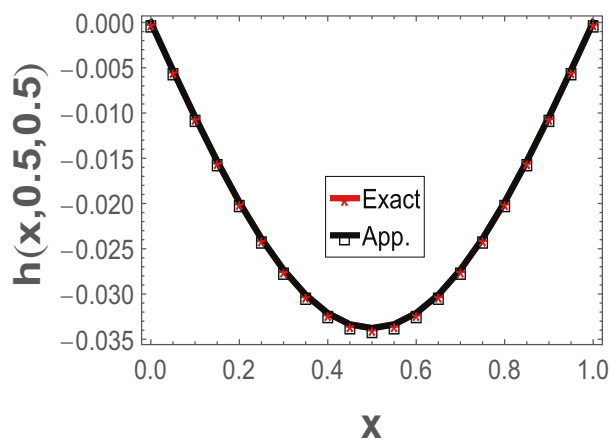
$$h_{xx}(x, y, z) + h_{yy}(x, y, z) + h_{zz}(x, y, z) - \sin(\pi x) \sin(\pi y) \sin(\pi z) = 0, \quad x, y, z \in [l, m]. \quad (24)$$

The following is the exact solution to that problem:

$$h(x, y, z) = -\frac{\sin(\pi x) \sin(\pi y) \sin(\pi z)}{2\pi^2}. \quad (25)$$

Table 5: Numerical results for the test problem are available at $y = z = 0.5$, $x, y, z \in [0, 1]$

x	Numerical results	Exact results	Absolute error	Maximum absolute error of our method	Maximum absolute error [20]	Maximum absolute error [9]
0.2	-0.019851	-0.01985	9.983×10^{-12}	1.615×10^{-11}	4.966×10^{-5}	8.922×10^{-4}
0.4	-0.032120	-0.03212	1.615×10^{-11}	—	—	—
0.6	-0.032120	-0.03210	1.615×10^{-11}	—	—	—
0.8	-0.019851	-0.01985	9.983×10^{-12}	—	—	—

Figure 9: The graph of numerical results and absolute error at $y = z = 0.5$.

We take the boundary conditions to the third problem in the following form:

$$\begin{aligned} h(l, y, z) &= h(x, l, z) = h(x, y, l) = \alpha, \\ h(m, y, z) &= h(x, m, z) = h(x, y, m) = \beta. \end{aligned} \quad (26)$$

By substituting from (9)–(12) into (24) with (26), we obtain the numerical results as in Table 5.

The results of the three-dimensional septic B-spline approach using mesh 15×15 are presented in Table 5. In terms of observation, the results appear to be acceptable. Figure 9 shows the numerical results and absolute errors at $y = z = 0.5$. Figure 10 depicts a three-dimensional graph of numerical results.

4 Conclusion

By the end of this study, we may have made a significant contribution to addressing some of the problems that most academics in various fields have when dealing with n -dimensional mathematical models. The study object is crucial, and we believe that the majority of academics are eagerly anticipating the results. We noted how difficult it is for researchers to cope with these models as the dimension expands after seeing various scholars present their discoveries on partial differential equation solutions in one-, two-, and three-dimensional. As a result, we decided to extend the basic B-spline method, which had hitherto only been used to solve one-dimensional mathematical problems, to two- and three-dimensional. To assess the correctness and efficacy of the developed schemes, we used numerical examples of various dimensions. When the numerical results are compared to the actual solution, we see that the formulas found are effective. From this perspective, we believe that a significant contribution has been made toward addressing problems involving partial differential equations in many dimensions. The proposed new structure provides accurate results than other methods because it deals with a larger number of points than the field. We would generalize a few other B-spline shapes to serve as solutions to n -dimensional differential equations as part of our long-term research.

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