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Approximate solution for fractional attractor one-dimensional Keller-Segel equations using homotopy perturbation sumudu transform method

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Abstract: In this paper, homotopy perturbation sumudu transform method (HPSTM) is proposed to solve fractional attractor one-dimensional Keller-Segel equations. The HPSTM is a combined form of homotopy perturbation method (HPM) and sumudu transform using He's polynomials. The result shows that the HPSTM is very efficient and simple technique for solving nonlinear partial differential equations. Test examples are considered to illustrate the present scheme.

Keywords: homotopy perturbation method; Sumudu transform; Keller-Segel equation; He's polynomial

1 Introduction

The fractional calculus deal with number of problem arising in the field of fluid mechanics, biology, diffusion, fractional signal, image processing and many other physical process. Fractional differential equations are used to model these types of the problems. In the various field of engineering and science, it is very important to find the approximate or the exact solution of some nonlinear partial differential equations [1]. There are several potent methods such as Homotopy perturbation [2, 3]; homotopy perturbation transformation method (HPTM) [4] and homotopy perturbation sumudu transformation method (HPSTM) have been proposed to obtain the approximate or the exact solutions of nonlinear equations [5–9].

In 1970, Keller and Segel presented a mathematical formulation of cellular slime mold aggregation process [10]. Recently many researchers use different methods to solve Keller- Segel equation [11–13]. Different types of numerical methods are used to solve nonlinear partial differential equations [14–19]. The solution of multidimensional linear and nonlinear partial differential equations are established by using combination of least square approximation and homotopy perturbation approximation [20]. A new semi-analytical method called the homotopy analysis Shehu transform method is used to solve multidimensional fractional diffusion equations and this method is combination of the homotopy analysis method and the Laplace-type integral transform transform [21]. Solution of Reaction-Diffusion-Convection Problem and nonlinear equation is discussed by homotopy perturbation technique [22, 23]. Non-linear Fisher equation is solved with help of homotopy perturbation method then solution is compared with solution from Variational Iteration Method (VIM) and Adomian Decomposition Method (ADM) [24]. In this paper, we propose HPSTM for the solution of fractional attractor one dimensional Keller Siegel equation. The simplified form of the Keller Segel equation in one dimension is given as [25]:

$$\frac{\partial U(x,t)}{\partial t} = a\frac{\partial^2 U(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\left(U(x,t)\frac{\partial \chi(\rho)}{\partial x}\right)$$

$$\frac{\partial \rho(x,t)}{\partial t} = b\frac{\partial^2 \rho(x,t)}{\partial x^2} + cU(x,t) - d\rho(x,t) \qquad (1)$$

Subject to the boundary conditions

$$\frac{\partial U(\alpha,t)}{\partial x} = \frac{\partial U(\beta,t)}{\partial x} = \frac{\partial \rho(\alpha,t)}{\partial x} = \frac{\partial \rho(\beta,t)}{\partial x} = 0$$

And the initial conditions

$$U(x,0) = U_0(x),\ \rho(x,0) = \rho_0(x),\ x \in I \qquad (2)$$

Where $I = (\alpha,\ \beta)$ is a bounded open interval and a, b, c and d are positive constants. The unknown functions $U(x,t)$ denote the concentration of amoebae where $\rho(x,t)$ denote the concentration of the chemical substance in $I \times (0,\infty)$. The chemo tactic term $\frac{\partial}{\partial x}\left(U(x,t)\frac{\partial \chi(\rho)}{\partial x}\right)$ indicates the sensitivity of the cells, $\chi(\rho)$ called the sensitivity function of $\rho \in (0,\infty)$. Different form of $\chi(\rho)$ like ρ, ρ^2 and

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$\log\rho$ have been suggested. But in this paper, we discuss the two cases of the fractional attractor one dimensional Keller- Siegel equations one with chemo tactic sensitivity function $\chi(\rho) = 1$ and other with $\chi(\rho) = \rho$.

