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# Lie symmetry analysis, optimal system, and new exact solutions of a (3 + 1) dimensional nonlinear evolution equation

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Abstract: Studies on Non-linear evolutionary equations have become more critical as time evolves. Such equations are not far-fetched in fluid mechanics, plasma physics, optical fibers, and other scientific applications. It should be an essential aim to find exact solutions of these equations. In this work, the Lie group theory is used to apply the similarity reduction and to find some exact solutions of a (3+1) dimensional nonlinear evolution equation. In this report, the groups of symmetries, Tables for commutation, and adjoints with infinitesimal generators were established. The subalgebra and its optimal system is obtained with the aid of the adjoint Table. Moreover, the equation has been reduced into a new PDE having less number of independent variables and at last into an ODE, using subalgebras and their optimal system, which gives similarity solutions that can represent the dynamics of nonlinear waves.

**Keywords:** (3 + 1) - dimensional nonlinear evolution equation; optimal system; Lie symmetry analysis; group invariant solutions

## 1 Introduction

Recently, non-linear governing equations suitable to analyze quartic autocatalysis were presented by Makinde and Animasaun in [1] and [2]. There has been an increasing interest in the study of NLEEs in the past few years. The (3+1)-dimensional nonlinear evolution equations was first introduced by Zhaqilao [3] in the study of algebraic-geometrical solutions. An evolution equation refers to a partial differ-

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ential equation having partial derivatives of the dependent variable **u** with respect to the time *t* and space variables  $x = (x_1, ..., x_n)$ , which are the independent variables. The (3+1)-dimensional equation possesses the KdV equation  $u_t - 6uu_x + u_{xxx} = 0$  as its main term under the transformations  $v(x, t) \rightarrow u(x', t'), x' \rightarrow \frac{1}{\sqrt{3}}x$  and  $t' \rightarrow \frac{1}{6\sqrt{3}}t$ . Based on this, the (3+1)-dimensional nonlinear evolution equation may be used to study the shallowwater waves and short waves in nonlinear dispersive models [4]. A physicist should be well aware of all new aspects of the nonlinear wave theory. It is always a good practice to study a new equation of the theory of nonlinear evolution equations. The proper understanding of qualitative significances of many incidents and procedures can be achieved by exact solitary wave solutions of NLEEs in different fields of applied mathematics, engineering, physics, biology, chemistry, and many more. So, to gain a clear understanding of the qualitative and quantitative properties of these equations, it is necessary to find some exact solutions to these equations. For illustration, the soliton pulse implies an ideal balance between nonlinearity and dispersion effects. The soliton is a crucial character of nonlinearity [5–16]. Soliton solutions are of special type PDEs solutions that model phenomena from the balance between nonlinear and dispersive effects in systems like light pulses propagation in optical fibers and water waves. For the nonlinear PDEs, the exact solutions graphically demonstrate and determine the structure of many nonlinear complex phenomena such as absence of multiplicity steady states under different conditions, spatial localization of transfer processes, presence of peaking regimes, and many others. First of all, Geng [13] introduced equation (1) in the algebraic geometrical solutions [17]. In [6] Nsoliton solutions of the (3+1)-d NEE was studied by Geng and Wazwaz [5, 18, 19] found some multiple soliton solutions and a collection of traveling wave solutions of the (3 + 1)-d NEE (1). Soliton and rogue wave solutions can be found in [3, 20-25]. There are many powerful methods to understand the nonlinear evolution equations that have been used, for instance, the Hietarinta approach [15], Hirota bilinear method [5–14], the Bäcklund transformation method, Pfaffian technique, Darboux transformation, the inverse scattering method, the generalized symmetry method, the Painlevé analysis, and other methods. To investigate nonlinear dynamical phenomena using a generalised model in shallow water, plasma and nonlinear optics, a generalized (2 + 1)- dimensional Hirota bilinear equation was proposed by Hua et al. [26]. Xin Zhao et al. [27] have investigated the generalized (2 + 1)-dimensional nonlinear wave equation in nonlinear optics, fluid mechanics and plasma physics. They have used the Hirota Bilinear method, and obtained bilinear Bäcklund transformation, to construct the Lax pair and obtained Mixed Rogue-Solitary Wave Solutions, Rogue-Periodic Wave Solutions and Lump-Periodic Wave Solutions. They have also explained the interactions between the rogue wave, periodic wave and the solitary wave. Here, we shall study such a (3+1)-dimensional nonlinear evolution equation [5] - [8] and [18].

$$3v_{xz} - (2v_t + v_{xxx} - 2vv_x)_v + 2(v_x \partial_x^{-1} v_y)_x = 0,$$
 (1)

where

$$(\partial_x^{-1} f)(x) = \int_{-\pi}^{x} f(t)dt. \tag{2}$$

Obviously,  $\partial_x \partial_x^{-1} f = \partial_x^{-1} \partial_x f = 1$  under the decaying condition at  $\infty$ . As per the coefficients of x, y and z, the multiple soliton solutions exist for equation (1) ([5] and [6]). Several soliton solutions, as well as singular soliton solutions, were obtained by the simpler form of the Hirota's method in [1]. An N-soliton solution of a (3+1)dimensional nonlinear evolution equation is obtained by using the Hirota bilinear method with the perturbation technique in [6]. A new Wronskian condition was set for equation (1), with the aid of the Hirota bilinear transformation, a novel Wronskian determinant solution is presented for the equation (1). The Wronskin determinant is different for both [5] and [6]. We aim to extend the work in [8], where the classical Lie symmetry of the (3+1)-dimensional nonlinear evolution equation (1) was found. Here, we obtained an optimal system for further results and then some new solutions which can explain new nonlinearity features with the approach applied in [28], [29] and [30]. To remove the integral term in equation (1) by introducing

$$v(x, y, z, t) = u_x(x, y, z, t),$$
 (3)

we get

the potential

$$\Delta := 3u_{xxz} - (2u_{xt} + u_{xxxx} - 2u_xu_{xx})_y + 2(u_{xx}u_y)_x = 0.$$
 (4)

Generally, it is not easy to get every possible combination of group generators to obtain the invariant solutions, as there may be infinitely many solutions. Researchers have always discussed relatively independent solutions, this inspires many other researchers to obtain a new system called an optimal system. Thus, in this paper, we constructed a one-dimensional optimal system of subalgebra for equation (4). The Norwegian mathematician Sophus Lie introduced the term invariant solutions and developed the Lie point symmetry analysis (1842–1899). The research conducted so far motivates us to obtain some new exact solutions using an optimal system, of equation (4), which has not been found in research yet.

One may find in this article some acceptable answers, as a result, shown in the graphs and solutions presented in the closed-form. Do the soliton solutions of the given equation exist? If so, how do they behave? Can one speculate the "soliton" nature of the solution even if solutions are not well known in some real systems? How can one find some precise solutions that can be useful "if the complexity of the methods affects the solution results"? Are there solutions to test stability and estimate errors for the newly proposed numerical algorithms"? The authors have tried to find the answers of the above mentioned questions in the present article.

This work has two main objectives. The first is to obtain an optimal system, and the second is to obtain several types of new exact solutions. In section (2), we have applied the Lie group approach to obtain the symmetries of equation (4). An optimal system of vector fields is established in section (3). In section (4), we investigated the reduced equations to find exact solutions, and in the end, some remarks are presented in the conclusion.

# 2 Lie point symmetries

Lie group of transformations with parameter ( $\epsilon$ ) acting on variables (dependent and independent) for equation (4) are as follows

$$\tilde{x} = x + \epsilon \psi^{x}(x, y, z, t, u) + O(\epsilon^{2}),$$

$$\tilde{y} = y + \epsilon \psi^{y}(x, y, z, t, u) + O(\epsilon^{2}),$$

$$\tilde{z} = z + \epsilon \psi^{z}(x, y, z, t, u) + O(\epsilon^{2}),$$

$$\tilde{t} = t + \epsilon \tau(x, y, z, t, u) + O(\epsilon^{2}),$$

$$\tilde{u} = u + \epsilon \eta(x, y, z, t, u) + O(\epsilon^{2}),$$

$$(5)$$

where  $\epsilon$  is a small Lie group parameter and  $\psi^x$ ,  $\psi^y$ ,  $\psi^z$ ,  $\tau$  and  $\eta$  are the infinitesimals of the transformation which are to be found for independent and dependent variables, respectively. Thus, the associated Lie algebra will be of the

form Olver [31]

$$\mathbb{P} = \psi^{x}(x, y, z, t, u) \frac{\partial}{\partial x} + \psi^{y}(x, y, z, t, u) \frac{\partial}{\partial y} + \psi^{z}(x, y, z, t, u) \frac{\partial}{\partial z} + \tau(x, y, z, t, u) \frac{\partial}{\partial t} + \eta(x, y, z, t, u) \frac{\partial}{\partial u}.$$
(6)

The above vector field generates a symmetry of equation (4). Also, for the invariance,  $pr^{(5)}\mathbb{P}(\Delta) = 0$ , when  $\Delta = 0$  for equation (4), where  $pr^{(5)}\mathbb{P}$  is the fifth prolongation of  $\mathbb{P}$ . To obtain an overdetermined system of the coupled PDEs, we applied  $pr^{(5)}\mathbb{P}$  to equation (4)

$$pr^{(5)}\mathbb{P} = \mathbb{P} + \eta^{xxz} \frac{\partial}{\partial u_{xxz}} + \eta^{xyt} \frac{\partial}{u_{xyt}} + \eta^{xxxxy} \frac{\partial}{u_{xxxxy}} + \eta^{xy} \frac{\partial}{u_{xy}} + \eta^{xx} \frac{\partial}{u_{xx}} + \eta^{x} \frac{\partial}{u_{x}} + \eta^{xxy} \frac{\partial}{u_{xxy}} + \eta^{xxx} \frac{\partial}{u_{xxx}} + \eta^{y} \frac{\partial}{u_{y}},$$
(7)

and get

$$3\eta^{xxz} - 2\eta^{xyt} - \eta^{xxxxy} + 4u_{xx}\eta^{xy} + 4u_{xy}\eta^{xx} + 2u_{x}\eta^{xxy} + 2u_{xxy}\eta^{x} + 2u_{xxx}\eta^{y} + 2u_{y}\eta^{xxx} = 0.$$
 (8)

After this, we use a computer algebra software (Maple) to obtain the following system of PDEs:

$$3\eta_{u} = -\psi_{t}^{t}, 2\eta_{x} = 3\psi_{z}^{y} - 2\psi_{t}^{x}, 2\eta_{y} = 3\psi_{z}^{x}, \psi_{u}^{t} = 0, \psi_{x}^{t} = 0,$$

$$\psi_{y}^{t} = 0, \psi_{z}^{t} = 0, \psi_{tt}^{t} = 0, \psi_{u}^{x} = 0, 3\psi_{x}^{x} = \psi_{t}^{t}, \psi_{y}^{x} = 0, \psi_{t}^{y} = 0,$$

$$\psi_{u}^{y} = 0, \psi_{x}^{y} = 0, 3\psi_{y}^{y} = 3\psi_{z}^{z} - 2\psi_{t}^{t}, \psi_{t}^{z} = 0, \psi_{u}^{z} = 0, \psi_{x}^{z} = 0,$$

$$\psi_{y}^{z} = 0, \psi_{zz}^{z} = 0,$$

$$(9)$$

and, thus, we obtained the required infinitesimal generator as follows:

$$\psi^{x} = \frac{1}{3}c_{1}x + f_{2}(z, t), \ \psi^{y} = c_{3}y + f_{1}(z),$$

$$\psi^{z} = \frac{1}{3}(3c_{3} + 2c_{1})z + c_{4}, \ \tau = c_{1}t + c_{2},$$

$$\eta = -\frac{1}{3}c_{1}u + \frac{3}{2}y\frac{\partial}{\partial z}f_{2}(z, t) - x\frac{\partial}{\partial t}f_{2}(z, t)$$

$$+ \frac{3}{2}x\frac{\partial}{\partial z}f_{1}(z) + f_{3}(z, t),$$
(10)

where  $c_i's$ , (i = 1, 2, 3, 4) and  $f_j's$ , (j = 1, 2, 3) are arbitrary. Following the Lie symmetry method explained in [31], we get the Lie algebra of symmetries for equation (4) as follows:

$$\mathbb{P}_{1} = \frac{1}{3}x\frac{\partial}{\partial x} + \frac{2}{3}z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} - \frac{1}{3}u\frac{\partial}{\partial u},$$

$$\mathbb{P}_{2} = \frac{\partial}{\partial t}, \quad \mathbb{P}_{3} = y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \quad \mathbb{P}_{4} = \frac{\partial}{\partial z},$$

$$\mathbb{P}_{5} = \frac{\partial}{\partial y}, \quad \mathbb{P}_{6} = \frac{\partial}{\partial x}, \quad \mathbb{P}_{7} = \frac{\partial}{\partial u}.$$
(11)

Table 1: Commutator Table

*	$\mathbb{P}_1$	$\mathbb{P}_2$	$\mathbb{P}_3$	$\mathbb{P}_4$	$\mathbb{P}_5$	$\mathbb{P}_6$	$\mathbb{P}_7$
$\mathbb{P}_1$	0	$-\mathbb{P}_2$	0	$-\frac{2}{3}\mathbb{P}_4$	0	$-\frac{1}{3}\mathbb{P}_6$	$\frac{1}{3}\mathbb{P}_7$
$\mathbb{P}_2$	$\mathbb{P}_2$	0	0	0	0	0	0
$\mathbb{P}_3$	0	0	0	$-\mathbb{P}_4$	$-\mathbb{P}_5$	0	0
$\mathbb{P}_4$	$\frac{2}{3}\mathbb{P}_4$	0	$\mathbb{P}_4$	0	0	0	0
$\mathbb{P}_5$	0	0	$\mathbb{P}_5$	0	0	0	0
$\mathbb{P}_6$	$\frac{1}{3}\mathbb{P}_6$	0	0	0	0	0	0
$\mathbb{P}_7$	$-\frac{1}{3}\mathbb{P}_7$	0	0	0	0	0	0

Now, for convenience, we obtain the Table 1 of commutator with entries as  $[\mathbb{P}_i, \mathbb{P}_i] = \mathbb{P}_i \cdot \mathbb{P}_i - \mathbb{P}_i \cdot \mathbb{P}_i$  (see [31]).

Clearly, the infinite-dimensional Lie algebra spanned by vector fields (11) generates an infinite continuous group of transformations of equation (4). These generators are linearly independent. Thus, it is very much appropriate to represent any infinitesimal of equation (4) as a linear combination of  $\mathbb{P}_i$ , given as

$$\mathfrak{X} = \mathfrak{c}_1 \mathbb{P}_1 + \mathfrak{c}_2 \mathbb{P}_2 + \mathfrak{c}_3 \mathbb{P}_3 + \mathfrak{c}_4 \mathbb{P}_4 + \mathfrak{c}_5 \mathbb{P}_5 + \mathfrak{c}_6 \mathbb{P}_6 + \mathfrak{c}_7 \mathbb{P}_7.$$

The group of transformation  $G_i$ :  $(x, y, z, t, u) \rightarrow (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u})$  which is generated by the infinitesimal generator  $\mathbb{P}_i$  for i = 1, 2, 3, 4, 5, 6, 7 are as follows [31]:

$$G_{1}:(xe^{\epsilon/3}, y, ze^{2\epsilon/3}, te^{\epsilon}, ue^{-\epsilon/3}),$$

$$G_{2}:(x, y, z, t + \epsilon, u),$$

$$G_{3}:(x, ye^{\epsilon}, ze^{\epsilon}, t, u),$$

$$G_{4}:(x, y, z + \epsilon, t, u),$$

$$G_{5}:(x, y + \epsilon, z, t, u),$$

$$G_{6}:(x + \epsilon, y, z, t, u),$$

$$G_{7}:(x, y, z, t, u + \epsilon).$$

$$(12)$$

The right hand side gives the transformed point  $exp(\varepsilon\mathbb{P}_i)(x,y,z,t,u) = (\tilde{x},\tilde{y},\tilde{z},\tilde{t},\tilde{u})$ . As each group  $G_i$  is a symmetry group (by [31]), if u = f(x,y,z,t) is a solution of equation (4) so are the functions

$$u^{(1)} = e^{-\frac{\epsilon}{3}} f(e^{-\frac{\epsilon}{3}} x, y, e^{-\frac{2}{3}} z, e^{-\epsilon} t),$$

$$u^{(2)} = f(x, y, z, t - \epsilon),$$

$$u^{(3)} = f(x, e^{-\epsilon} y, e^{-\epsilon} z, t),$$

$$u^{(4)} = f(x, y, z - \epsilon, t),$$

$$u^{(5)} = f(x, y - \epsilon, z, t),$$

$$u^{(6)} = f(x - \epsilon, y, z, t),$$

$$u^{(7)} = f(x, y, z, t) + \epsilon,$$

$$(13)$$

where  $\epsilon$  is any real no. For a detailed description, the reader can see [31].