Definition: A real function $f(t)$ is said to be in space C_μ, $\mu \in R$, if $\exists$ a real number $p(> \mu)$ such that $f(t) = t^p g(t)$, where $g(t) \in C[0, \infty)$ and it is said to be in the space C_μ^m iff $f^m \in C_\mu$, $m \in N$

Definition: The Riemann- Liouville fractional integral operator of order $\alpha > 0$ of a function $f(t) \in C_\mu$, $\mu \geq -1$ is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, J^0 f(t) = f(t)$$

Definition: The Caputo fractional derivative of $f(t)$ in the Caputo sense is given by Podlubny (1999) and Debnath (2003)

$$D_t^\alpha f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau,$$

For $m-1 < \alpha \leq m$, $m \in N$, $t > 0$, where D_t^α is Caputo derivative operator and $\Gamma(\alpha)$ is the Gamma function.

Sumudu Transformation: The Sumudu transformation over the set of function

$$A = \left\{ f(t) \exists M,\ \tau_1,\ \tau_2 > 0,\ |f(t)| < M e^{\frac{|t|}{\tau_j}} \text{ if } t \in (-1)^j \times [0, \infty) \right\}$$

is defined by Watugala (1993) as

$$S[f(t)] = \int_0^\infty \frac{1}{u} f(t)\, e^{-\frac{t}{u}}\, dt,\ u \in (-\tau_1,\ \tau_2)$$

Some properties of the Sumudu transformations are

$$S[1] = 1,\ S\left[\frac{t^m}{\Gamma(m+1)}\right] = u^m,\ m > 0$$

The Sumudu transformation of the Caputo fractional derivative is defined as

$$S[D_t^\alpha f(x,t)] = u^{-\alpha} S[f(x,t)] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0+),$$
$$m-1 < \alpha \leq m.$$

2 Homotopy perturbation Sumudu transformation method (HPSTM)

To illustrate the idea of HPSTM technique, we consider the fractional attractor one dimensional Keller-Segel equation

$$D_t^\mu U(x,t) = a \frac{\partial^2 U(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\left(U(x,t) \frac{\partial \chi(\rho)}{\partial x} \right)$$

$$D_t^\eta \rho(x,t) = b \frac{\partial^2 \rho(x,t)}{\partial x^2} + cU(x,t) - d\rho(x,t),$$
$$0 < \mu \leq 1, 0 < \eta \leq 1 \quad (3)$$

Subject to the conditions $U(x,0) = f(x)$ and $\rho(x,0) = g(x)$,

Where D_t^μ and D_t^η is the Caputo fractional derivative of the function $U(x,t)$ and $\rho(x,t)$ respectively. Apply sumudu transformation on both sides of the equation (3), we have

$$S\left[D_t^\mu U(x,t)\right] = S\left[a \frac{\partial^2 U(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\left(U(x,t) \frac{\partial \chi(\rho)}{\partial x} \right) \right]$$

$$S\left[D_t^\eta \rho(x,t)\right] = S\left[b \frac{\partial^2 \rho(x,t)}{\partial x^2} + cU(x,t) - d\rho(x,t) \right] \quad (4)$$

Using differentiation property of the sumudu transformation and the initial conditions, we get

$$S[U(x,t)] = f(x) + u^\alpha S\left[a \frac{\partial^2 U(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\left(U(x,t) \frac{\partial \chi(\rho)}{\partial x} \right) \right]$$

$$S\left[\rho(x,t)\right] = g(x) + u^\eta S\left[b \frac{\partial^2 \rho(x,t)}{\partial x^2} + cU(x,t) - d\rho(x,t) \right] \quad (5)$$

Operating with inverse sumudu transformation on both sides,

$$U(x,t) = f(x) + S^{-1} u^\alpha S\left[a \frac{\partial^2 U(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\left(U(x,t) \frac{\partial \chi(\rho)}{\partial x} \right) \right]$$

$$\rho(x,t) = g(x) + S^{-1} u^\eta S\left[b \frac{\partial^2 \rho(x,t)}{\partial x^2} + cU(x,t) - d\rho(x,t) \right] \quad (6)$$

Now we apply Homotopy perturbation method,

$$U(x,t) = \sum_{n=0}^\infty U_n(x,t) p^n \text{ and } \rho(x,t) = \sum_{n=0}^\infty \rho_n(x,t) p^n \quad (7)$$

where the nonlinear term can be decomposed as

$$\frac{\partial}{\partial x}\left(U(x,t) \frac{\partial \chi(\rho)}{\partial x} \right) = NY(x,t) = \sum_{n=0}^\infty p^n H_n \quad (8)$$

for some He's polynomial H_n given by

$$H_n = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{n=0}^{\infty} Y_n(x,t)\,p^n\right)\right]_{p=0}, \quad n = 0, 1, 2, 3, \ldots. \tag{9}$$

Substituting these values in (5), we have

$$\sum_{n=1}^{\infty} p^n U_n(x,t) = U(x,0) + pS^{-1}\left[u^{\mu} S\left\{a\left(\sum_{n=1}^{\infty} p^n U_n(x,t)\right)_{xx} - \sum_{n=1}^{\infty} p^n H_n\right\}\right]$$

$$\sum_{n=1}^{\infty} p^n \rho_n(x,t) = \rho(x,0) + pS^{-1}\left[u^{\eta} S\left\{b\left(\sum_{n=1}^{\infty} p^n \rho_n(x,t)\right)_{xx} + c\left(\sum_{n=1}^{\infty} p^n U_n(x,t)\right) - d\sum_{n=1}^{\infty} p^n \rho_n(x,t)\right\}\right] \tag{10}$$

On comparing the like powers of p on both sides,

$$U_0 = U(x,0) = f(x),\ U_1 = S^{-1}[u^{\mu} S\{aU_{0xx} - H_0\}],\quad U_2 = S^{-1}[u^{\mu} S\{aU_{1xx} - H_1\}] \tag{11}$$

$$\rho_0 = \rho(x,0) = g(x),\ \rho_1 = S^{-1}\left[u^{\eta} S\left\{b\rho_{0xx} + cU_o - d\rho_o\right\}\right]$$

$$\rho_2 = S^{-1}\left[u^{\eta} S\left\{b\rho_{1xx} + cU_1 - d\rho_1\right\}\right] \tag{12}$$

Similarly we can find all the values of U_0, U_1, U_2, $U_3, \ldots.$ and $\rho_0, \rho_1, \rho_2, \ldots.$

The approximate solution of equation (3) can be calculated by setting $p \to 1$.

$$U(x,t) = U_0 + U_1 + U_3 + \cdots,\ \rho(x,0) = \rho_0 + \rho_1 + \rho_2 + \cdots \tag{13}$$

2.1 Application of HPSTM

In order to understand the solution procedure of the homotopy perturbation sumudu transform method, we consider the following examples:

Solution of fractional attractor 1-D Keller Segel equation

Example: Consider the following coupled system:

$$\frac{\partial^{\beta} v}{\partial t} = a\frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x}\left(v\frac{\partial \chi(\rho)}{\partial x}\right)$$

$$\frac{\partial^{\beta} \rho}{\partial t} = b\frac{\partial^2 \rho}{\partial x^2} + cv - d\rho,\ 0 < \beta \le 1 \tag{14}$$

Subject to the boundary conditions

$$v(x,0) = m\,exp(-x^2),\quad \rho(x,0) = n\,exp(-x^2) \tag{15}$$

Case-I: Consider the sensitivity function $\chi(\rho) = 1$, then the chemo-tactic term i.e. $\frac{\partial}{\partial x}\left(v\frac{\partial \chi(\rho)}{\partial x}\right) = 0$. Hence Keller Segel equation reduces to:

$$\frac{\partial^{\beta} v}{\partial t} = a\frac{\partial^2 v}{\partial x^2},$$

$$\frac{\partial^{\beta} \rho}{\partial t} = b\frac{\partial^2 \rho}{\partial x^2} + cv - d\rho,\ 0 < \beta \le 1 \tag{16}$$