Generally, there is an infinite number of subalgebras for this Lie algebra formed from linear combinations of generators  $\mathbb{P}_1$ ,  $\mathbb{P}_2$ ,  $\mathbb{P}_3$ ,  $\mathbb{P}_4$ ,  $\mathbb{P}_5$ ,  $\mathbb{P}_6$  and  $\mathbb{P}_7$ . If two subalgebras are equivalent, i.e., each has conjugate in the symmetry group, then their corresponding invariant solutions are connected by the same transformation. Thus, it is sufficient to place all similar subalgebras in one class and select a representative for every class. The set of all these representatives is called an optimal system (for details, see [31] and [32]). A detailed discussion is given in the next section.

# 3 Optimal system of subalgebra

Now, we find an optimal system of one dimensional Lie subalgebra. As an application of Lie group analysis, the primary use of an optimal system is to classify the group invariant solutions of partial differential equations to shorten the problem of categorizing subgroups of the complete symmetry group. A set of subalgebras forms an optimal system if each subalgebra of the Lie algebra is equivalent to a unique member of the set of subalgebras under some element of adjoint representation. Ovsiannnikov and Olver [31, 32] suggested the construction of an optimal system for the Lie subalgebra. The method made useful progress under the work of Petera, Winternitz, and Zassenhaus [33, 34], where various illustrations of an optimal system of subgroups can be seen for the Lie groups of mathematical physics. Based on the systematic algorithm [35], we find an optimal system of one-dimensional subalgebras of the equation (4). The symmetry Lie algebra having a basis  $\{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3, \mathbb{P}_4, \mathbb{P}_5, \mathbb{P}_6, \mathbb{P}_7\}$  of section (2) and identify this with  $\mathbb{R}^7$  as a vector space using the map  $\mathbb{P}_i \to \mathfrak{e}_i$ where  $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$  is the standard basis of  $\mathbb{R}^7$ . Then, from the Table 1, we obtain the following matrix description of  $Ad(\mathbb{P}_i)$ :

$$Ad\left(exp(\epsilon\mathbb{P}_i)\right)\mathbb{P}_j = \mathbb{P}_j - \epsilon[\mathbb{P}_i, \mathbb{P}_j] + \frac{1}{2!}\epsilon^2[\mathbb{P}_i, [\mathbb{P}_i, \mathbb{P}_j]] - \dots,$$

where  $[\mathbb{P}_i, \mathbb{P}_j]$  is the commutator of the two operators. A real function  $\phi$  on the Lie algebra  $\mathfrak{g}$  is called an invariant if it satisfies the following condition:

$$\phi(Ad_g(\mathcal{M})) = \phi(\mathcal{M})$$
 for all  $\mathcal{M} \in \mathfrak{g}$ .

For the Lie algebra  $\mathfrak{g}$ , we consider any subgroup  $g = exp(\epsilon S)$ , where  $S = \sum_{i=1}^{7} b_i \mathbb{P}_i$  to act on  $\mathcal{M} = \sum_{i=1}^{7} a_i \mathbb{P}_i$ , we

get

$$Ad_{g}(\mathcal{M}) = e^{-\epsilon S} \mathcal{M} e^{\epsilon S}$$

$$= (a_{1} \mathbb{P}_{1} + a_{2} \mathbb{P}_{2} + a_{3} \mathbb{P}_{3} + a_{4} \mathbb{P}_{4} + a_{5} \mathbb{P}_{5} + a_{6} \mathbb{P}_{6} + a_{7} \mathbb{P}_{7})$$

$$- \epsilon (\theta_{1} \mathbb{P}_{1} + \theta_{2} \mathbb{P}_{2} + \theta_{3} \mathbb{P}_{3} + \theta_{4} \mathbb{P}_{4}$$

$$+ \theta_{5} \mathbb{P}_{5} + \theta_{6} \mathbb{P}_{6} + \theta_{7} \mathbb{P}_{7}) + O(\epsilon^{2}),$$

$$(14)$$

where  $\theta_i = \theta_i(a_1, a_2, a_3, a_4, a_5, a_6, a_7, b_1, b_2, b_3, b_4, b_5, b_6, b_7)$ , i = 1, 2, 3, 4, 5, 6, 7 can be obtained from the commutator table (1), and for invariance

$$\phi(a_1, a_2, a_3, a_4, a_5, a_6, a_7) = \phi(a_1 - \epsilon \theta_1, a_2 - \epsilon \theta_2, a_3 - \epsilon \theta_3, a_4 - \epsilon \theta_4, a_5 - \epsilon \theta_5, a_6 - \epsilon \theta_6, a_7 - \epsilon \theta_7).$$
(15)

Expanding the right-hand side of eq. (15), we obtain

$$\theta_{1} \frac{\partial \phi}{\partial a_{1}} + \theta_{2} \frac{\partial \phi}{\partial a_{2}} + \theta_{3} \frac{\partial \phi}{\partial a_{3}} + \theta_{4} \frac{\partial \phi}{\partial a_{4}} + \theta_{5} \frac{\partial \phi}{\partial a_{5}} + \theta_{6} \frac{\partial \phi}{\partial a_{6}} + \theta_{7} \frac{\partial \phi}{\partial a_{7}} = 0,$$
(16)

where

$$\theta_{1} = 0, \ \theta_{2} = -b_{1}a_{2} + b_{2}a_{1}, \ \theta_{3} = 0,$$

$$\theta_{4} = -\frac{2}{3}b_{1}a_{4} - b_{3}a_{4} + \frac{2}{3}b_{4}a_{1} + b_{4}a_{3},$$

$$\theta_{5} = -b_{3}a_{5} + b_{5}a_{3},$$

$$\theta_{6} = -\frac{1}{3}b_{1}a_{6} + \frac{1}{3}b_{6}a_{1},$$

$$\theta_{7} = \frac{1}{3}b_{1}a_{7} - \frac{1}{3}b_{7}a_{1}.$$
(17)

Substitution of equations (17) into equation (16) and collection of the coefficients of all  $b_i^{'s}$  gives the following linear over determined system of PDEs in  $\phi$ :

$$b_{1}: -a_{2} \frac{\partial \phi}{\partial a_{2}} - \frac{2}{3} a_{4} \frac{\partial \phi}{\partial a_{4}} - \frac{1}{3} a_{6} \frac{\partial \phi}{\partial a_{6}} + \frac{1}{3} a_{7} \frac{\partial \phi}{\partial a_{7}} = 0,$$

$$b_{2}: a_{1} \frac{\partial \phi}{\partial a_{2}} = 0,$$

$$b_{3}: -a_{4} \frac{\partial \phi}{\partial a_{4}} - a_{5} \frac{\partial \phi}{\partial a_{5}} = 0,$$

$$b_{4}: \frac{2}{3} a_{1} \frac{\partial \phi}{\partial a_{4}} + a_{3} \frac{\partial \phi}{\partial a_{4}} = 0,$$

$$b_{5}: a_{3} \frac{\partial \phi}{\partial a_{5}} = 0,$$

$$b_{6}: \frac{1}{3} a_{1} \frac{\partial \phi}{\partial a_{6}} = 0,$$

$$b_{7}: -\frac{1}{3} a_{1} \frac{\partial \phi}{\partial a_{7}} = 0.$$

$$(18)$$

Looking the solutions of the above system, we get the invariant form given as,  $\phi(a_1, a_2, a_3, a_4, a_5, a_6, a_7) = F(a_1, a_3)$ , where F can be chosen as an arbitrary function.