By applying HPSTM on Eq. (36), we have

$$\sum_{n=0}^{\infty} p^n v_n = v(x,0) + pS^{-1}\left[u^{\beta} S\left\{a\left(\sum_{n=1}^{\infty} p^n v_n\right)_{xx}\right\}\right] \tag{17}$$

$$\sum_{n=0}^{\infty} p^n \rho_n = \rho(x,0) + pS^{-1}\left\{u^{\beta} S\left\{b\left(\sum_{n=0}^{\infty} p^n \rho_n\right)_{xx}\right\}\right\} + pS^{-1}\left\{u^{\beta} S\left\{c\left(\sum_{n=0}^{\infty} p^n v_n\right) - d\left(\sum_{n=0}^{\infty} p^n \rho_n\right)\right\}\right\} \tag{18}$$

On looking at the like terms of p of Eq. (37) & (38) and using (35), we have

$$p^0 : v_0 = me^{-x^2};$$

$$p^0 : \rho_0 = ne^{-x^2};$$

$$p^1 : v_1 = 2ame^{-x^2}(2x^2 - 1)\frac{t^{\beta}}{\Gamma(1+\beta)},$$

$$p^1 : \rho_1 = \frac{t^{\beta}}{\Gamma(1+\beta)}\left[(cm - nd) - 2bn\left(2x^2 - 1\right)e^{-x^2}\right];$$

$$p^1 : \rho_1 = \frac{a^2 m t^{2\beta}}{\Gamma(1+2\beta)}e^{-x^2}\left[12 - 48x^2 + 16x^4\right];$$

$$p^2 : v_2 = \frac{t^{2\beta}}{\Gamma(1+2\beta)}e^{-x^2}\left[d(-cm + dn) + 2acm\left(-1 + 2x^2\right) + 2b\left(-1 + 2x^2\right)(cm - 2dn) + 4b^2(3 - 12x^2 + 4x^4)\right];$$

$$p^3 : v_3 = \frac{a^3 m t^{3\beta}}{\Gamma(1+3\beta)}e^{-x^2}\left[-120 + 720x^2 - 480x^4 + 64x^6\right];$$

$$p^3 : \rho_3 = \frac{t^{3\beta}}{\Gamma(1+3\beta)}e^{-x^2}\left[d^2(cm - dn) + b\left(6 - 24x^2 + 8x^4\right) + 2acm\left(d - 2dx^2\right) + 4b^2(cm - 3dn)\left(3 - 12x^2 + 4x^4\right) + 8b^3 n\left(-15 + 90x^2 - 60x^4 + 8x^6\right) + 4a^2 cm\left(3 - 12x^2 + 4x^4\right) + 2bd(-2cm + 3dn)\left(-1 + 2x^2\right)\right];$$

The approximation solution of Eq. (36) obtained as $p \to 1$, i.e.

$$v(x,t) = v_0 + v_1 + v_2 + \dots$$

$$\rho(x,t) = \rho_0 + \rho_1 + \rho_2 + \dots$$

$$v(x,t) = me^{-x^2}\left(1 + a\left(-2+4x^2\right)\frac{t^\beta}{\Gamma(1+\beta)} + a^2(12 - 48x^2 + 16x^4)\frac{t^{2\beta}}{\Gamma(1+2\beta)}\right) + me^{-x^2}\left(\left(a^3(-120 + 720x^2 - 480x^4 + 64x^6)\right)\frac{t^{3\beta}}{\Gamma(1+3\beta)}\right);$$

$$\rho(x,t) = ne^{-x^2} + \frac{t^\beta}{\Gamma(1+\beta)}\left[(cm-nd) - 2bn\left(2x^2-1\right)e^{-x^2}\right] + \frac{t^{2\beta}}{\Gamma(1+2\beta)}\left[d(-cm+dn) + 2acm\left(-1+2x^2\right) + 2b\left(-1+2x^2\right)(cm-2dn) + 4b^2(3-12x^2+4x^4)\right] + \frac{t^{3\beta}}{\Gamma(1+3\beta)}e^{-x^2}\left[d^2(cm-dn) + b\left(6-24x^2+8x^4\right) + 2acm\left(d-2dx^2\right) + 4b^2(cm-3dn)\left(3-12x^2+4x^4\right) + 8b^3n\left(-15+90x^2-60x^4+8x^6\right) + 4a^2cm\left(3-12x^2+4x^4\right) + 2bd(-2cm+3dn)\left(-1+2x^2\right)\right]$$

Case-II: Consider the sensitivity function $\chi(\rho) = \rho$, then the chemo-tactic term i.e. $\frac{\partial}{\partial x}\left(v\frac{\partial \chi(\rho)}{\partial x}\right) = \frac{\partial v}{\partial x}\frac{\partial \rho}{\partial x} + v\frac{\partial^2 \rho}{\partial x^2}$; hence Keller –Segel equation reduces to

$$\frac{\partial^\beta v}{\partial t} = a\frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\frac{\partial \rho}{\partial x} + v\frac{\partial^2 \rho}{\partial x^2}\right),$$

$$\frac{\partial^\beta \rho}{\partial t} = b\frac{\partial^2 \rho}{\partial x^2} + cv - d\rho, 0 < \beta \le 1 \tag{19}$$

Now, for the solution of Eq. (39), we apply HPSTM on Eq. (37), we have

$$\sum_{n=0}^{\infty} p^n v_n = v(x,0) + pS^{-1}\left[u^\beta S\left\{a\left(\sum_{n=0}^{\infty} p^n v_n\right)_{xx} - \left(\sum_{n=0}^{\infty} p^n H_n(x,t)\right)\right\}\right] \tag{20}$$

$$\sum_{n=0}^{\infty} p^n \rho_n = \rho(x,0) + pS^{-1}\left[u^\beta S\left\{b\left(\sum_{n=0}^{\infty} p^n \rho_n\right)_{xx}\right\}\right] + pS^{-1}\left[u^\beta S\left\{c\left(\sum_{n=0}^{\infty} p^n v_n\right) - d\left(\sum_{n=0}^{\infty} p^n \rho_n\right)\right\}\right]. \tag{21}$$

Where

$$\sum_{n=0}^{\infty} p^n H_n(x,t) = \frac{\partial v}{\partial x}\frac{\partial \rho}{\partial x} + v\frac{\partial^2 \rho}{\partial x^2}$$

An initial couple of terms of He's polynomial i.e. $H_n(x,t)$ are given below:

$H_o(x,t) = v_{0x}\rho_{0x} + v_0\rho_{0xx}$;
$H_1(x,t) = v_{0x}\rho_{1x} + v_{1x}\rho_{0x} + v_0\rho_{1xx} + v_1\rho_{0xx}$;
$H_2(x,t) = v_{0x}\rho_{2x} + v_{1x}\rho_{1x} + v_{2x}\rho_{0x} + v_0\rho_{2xx} + v_1\rho_{1xx} + v_2\rho_{0xx}$;
$\vdots$

On looking at the like term of $p$of Eq. (40) and (41) and using Eq. (35) and He's polynomial we get

$$p^0 : v_0(x,t) = me^{-x^2};$$

$$p^0 : \rho_0(x,t) = ne^{-x^2};$$

$$p^1 : v_1(x,t) = 2m\frac{t^\beta}{\Gamma(1+\beta)}\left\{a\left(2x^2-1\right) - ne^{-x^2}\left(4x^2-1\right)\right\};$$

$$p^1 : \rho_1(x,t) = \frac{t^\beta}{\Gamma(1+\beta)}e^{-x^2}\left[2bn\left(2x^2-1\right) + (cm-nd)\right];$$

$$p^2 : v_2(x,t) = 2m\frac{t^{2\beta}e^{-3x^2}}{\Gamma(1+2\beta)}\left\{-ce^{x^2}m\left(-1+4x^2\right) + 2a^2e^{2x^2}\left(3-12x^2+4x^4\right) - 2ae^{x^2}n\left(7-58x^2+40x^4\right) + nde^{x^2}\left(-1+4x^2\right) - 2nbe^{x^2}\left(3-18x^2+8x^4\right) + 2n^2(1-18x^2+24x^4)\right\};$$

$$p^2 : \rho_2(x,t) = \frac{t^{2\beta}e^{-2x^2}}{\Gamma(1+2\beta)}\left\{e^{x^2}d(-cm+nd) + 2cmn\left(1-4x^2\right) + 2e^{x^2}(acm + b(cm-2dn)\left(-1+2x^2\right)) + 4b^2ne^{x^2}\left(3-12x^2+4x^4\right)\right\};$$