Thus, the following two basic invariants of the Lie algebra g exist:

$$\Gamma_1 = a_1$$
 and  $\Gamma_2 = a_3$ ,

also the function  $\eta(\mathbb{P}) = \frac{5}{3}a_1^2 + 2a_3^2 + \frac{4}{3}a_1a_3$ , is invariant of the full adjoint action known as the Killing's form for  $\mathfrak{g}$  ([31] and [36]). It can be seen that the Killing form is a combination of the basic invariants of the Lie Algebra  $\mathfrak{g}$ . Thus, the basic invariants of the Lie algebra  $\mathfrak{g}$  are used to find the one-dimensional optimal system of the equation (4).

Now, we need to prepare the general adjoint transformation matrix A, which is obtained by the product of the individual matrices of the adjoint actions  $A_1, A_2, A_3, A_4, A_5, A_6, A_7$ , which are the adjoint action of  $\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3, \mathbb{P}_4, \mathbb{P}_5, \mathbb{P}_6, \mathbb{P}_7$  to A.

Let  $\epsilon_i$ , i = 1, 2, 3, 4, 5, 6, 7 be real constants and  $g = e^{\epsilon_i \mathbb{P}_i}$ , then we get

$$A_{1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\epsilon_{1}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\frac{2}{3}\epsilon_{1}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\frac{\epsilon_{1}}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\frac{\epsilon_{1}}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{\frac{\epsilon_{1}}{3}} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & e^{\epsilon_{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\epsilon_{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\epsilon_{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\epsilon_{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\epsilon_{3}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\epsilon_{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \end{pmatrix},$$

The adjoint action of  $\mathbb{P}_j$  on  $\mathbb{P}_i$  can be obtained from the adjoint representation, (see Table 2) for more detail, one may refer to Hu et al. [35].

The formation of an optimal system of subalgebras of a Lie algebra is not an easy assignment. An optimal system of Lie subalgebras can be obtained by solving the system of algebraic equations, and the equivalent Lie subalgebras can be identified by the use of adjoint action on the set of these Lie subalgebras. Let

$$X = c_1 \mathbb{P}_1 + c_2 \mathbb{P}_2 + c_3 \mathbb{P}_3 + c_4 \mathbb{P}_4 + c_5 \mathbb{P}_5 + c_6 \mathbb{P}_6 + c_7 \mathbb{P}_7,$$
 (19)

where  $c_1$ ,  $c_2$ ,  $c_3$ ,  $c_4$ ,  $c_5$ ,  $c_6$ ,  $c_7$  are the real constants. Here, X can be considered as a column vector with entries  $c_1$ ,  $c_2$ ,  $c_3$ ,  $c_4$ ,  $c_5$ ,  $c_6$ ,  $c_7$ . Let  $A(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7) = A_7 A_6 A_5 A_4 A_3 A_2 A_1$ , which gives

$$A = \begin{bmatrix} 1 & -\epsilon_2 e^{\epsilon_1} & 0 & -\frac{2}{3} \epsilon_4 e^{\frac{2}{3} \epsilon_1 \epsilon_5} & 0 & -\frac{1}{3} \epsilon_6 e^{\frac{1}{3} \epsilon_1} & \frac{1}{3} \epsilon_7 e^{-\frac{1}{3} \epsilon_1} \\ 0 & e^{\epsilon_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\epsilon_4 e^{\frac{1}{3} \epsilon_1 \epsilon_5} & -\epsilon_5 e^{\epsilon_5} & 0 & 0 \\ 0 & 0 & 0 & e^{\frac{1}{3} \epsilon_1 \epsilon_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\epsilon_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\epsilon_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{\frac{\epsilon_5}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{\frac{\epsilon_5}{3}} \end{bmatrix}.$$

Now, to construct an optimal system of equation (4), we consider  $X = \sum_{j=1}^{7} c_j \mathbb{P}_j$  and  $Y = \sum_{j=1}^{7} d_j \mathbb{P}_j$  as two elements of Lie algebra  $\mathfrak{g}$ . Adjoint transformation equation for equation (4) is

$$(d_1, d_2, d_3, d_4, d_5, d_6, d_7) = (c_1, c_2, c_3, c_4, c_5, c_6, c_7) \cdot A$$
(20)

Table 2: Adjoint Table

Ad	$\mathbb{P}_1$	$\mathbb{P}_2$	$\mathbb{P}_3$	$\mathbb{P}_4$	$\mathbb{P}_5$	$\mathbb{P}_6$	$\mathbb{P}_7$
$\mathbb{P}_1$	$\mathbb{P}_1$	$e^{\epsilon}\mathbb{P}_2$	$\mathbb{P}_3$	$e^{rac{2}{3}}\mathbb{P}_4$	$\mathbb{P}_5$	$e^{rac{1}{3}\epsilon}\mathbb{P}_6$	$e^{-rac{1}{3}\epsilon}\mathbb{P}_7$
$\mathbb{P}_2$	$\mathbb{P}_1 - \epsilon \mathbb{P}_2$	$\mathbb{P}_2$	$\mathbb{P}_3$	$\mathbb{P}_4$	$\mathbb{P}_5$	$\mathbb{P}_6$	$\mathbb{P}_7$
$\mathbb{P}_3$	$\mathbb{P}_1$	$\mathbb{P}_2$	$\mathbb{P}_3$	$e^{\epsilon}\mathbb{P}_{4}$	$e^{\epsilon}\mathbb{P}_{5}$	$\mathbb{P}_6$	$\mathbb{P}_7$
$\mathbb{P}_4$	$\mathbb{P}_1 - \frac{2}{3} \epsilon \mathbb{P}_4$	$\mathbb{P}_2$	$\mathbb{P}_3 - \boldsymbol{\epsilon} \mathbb{P}_4$	$\mathbb{P}_4$	$\mathbb{P}_5$	$\mathbb{P}_6$	$\mathbb{P}_7$
$\mathbb{P}_5$	$\mathbb{P}_1$	$\mathbb{P}_2$	$\mathbb{P}_3 - \boldsymbol{\epsilon} \mathbb{P}_5$	$\mathbb{P}_4$	$\mathbb{P}_5$	$\mathbb{P}_6$	$\mathbb{P}_7$
$\mathbb{P}_6$	$\mathbb{P}_1 - \frac{1}{3}\epsilon \mathbb{P}_6$	$\mathbb{P}_2$	$\mathbb{P}_3$	$\mathbb{P}_4$	$\mathbb{P}_5$	$\mathbb{P}_6$	$\mathbb{P}_7$
$\mathbb{P}_7$	$\mathbb{P}_1 + \frac{1}{3}\epsilon \mathbb{P}_7$	$\mathbb{P}_2$	$\mathbb{P}_3$	$\mathbb{P}_4$	$\mathbb{P}_5$	$\mathbb{P}_6$	$\mathbb{P}_7$

In addition,  $A(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7)$  transform X as follows

$$A(\epsilon_{1}, \epsilon_{2}, \epsilon_{3}, \epsilon_{4}, \epsilon_{5}, \epsilon_{6}, \epsilon_{7}) \cdot X = c_{1} \mathbb{P}_{1}$$

$$+ (-c_{1}\epsilon_{2}e^{\epsilon_{1}} + c_{2}e^{\epsilon_{1}}) \mathbb{P}_{2} + c_{3} \mathbb{P}_{3} + \left(-\frac{2}{3}c_{1}\epsilon_{4}e^{\frac{2}{3}\epsilon_{1}+\epsilon_{3}}\right)$$

$$-c_{3}\epsilon_{4}e^{\frac{2}{3}\epsilon_{1}+\epsilon_{3}} + c_{4}e^{\frac{2}{3}\epsilon_{1}+\epsilon_{3}}) \mathbb{P}_{4} + \left(-c_{3}\epsilon_{5}e^{\epsilon_{3}} + c_{5}e^{\epsilon_{3}}\right) \mathbb{P}_{5}$$

$$+ \left(-\frac{1}{3}c_{1}\epsilon_{6}e^{\frac{1}{3}\epsilon_{1}} + c_{6}e^{\frac{1}{3}\epsilon_{1}}\right) \mathbb{P}_{6}$$

$$+ \left(\frac{1}{3}c_{1}\epsilon_{7}e^{-\frac{1}{3}\epsilon_{1}} + c_{7}e^{-\frac{1}{3}\epsilon_{1}}\right) \mathbb{P}_{7}.$$

$$(21)$$

By definition, X and  $A(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7) \cdot X$  generate equivalent one dimensional Lie subalgebras for any  $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7$ . This provides the liberty of choosing various values of  $\varepsilon_i$  to represent the equivalence class of X that might be much simpler than X. In order to distinguish the one dimensional Lie subalgebras of equation (4), we consider the cases as follows:

#### Case-1

 $c_1=1$ ,  $c_3=l_1$ . Here,  $l_1\notin \left\{-\frac{2}{3},0\right\}$  is an arbitrary real constant. Now, choosing a representative element  $\tilde{X}=\mathbb{P}_1+c_2\mathbb{P}_2+c_3\mathbb{P}_3+c_4\mathbb{P}_4+c_5\mathbb{P}_5+c_6\mathbb{P}_6+c_7\mathbb{P}_7$ , and putting  $d_1=1$ ,  $d_2=d_3=d_4=d_5=d_6=d_7=0$ ,  $d_3=l_1$  in equation (20), we get the solution as

$$\epsilon_2 = c_2$$
,  $\epsilon_4 = \frac{c_4}{\frac{2}{3} + c_3}$ ,  $\epsilon_5 = \frac{c_5}{c_3}$ ,  $\epsilon_6 = 3a_6$ ,  $\epsilon_7 = -3a_7$ .

Thus, the action of adjoint maps  $Ad(exp(\epsilon_2\mathbb{P}_2))$ ,  $Ad(exp(\epsilon_4\mathbb{P}_4))$ ,  $Ad(exp(\epsilon_5\mathbb{P}_5))$ ,  $Ad(exp(\epsilon_6\mathbb{P}_6))$  and

 $Ad(exp(\epsilon_7\mathbb{P}_7))$  will eliminate the coefficients of  $\mathbb{P}_2$ ,  $\mathbb{P}_4$ ,  $\mathbb{P}_5$ ,  $\mathbb{P}_6$  and  $\mathbb{P}_7$ , respectively, from  $\tilde{X}$ . Thus,  $\tilde{X} = \mathbb{P}_1 + c_3\mathbb{P}_3$  is equivalent to  $\mathbb{P}_1 + c_2\mathbb{P}_2 + l_1\mathbb{P}_3 + c_4\mathbb{P}_4 + c_5\mathbb{P}_5 + c_6\mathbb{P}_6 + c_7\mathbb{P}_7$ .

#### Case-2

 $c_1=1$ ,  $c_3=0$ . Now, choosing a representative element  $\tilde{X}=\mathbb{P}_1+c_2\mathbb{P}_2+c_4\mathbb{P}_4+c_5\mathbb{P}_5+c_6\mathbb{P}_6+c_7\mathbb{P}_7$ , and putting  $d_1=1$ ,  $d_2=d_3=d_4=d_5=d_6=d_7=0$  in equation (20), we get the solution as

$$\epsilon_2 = c_2, \ \epsilon_4 = \frac{3}{2}c_4, \ \epsilon_6 = 3a_6, \ \epsilon_7 = -3a_7.$$
 (23)

Thus, the actions of adjoint maps  $Ad(exp(\epsilon_2\mathbb{P}_2))$ ,  $Ad(exp(\epsilon_4\mathbb{P}_4))$ ,  $Ad(exp(\epsilon_6\mathbb{P}_6))$  and  $Ad(exp(\epsilon_7\mathbb{P}_7))$  will eliminate the coefficients of  $\mathbb{P}_2$ ,  $\mathbb{P}_3$ ,  $\mathbb{P}_4$ ,  $\mathbb{P}_6$  and  $\mathbb{P}_7$  from  $\tilde{X}$ . Thus,  $\tilde{X} = \mathbb{P}_1 + c_5\mathbb{P}_5$  is equivalent to  $\mathbb{P}_1 + c_2\mathbb{P}_2 + c_4\mathbb{P}_4 + c_5\mathbb{P}_5 + c_6\mathbb{P}_6 + c_7\mathbb{P}_7$ .

#### Case-3

 $c_1=0$ ,  $c_3=1$ . Now, choosing a representative element  $\tilde{X}=c_2\mathbb{P}_2+\mathbb{P}_3+c_4\mathbb{P}_4+c_5\mathbb{P}_5+c_6\mathbb{P}_6+c_7\mathbb{P}_7$ , and putting  $d_1=0$ ,  $d_2=d_4=d_5=d_6=d_7=0$ ,  $d_3=1$  in equation (20), we get the solution as

$$\epsilon_4 = c_4, \ \epsilon_5 = a_5.$$
 (24)

Thus, the action of adjoint maps  $Ad(exp(\epsilon_4\mathbb{P}_4))$ ,  $Ad(exp(\epsilon_5\mathbb{P}_5))$  will eliminate the coefficients of  $\mathbb{P}_4$  and  $\mathbb{P}_5$  fron  $\tilde{X}$ . Thus,  $\tilde{X} = c_2\mathbb{P}_2 + \mathbb{P}_3 + c_6\mathbb{P}_6 + c_7\mathbb{P}_7$ .

#### Case-4

 $c_1=0$ ,  $c_3=0$ . Now, choosing a representative element  $\tilde{X}=c_2\mathbb{P}_2+c_4\mathbb{P}_4+c_5\mathbb{P}_5+c_6\mathbb{P}_6+c_7\mathbb{P}_7$ , and putting  $d_1=d_2=d_3=d_4=d_5=d_6=d_7=0$  in equation (20), we get,  $\tilde{X}=c_2\mathbb{P}_2+c_4\mathbb{P}_4+c_5\mathbb{P}_5+c_6\mathbb{P}_6+c_7\mathbb{P}_7$ . To summarize, an optimal system of one-dimensional subalgebras of equation (4) is obtained to be those generated by

$$\begin{split} \tilde{\mathbb{P}_{1}} &= \mathbb{P}_{1} + c_{3}\mathbb{P}_{3}, \\ \tilde{\mathbb{P}_{2}} &= \mathbb{P}_{1} + c_{5}\mathbb{P}_{5}, \\ \tilde{\mathbb{P}_{3}} &= c_{2}\mathbb{P}_{2} + \mathbb{P}_{3} + c_{6}\mathbb{P}_{6} + c_{7}\mathbb{P}_{7}, \\ \tilde{\mathbb{P}_{4}} &= c_{2}\mathbb{P}_{2} + c_{4}\mathbb{P}_{4} + c_{5}\mathbb{P}_{5} + c_{6}\mathbb{P}_{6} + c_{7}\mathbb{P}_{7}. \end{split}$$

i.e., any subalgebra spanned by  $\mathbb{P}_1$ ,  $\mathbb{P}_2$ ,  $\mathbb{P}_3$ ,  $\mathbb{P}_4$ ,  $\mathbb{P}_5$ ,  $\mathbb{P}_6$ ,  $\mathbb{P}_7$  is equivalent to some  $\tilde{\mathbb{P}}_i$  in the set  $\{\tilde{\mathbb{P}}_1, \tilde{\mathbb{P}}_2, \tilde{\mathbb{P}}_3, \tilde{\mathbb{P}}_4\}$ .

## 4 Invariant solutions

After the formation of one-dimensional optimal system of equation (4), we reach the equivalence class of group invariant solutions of equation (4). We will present the details of the calculation for some of the vector fields and directly give the calculation results for the remaining vector fields.