$$p^3 : v_3(x,t) = 2m\frac{t^{3\beta}e^{-4x^2}}{\Gamma(1+3\beta)(\Gamma(1+\beta))^2}\left\{-cdme^{2x^2} + d^2e^{2x^2}n + 14ce^{x^2}mn - 2de^{x^2}n^2 + 4n^3 + 4cde^{2x^2}mx^2 - 4d^2nx^2e^{2x^2} + n^3x^4\left(1056-768x^2\right) - 156cmnx^2e^{x^2} + 36dn^2x^2e^{x^2} - 248n^3x^2 + 144cmnx^4e^{x^2} - 48dn^2x^4e^{x^2} + 4a^3e^{3x^2}\left(-15+90x^2-60x^4+8x^6\right) - 4b^2e^{2x^2}\left(-15+120x^2-100x^4+16x^6\right) - 4a^2ne^{2x^2}\left(-75+924x^2-1252x^4+336x^6\right) - 2bcme^{2x^2}\left(3-18x^2+8x^4\right)$$

$$
\begin{aligned}
&+4nbe^{x^2}\left(de^{x^2}\left(3-18x^2+8x^4\right)\right.\\
&\left.-2bne^{x^2}\left(-3+72x^2-148x^4+48x^6\right)\right)\\
&-2ae^{2x^2}cm\left(9-66x^2+40x^4\right)\\
&+4ane^{x^2}\left(de^{x^2}\left(3-24x^2+16x^4\right)\right.\\
&\left.-4be^{x^2}\left(-6+63x^2-72x^4+16x^6\right)\right)\\
&-8ane^{x^2}\left(-7+162x^2-380x^4+168x^6\right)(\Gamma(1+\beta))^2\\
&-2e^{x^2}(n(-cm+dn)\left(1-18x^2+24x^4\right)\\
&+bn^2\left(6-120x^2+248x^4-96x^6\right)\\
&+ae^{x^2}(cm-dn)\left(1-10x^2+8x^4\right)\\
&\left.+2bn\left(-3+36x^2-52x^4+16x^6\right))\Gamma(1+2\beta)\right\};
\end{aligned}
$$

$$
\begin{aligned}
p^3:\rho_3(x,t)=&\frac{t^{3\beta}e^{-3x^2}}{\Gamma(1+3\beta)}\left(cd^2e^{x^2}+2c^2m^2e^{x^2}\right.\\
&\left.-nd^2e^{x^2}-4cdmne^{x^2}\right)+4cmn^2-8c^2m^2x^2e^{x^2}\\
&-16cdmnx^2e^{x^2}-72cmn^2x^2+96cmn^2x^4\\
&+6bd^2ne^{2x^2}n\left(-1+2x^2\right)\left(-15+90x^2-60x^4+80x^6\right)\\
&+4a^2cme^{2x^2}\left(3-12x^2+4x^4\right)+8b^3ne^{2x^2}\\
&-4bcme^{x^2}\left(de^{x^2}\left(-1+2x^2\right)+n\left(9-66x^2+40x^4\right)\right)\\
&+2acme^{x^2}(-de^{x^2}\left(-1+2x^2\right)+2be^{x^2}\left(3-12x^2+4x^4\right)\\
&-2n\left(7-58x^2+40x^4\right)))
\end{aligned}
\tag{22}
$$

On using Eq. (42) and as $p \to 1$, the approximation solution of Eq. (39) is

$$v(x,t)=v_0+v_1+v_2+\ldots$$

$$\rho(x,t)=\rho_0+\rho_1+\rho_2+\ldots$$

3 Homotopy perturbation transform method (HPTM)

To elucidate the basic idea of this method, we consider coupled attractor for one-dimensional Keller-Segel equation:

$$U_t(x,t)=aU_{xx}(x,t)-(U(x,t)\chi_x(\rho))_x$$

$$\rho_t(x,t)=b\rho_{xx}+cU(x,t)-d\rho(x,t) \tag{20}$$

Subjected to initial condition:

$$U(x,0)=U_0(x) \tag{21}$$

Taking Laplace transform on both sides of equation (20)

$$L[U_t(x,t)]=aL[U_{xx}(x,t)]-L[(U(x,t)\chi_x(\rho))_x] \tag{22}$$

$$L[\rho_t(x,t)]=L[b\rho_{xx}+U(x,t)-d\rho(x,t)] \tag{23}$$

Applying the differentiation property of Laplace transform, we have

$$U(x,\ s)=\frac{U(x,0)}{s}+\frac{1}{s}L\left[aU_{xx}(x,t)-(U(x,t)\chi_x(\rho))_x\right] \tag{24}$$

$$\rho(x,\ s)=\frac{\rho(x,0)}{s}+\frac{1}{s}L\left[b\rho_{xx}+U(x,t)-d\rho(x,t)\right] \tag{25}$$

Taking the inverse Laplace transform on both sides of equation (24) and (25)

$$U(x,t)=U(x,0)+L^{-1}\left\{\frac{1}{s}L\left[aU_{xx}(x,t)-(U(x,t)\chi_x(\rho))_x\right]\right\} \tag{26}$$

$$\rho(x,\ t)=\frac{\rho(x,0)}{s}+L^{-1}\left\{\frac{1}{s}L\left[b\rho_{xx}+U(x,t)-d\rho(x,t)\right]\right\} \tag{27}$$

Now, apply homotopy perturbation method, with

$$U(x,t)=\sum_{n=0}^{\infty}p^nU_n(x,t),\ NU(x,t)=\sum_{n=0}^{\infty}p^nH_n(U) \tag{28}$$

Where $H_n(U)$ is He's polynomial use to decompose the nonlinear terms. This polynomial is of the form:

$$H_n(U_0,U_1,\ldots,U_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{n}p^iU_i(x,t)\right)\right],\quad n=0,1,2,\ldots \tag{29}$$

Substituting equation (28) in equation (27) and (26), we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty}p^nU_n(x,t)=U(x,0)\\
&+pL^{-1}\left\{\frac{1}{s}L\left[a\sum_{n=0}^{\infty}(U_np^n)_{xx}-\sum_{n=0}^{\infty}(H_np^n)\right]\right\}
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
&\sum_{n=0}^{\infty}p^n\rho_n(x,t)=\rho(x,0)\\
&+pL^{-1}\left[\frac{1}{s}\left\{b(\sum_{n=1}^{\infty}p^n\rho_n(x,t))_{xx}+c(\sum_{n=1}^{\infty}p^nU_n(x,t))\right.\right.\\
&\left.\left.-d\sum_{n=0}^{\infty}p^n\rho_n(x,t)\right\}\right]
\end{aligned}
\tag{31}
$$

The Laplace transform and the homotopy perturbation method are coupled here by using He's polynomials. Comparing the coefficients of like powers of p, the following approximations are obtained

$$p^0 : U_0 = U(x,0),\ \rho_0 = \rho(x,0)$$

$$p^1 : U_1 = L^{-1}\left\{\frac{1}{s}L[a\,U_{0xx} - H_0]\right\},$$

$$\rho_1 = L^{-1}\left[\frac{1}{s}L\left\{b\rho_{0xx} + cU_o - d\rho_o\right\}\right]$$

$$p^2 : U_2 = L^{-1}\left\{\frac{1}{s}L[a\,U_{1xx} - H_1]\right\},$$

$$\rho_2 = L^{-1}\left[\frac{1}{s}L\left\{b\rho_{1xx} + cU_1 - d\rho_1\right\}\right],$$

$$p^3 : U_3 = L^{-1}\left\{\frac{1}{s}L[a\,U_{2xx} - H_2]\right\} \tag{32}$$

And so on. Setting, $p = 1$ results the approximate solution of equation (20)

$$U(x,t) = U_0 + U_1 + U_2 + \ldots,\quad \rho(x,t) = \rho_0 + \rho_1 + \rho_2 + \ldots, \tag{33}$$