**4.1** 
$$\tilde{\mathbb{P}}_1 = \mathbb{P}_1 + c_3 \mathbb{P}_3$$

Solving the characteristic equation, similarity variables can be obtained as

$$\frac{dx}{\frac{x}{3}} = \frac{dy}{c_3 y} = \frac{dz}{(c_3 + \frac{2}{3})z} = \frac{dt}{t} = \frac{du}{-\frac{u}{3}},$$
 (25)

then, we get the invariants as  $X = \frac{X}{t^{\frac{1}{3}}}$ ,  $Y = \frac{Y}{t^{c_3}}$ ,  $Z = \frac{Z}{t^{c_3+\frac{2}{3}}}$ ,  $u = t^{-\frac{1}{3}}F(X,Y,Z)$  with X,Y,Z as new independent variables and F as a new dependent variable. Thus, equation (4) transforms to

$$6c_{3}YF_{XYY} + 6c_{3}ZF_{XYZ} + 2XF_{XXY} + 4ZF_{XYZ} + 6F_{X}F_{XXY} + 12F_{XY}F_{XX} + 6c_{3}F_{XY} + 6F_{XXX}F_{Y} + 9F_{XXZ} + 4F_{XY} - 3F_{XXXXY} = 0.$$
 (26)

Again, reducing the equation (26) by point symmetries, the following vector fields are found to span the symmetry group of equation (26):

$$\begin{split} \xi_{X} &= f_{1}(Z), \; \xi_{Y} = n_{1}Y + n_{2}Z^{\left(\frac{1}{1-\frac{2}{1-\alpha_{2}}}\right)}, \; \xi_{Z} = n_{1}Z, \\ \eta_{F} &= \frac{\left(27^{\frac{3-c_{1}}{2^{3-c_{1}+2}}}n_{2}\;c_{3}\;x + 18\;z\left(c_{3} + \frac{2}{3}\right)\left(\left(x\left(c_{3} + \frac{2}{3}\right)z + \frac{3}{2}y\right)\frac{d}{dz}f_{1}\left(z\right) - \frac{1}{3}\;xf_{1}\left(z\right) + f_{2}\left(z\right)\right)\right)}{(18\;c_{3} + 12)\;z}, \end{split}$$

where  $f_1(Z)$  and  $f_2(Z)$  are the arbitrary functions. By the appropriate choice of the arbitrary functions of the above equation, if  $f_1(Z) = Z$ ,  $f_2(Z) = f_1'(Z)$  and  $c_3 = 1$ , it leads to the following characteristic equations:

$$\frac{dX}{Z} = \frac{dY}{n_1 YZ + n_2 Z^{\frac{3}{5}}} = \frac{dZ}{n_1 Z}$$

$$= \frac{dF}{\frac{27 n_2 Z^{\frac{3}{5}} X + 30 Z((\frac{5}{3} XZ + \frac{3}{2} Y) \frac{d}{dZ} f_1(Z) - \frac{1}{3} X f_1(Z) + f_2(Z))}{30 Z}}, (28)$$

which yields

$$F = \frac{2}{3} \frac{Z^2}{n_1^2} + \frac{4}{3} \frac{rZ}{n_1} + \frac{3}{2} \frac{sZ}{n_1} - \frac{19 n_2 Z^{3/5}}{4 n_1^2} + \frac{\ln(Z)}{n_1} - \frac{9}{4} \frac{n_2 r}{n_1 Z^{\frac{2}{5}}} + G(r, s),$$
(29)

where G(r, s) is a similarity function of variables r and s, which are given by

$$r = X + \frac{Z}{n_1}, \ s = \frac{Y}{Z} + \frac{5}{2} \frac{n_2}{n_1 Z^{\frac{2}{5}}}.$$
 (30)

Thus, the second reduction by similarity of equation (4) gives

$$12G_{rrrs}G_r + (4r - 18s)G_{rrs} + 24G_{rs}G_{rr} + 12G_{rrr}G_s$$
$$-6G_{rrrrs} - 8sG_{rss} = 0.$$
(31)

We can see that, this is a nonlinear PDE with two independent and one dependent variable. After applying the similarity transform again, the following vector fields are found to span the symmetry group of equation (31):

$$\xi_r = 0, \ \xi_s = 0, \ \eta_G = m_1.$$
 (32)

Thus, the characteristic equation for the second reduction of equation (4) is

$$\frac{dr}{0} = \frac{ds}{0} = \frac{dG}{m_1},\tag{33}$$

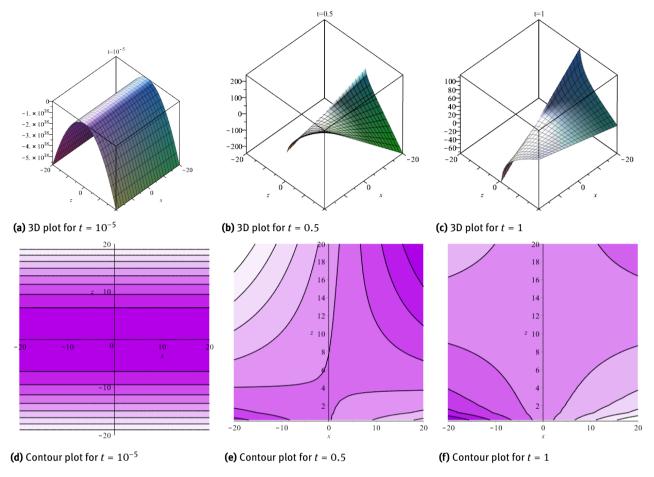
which leads to  $G = m_2$ , where  $m_1$  and  $m_2$  are some real constants. Thus, the invariant solution of equation (4) is given by

$$u(x, y, z, t) = \frac{1}{12 n_1^2 t^{\frac{11}{3}} z^{\frac{2}{5}}} \left( 12 m_2 n_1^2 t^{\frac{10}{3}} z^{\frac{2}{5}} \right)$$

$$+ 12 n_1 t^{\frac{10}{3}} z^{\frac{2}{5}} \ln(z) - 20 n_1 t^{\frac{10}{3}} z^{\frac{2}{5}} \ln(t) + 16 n_1 z^{\frac{7}{5}} t^{\frac{4}{3}} x$$

$$-8 z^{\frac{12}{5}} + 18 n_1 z^{\frac{2}{5}} t^{\frac{7}{3}} y + 15 n_2 t^{\frac{7}{3}} z - 27 n_1 n_2 t^{\frac{11}{3}} x \right).$$

$$(34)$$



**Figure 1:** Solution profiles for equation (34) with  $m_2 = 20$ ,  $n_1 = 10$ ,  $n_2 = 20$  and y = 5.

**4.2** 
$$\tilde{\mathbb{P}}_2 = \mathbb{P}_1 + c_5 \mathbb{P}_5$$

For this subalgebra, the similarity variables can be obtained by the following characteristic equation:

$$\frac{dx}{\frac{x}{3}} = \frac{dy}{c_5} = \frac{dz}{\frac{2}{3}z} = \frac{dt}{t} = \frac{du}{-\frac{u}{3}},$$
 (35)

then, we get the invariants as  $X = \frac{X}{t^{\frac{1}{3}}}$ ,  $Y = y - c_5 \ln(t)$ ,  $Z = \frac{Z}{t^{\frac{2}{3}}}$  and  $u = t^{-\frac{1}{3}} F(X, Y, Z)$  with X, Y, Z as new independent variables and F as new dependent variable. Thus, equation (4) is then transformed to

$$3F_{XXZ} + \frac{2}{3}XF_{XXY} + 2c_5F_{XYY} + \frac{4}{3}ZF_{XYZ} + \frac{4}{3}F_{XY} - F_{XXXXY} + 4F_{XY}F_{XX} + 2F_XF_{XXY} + 2F_YF_{XXX} = 0.$$
 (36)

Again, reducing the equation (36) by point symmetries, the following vector fields are found to span the symmetry group of equation (36):

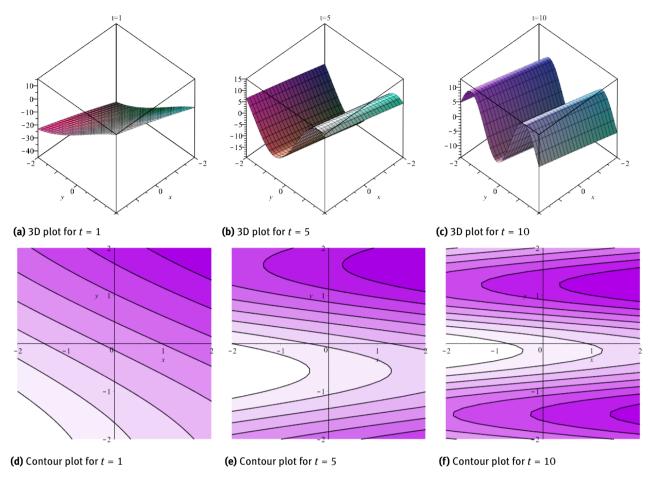
$$\xi_{X} = f_{1}(Z), \ \xi_{Y} = n_{1}Y - \frac{3}{2} n_{1} c_{5} \ln(Z) + n_{2}, \ \xi_{Z} = n_{1}Z,$$

$$\eta_{F} = \frac{3}{2} Y f_{1}'(Z) + \frac{1}{12} \frac{X}{Z} (8 Z^{2} f_{1}'(Z) - 4 Z f_{1}(Z) - 27 n_{1} c_{5}) + f_{2}(Z),$$
(37)

where  $f_1(Z)$  and  $f_2(Z)$  are the arbitrary functions. It leads to the following characteristic equations:

$$\frac{dX}{f_1(Z)} = \frac{dY}{n_1Y - \frac{3}{2} n_1 c_5 \ln(Z) + n_2} = \frac{dZ}{n_1Z}$$

$$= \frac{dF}{\frac{3}{2} Y f_1'(Z) + \frac{X}{12Z} (8 Z^2 f_1'(Z) - 4 Z f_1(Z) - 27 n_1 c_5) + f_2(Z)}$$
(38)



**Figure 2:** Solution profiles for equation (44) with  $n_1 = 1$ ,  $n_2 = 2$ ,  $m_3 = 3$ , z = 2 and R = 20sin(w).

By the appropriate choice of the arbitrary functions of the above equation, if  $f_1(Z) = Z$ ,  $f_2(Z) = f_1^{'}(Z)$  and  $c_5 = 1$ , we obtain

$$F = \frac{9 \left(\ln (Z)\right)^{2}}{8 n_{1}} + \frac{9r}{4Z} + \frac{rZ}{3n_{1}} + \frac{3sZ}{2n_{1}} + \frac{Z^{2}}{6n_{1}^{2}} + \frac{\ln (Z)}{n_{1}} - \frac{3n_{2} \ln (Z)}{2n_{1}^{2}} + G(r, s),$$
(39)

where G(r, s) is a similarity function of variables r and s, which are given by

$$r = X - \frac{Z}{n_1}, \ s = \left(Y - \frac{3}{2}\ln(Z) - \frac{3}{2} + \frac{n_2}{n_1}\right) \cdot \frac{1}{Z}.$$
 (40)

Thus, the second reduction by similarity of equation (4) gives

$$3sG_{rrs} + \frac{2}{3}rG_{rrs} - \frac{4}{3}sG_{rss} - G_{rrrrs} + 4G_{rs}G_{rr} + 2G_{r}G_{rrs} + 2G_{rrr}G_{s} = 0.$$
 (41)

We can see that, this is a nonlinear PDE with two independent and one dependent variable. After applying the similarity transform again, the following vector fields are found to span the symmetry group of equation (41),

$$\xi_r = m_1, \ \xi_s = 0, \ \eta_G = -\frac{1}{3} m_1 r + m_2.$$
 (42)

Thus, the characteristic equation for the second reduction of equation (4) is

$$\frac{dr}{m_1} = \frac{ds}{0} = \frac{dG}{-\frac{1}{2}m_1r + m_2},\tag{43}$$

where  $m_1$  and  $m_2$  are some real constants.

This leads to  $G = -\frac{r^2}{6} + m_3 r + R(w)$ , where R(w) is an arbitrary function of w with w = s and  $m_3 = \frac{m_2}{m_1}$ . Thus, an invariant solution of equation (4) is given by

$$u(x, y, z, t) = \frac{1}{\sqrt[3]{t}} \left( \frac{9}{8 n_1} \left( \ln \left( \frac{z}{t^{\frac{2}{3}}} \right) \right)^2 + \frac{9}{4} \frac{t^{\frac{2}{3}}}{z} \left( \frac{x}{\sqrt[3]{t}} - \frac{z}{n_1 t^{\frac{2}{3}}} \right) + \frac{1}{3} \frac{z}{n_1 t^{\frac{2}{3}}} \left( \frac{x}{\sqrt[3]{t}} - \frac{z}{n_1 t^{\frac{2}{3}}} \right) \right)$$

$$+ \frac{3}{2} \frac{1}{n_1} \left( y - \ln(t) - \frac{3}{2} \ln \left( \frac{z}{t^{\frac{2}{3}}} \right) - \frac{3}{2} + \frac{n_2}{n_1} \right) + \frac{1}{6} \frac{z^2}{n_1^2 t^{\frac{4}{3}}} + \frac{1}{n_1} \ln \left( \frac{z}{t^{\frac{2}{3}}} \right)$$

$$- \frac{3}{2} \frac{n_2}{n_1^2} \ln \left( \frac{z}{t^{\frac{2}{3}}} \right) - \frac{1}{6} \left( \frac{x}{\sqrt[3]{t}} - \frac{z}{n_1 t^{\frac{2}{3}}} \right)^2 + m_3 \left( \frac{x}{\sqrt[3]{t}} - \frac{z}{n_1 t^{\frac{2}{3}}} \right) + R(w) \right),$$

$$(44)$$

where

$$w = \frac{t^{\frac{2}{3}}}{z} \left( y - \ln(t) - \frac{3}{2} \ln\left(\frac{z}{t^{\frac{2}{3}}}\right) - \frac{3}{2} + \frac{n_2}{n_1} \right).$$

**4.3** 
$$\tilde{\mathbb{P}}_3 = c_2 \mathbb{P}_2 + \mathbb{P}_3 + c_6 \mathbb{P}_6 + c_7 \mathbb{P}_7$$

For this subalgebra the similarity variables can be obtained by the following characteristic equations:

$$\frac{dx}{c_6} = \frac{dy}{y} = \frac{dz}{z} = \frac{dt}{c_2} = \frac{du}{c_7},$$
 (45)

then we get the invariants as  $X = x - c_6 \ln(z)$ ,  $Y = \frac{y}{z}$ ,  $T = t + c_2 \ln(z)$  and  $u = c_7 \ln(z) + F(X, Y, T)$  with X, Y, T as new independent variables and F as new dependent variable. Thus, equation (4) gets transformed to

$$-3c_{6}F_{XXX} - 3YF_{XXY} - 3c_{2}F_{XXZ} - 2F_{XYZ} - F_{XXXXY} + 4F_{XY}F_{XX} + 2F_{X}F_{XXY} + 2F_{XXX}F_{Y} = 0.$$
 (46)

Again, reducing the equation (46) by point symmetries, the following vector fields are found to span the symmetry group of equation (46):

$$\xi_X = -\frac{1}{2} f_2(T)X + f_4(T), \ \xi_Y = f_2(T) Y + f_3(T), \ \xi_T = f_1(T),$$

$$\eta_F = \frac{1}{4} (3 c_6 Y + 2F) f_2(T) + \frac{3}{2} X f_3(T) + f_5(T),$$
(47)

where  $f_1(T)$ ,  $f_2(T)$ ,  $f_3(T)$ ,  $f_4(T)$  and  $f_5(T)$  are the arbitrary functions. By the appropriate choice of the arbitrary functions in the above equation, If  $f_1(T) = T$ ,  $f_2(T) = f_1'(T)$ ,  $f_3(T) = m_1$ ,  $f_4(T) = m_2$ , and  $f_5(T) = m_3$ , where  $m_1$ ,  $m_2$  and  $m_3$  are the arbitrary constants, it leads to the following characteristic equations:

$$\frac{dX}{-\frac{1}{2}X + m_2} = \frac{dY}{Y + m_1} = \frac{dT}{T} = \frac{dF}{\frac{1}{4}(3c_6Y + 2F) + \frac{3}{2}m_1X + m_3},$$
(48)

which yields,

$$F = -\frac{3}{2} \frac{m_1}{\sqrt{T}} r + \frac{3}{2} s c_6 T + \frac{3}{2} m_1 c_6 - 6 m_1 m_2 - 2 m_3 + \sqrt{T} R(r, s), \tag{49}$$

where G(r, s) is a similarity function of variables r and s, which are given by

$$r = \sqrt{T}(X - 2 m_2), \ s = \frac{Y + m_1}{T},$$
 (50)

Thus, the second reduction by similarity of equation (4) gives

$$2G_{rrs}G_r + 4G_{rs}G_{rr} + 2G_{rrr}G_s - 3sG_{rrs} - G_{rrrs} = 0. ag{51}$$

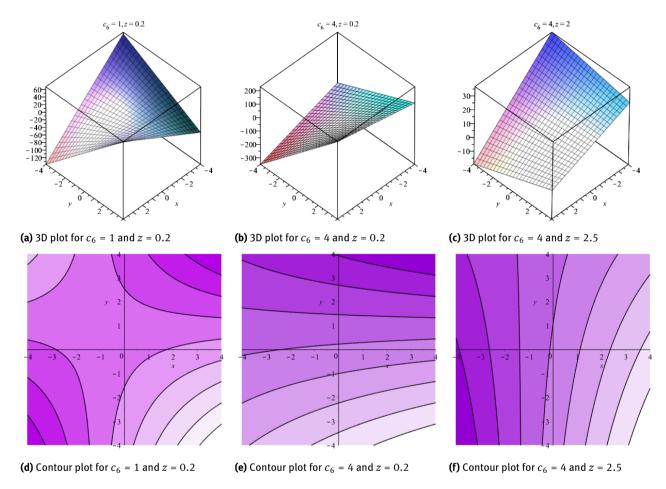
We can see that, this is a nonlinear PDE with two independent and one dependent variable. After using the similarity transform again, the following vector fields are found to span symmetry group of equation (51):

$$\xi_r = -\frac{1}{2} n_1 r + n_3, \ \xi_s = n_1 s + n_2, \ \eta_G = \frac{3}{2} n_2 r + \frac{1}{2} n_1 G + n_4,$$
 (52)

where  $n_1$ ,  $n_2$ ,  $n_3$  and  $n_4$  are the arbitrary constants. By the appropriate choice of these constants, If  $n_1 = 1$ ,  $n_2 = n_3 = n_4 = 0$ , it leads to the following characteristic equations:

$$\frac{dr}{-\frac{r}{2}} = \frac{ds}{s} = \frac{dG}{\frac{1}{2}G}.$$
 (53)

which leads to  $G = \frac{R(w)}{r}$ , where R(w) is an arbitrary function of  $w = s r^2$ .



**Figure 3:** Solution profiles for equation (55) with  $m_1 = 2$ ,  $m_2 = 1$ ,  $m_3 = 1$  and  $c_7 = 2$ .

Thus, the second reduction by similarity of equation (4) gives

$$-8 w^{4} R^{'''''} + 16 w^{3} R^{'''} R^{'} + 16 w^{3} (R^{''})^{2} - 6 w^{3} R^{'''} - 40 w^{3} R^{''''} - 4 w^{2} R^{'''} R + 12 w^{2} R^{''} R^{'}$$

$$-30 w^{2} R^{'''} - 9 w^{2} R^{''} + 2 w R^{''} R + 2 w (R^{'})^{2} - 2 R^{'} R = 0.$$
(54)

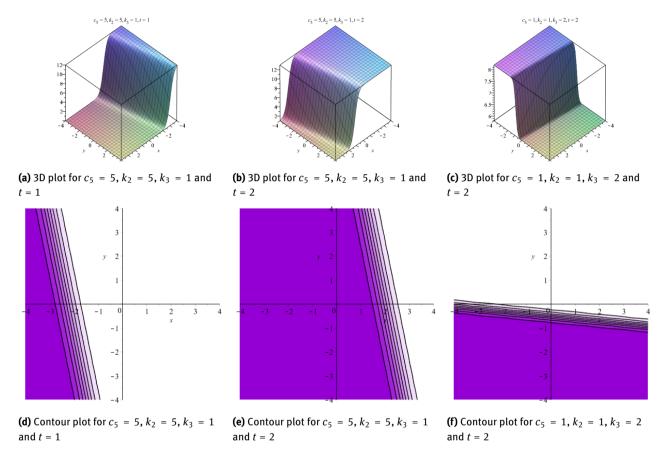
Now, with a particular solution for equation (54) as R(w) = w, an invariant solution of equation (4) is given by

$$u(x, y, z, t) = c_7 \ln(z) - \frac{3}{2} (x - c_6 \ln(z) - 2 m_2) m_1 + \frac{3}{2} (\frac{y}{z} - m_1) c_6 + \frac{3}{2} c_6 m_1 - 6 m_1 m_2 - 2 m_3 + (\frac{y}{z} - m_1) (x - a_6 \ln(z) - 2 m_2).$$
(55)

Solving the characteristic equations:

$$\frac{dx}{c_6} = \frac{dy}{c_5} = \frac{dz}{c_4} = \frac{dt}{c_2} = \frac{du}{c_7},\tag{56}$$

we obtain the invariants as  $X = \frac{c_2}{c_6}x - t$ ,  $Y = \frac{c_2}{c_5}y - t$ ,  $Z = \frac{c_2}{c_4}z - t$  and  $u = \frac{c_7}{c_2}t + F(X, Y, Z)$  with X, Y, Z as new independent



**Figure 4:** Solution profiles for equation (58) with  $c_2 = 2$ ,  $c_4 = 5$ ,  $c_6 = 5$ ,  $c_7 = 2$ ,  $k_1 = 8$ ,  $k_5 = 5$ , and z = 5.

variables and F as a new dependent variable. Thus, equation (4) gets transformed to

$$2c_{2}^{2}c_{4}c_{6}F_{X}F_{XXY} + 4c_{2}^{2}c_{4}c_{6}F_{XY}F_{XX} + 2c_{2}^{2}c_{4}c_{6}F_{XXX}F_{Y} + 3c_{2}c_{6}^{2}c_{5}F_{XXZ} + 2c_{4}c_{6}^{3}F_{XXY} + 2c_{4}c_{6}^{3}F_{XYY} + 2c_{4}c_{6}^{3}F_{XYZ} - c_{2}^{2}c_{4}F_{XXXXY} = 0.$$

$$(57)$$

This is a nonlinear PDE in three variables X, Y, Z having the general solution  $F(X, Y, Z) = -3 \frac{k_2 c_2}{c_6} \tanh \left( k_2 X + k_3 Y + \frac{2 k_3 c_4 \left( 2 k_2^3 c_2^3 - k_2 c_6^3 - k_3 c_6^3 \right) Z}{c_6^2 (3 k_2 c_2 c_5 + 2 k_3 c_4 c_6)} + k_1 \right) + k_4$ , which provides an invariant solution for equation (4) as

$$u(x, y, z, t) = \frac{c_7 t}{c_2} - \frac{3 k_2 c_2}{c_6} \tanh \left( k_2 \left( \frac{c_2 x}{c_6} - t \right) + k_3 \left( \frac{c_2 y}{c_5} - t \right) + \frac{2 k_3 c_4 \left( 2 c_2^3 k_2^3 - c_6^3 k_2 - c_6^3 k_3 \right)}{c_6^2 \left( 3 c_2 c_5 k_2 + 2 c_4 c_6 k_3 \right)} \left( \frac{c_2 z}{c_4} - t \right) + k_1 \right) + k_4.$$
(58)

## 5 Discussion and conclusion

In the previous sections, we have made a possible attempt to analyze a (3+1)-dimensional nonlinear evolution equation ([5–8, 18]) by a well-organized Lie Symmetry method to find the group invariant solutions of the equation so that different types of solitary solutions can be obtained for the same. We acquired the geometric symmetry encompassed by seven basic symmetry algebra. For the classification of all the subalgebra, an optimal system of subalgebras is entrenched. Moreover, similarity solutions are also presented, along with solutions in terms of hypergeometric function. Thus, we obtained a variety of different kinds of multiple soliton solutions for the (3+1)-dimensional nonlinear equation, where significant features and distinct physical structures can be noticed for each set of specific solutions. To the best of our knowledge, the similarity solutions through an optimal system for the same nonlinear equation have not been obtained before. A different variety of soliton solutions has been obtained, and in further work, it can be considered for other nonlinear models by the same systematic approach. The results would be of more importance in understanding different phenomena of different types of nonlinear waves in nonlinear systems, optics, fluid dynamics, including water waves. Also, in view of the availability of programming languages like Mathematica or Maple (which makes tedious algebraic calculations easy), we observed that the Lie Symmetry method is a direct, standard, and computerbased method. The properties of new solutions for the (3+1)-dimensional nonlinear equation are easy to observe by the given figures.

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