3.1 Application of HPTM

In the order to understand solution of the homotopy perturbation transform method, we consider the following example:

Example: The simplified form of the Keller Segel equation in one dimension in given as

$$\frac{\partial^\beta v}{\partial t^\beta} = a\frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x}\left(v\frac{\partial}{\partial x}\chi(\rho)\right)$$

$$\frac{\partial^\beta \rho}{\partial t^\beta} = b\frac{\partial^2 \rho}{\partial x^2} + cv - d\rho \tag{34}$$

Subject to the boundary conditions

$$v(x,0) = m\,exp(-x^2),\ \rho(x,0) = nexp(-x^2) \tag{35}$$

Case-I: Consider $\chi(\rho) = 1$, then $\frac{\partial}{\partial x}\left(v\frac{\partial\chi(\rho)}{\partial x}\right) = 0$
Hence Keller-Segel equation (14) reduced to

$$\frac{\partial^\beta v}{\partial t^\beta} = a\frac{\partial^2 v}{\partial x^2} \tag{36}$$

$$\frac{\partial^\beta \rho}{\partial t^\beta} = b\frac{\partial^2 \rho}{\partial x^2} + cv - d\rho$$

By applying HPTM on equation (16), we have

$$\sum_{n=0}^{\infty} p^n v_n = v(x,0) + pL^{-1}\left\{\frac{1}{s^\beta}L\left\{a\left(\sum_{n=0}^{\infty} p^n v_n\right)_{xx}\right\}\right\} \tag{37}$$

$$\sum_{n=0}^{\infty} p^n \rho_n = \rho(x,0) + pL^{-1}\left\{\frac{1}{s^\beta}L\left\{b\left(\sum_{n=0}^{\infty} p^n \rho_n\right)_{xx}\right\}\right\}$$
$$+ pL^{-1}\left\{\frac{1}{s^\beta}L\left\{c\left(\sum_{n=0}^{\infty} p^n v_n\right) - d(\sum_{n=0}^{\infty} p^n \rho_n)\right\}\right\} \tag{38}$$

On looking at the coefficients of like powers of p of Eq. (17) and (18) and using (15), we have:

$$p^0 : v_0 = me^{-x^2};$$

$$p^0 : \rho_0 = ne^{-x^2};$$

$$p^1 : v_1 = \frac{amt^\beta}{\Gamma(1+\beta)}(-2e^{-x^2} + 4x^2e^{-x^2});$$

$$p^1 : \rho_1 = \frac{t^\beta}{\Gamma(1+\beta)}e^{-x^2}\left[2bn\left(2x^2 - 1\right) + (cm - nd)\right];$$

$$p^2 : v_2 = \frac{4a^2me^{-x^2}\,t^{2\beta}}{\Gamma(1+2\beta)}(3 - 12x^2 + 4x^4);$$

$$p^2 : \rho_2 = e^{-x^2}\,\frac{t^{2\beta}}{\Gamma(1+2\beta)}[d(-cm + dn)$$
$$+ 2acm\left(-1 + 2x^2\right) + 2b\left(-1 + 2x^2\right)(cm - 2dn)$$
$$+ 4b^2n(3 - 12x^2 + 4x^4)\,];$$

$$p^3 : v_3 = 8a^3me^{-x^2}\,(8x^6 - 60x^4 + 90x^2 - 15)\frac{t^{3\beta}}{\Gamma(1+3\beta)};$$

$$p^3 : \rho_3 = e^{-x^2}\,\frac{t^{3\beta}}{\Gamma(1+3\beta)}[d^2(cm - dn) + b\left(6 - 24x^2 + 8x^4\right)$$
$$+ 2acm\left(d - 2dx^2\right) + 4b^2(cm - 3dn)\left(3 - 12x^2 + 4x^4\right)$$
$$+ 8b^3n\left(-15 + 90x^2 - 60x^4 + 8x^6\right) + 4a^2cm\left(3 - 12x^2 + 4x^4\right)$$
$$+ 2bd(-2cm + 3dn)\left(-1 + 2x^2\right)];$$

The approximate solution is obtained by letting $p \to 1$, & $\beta \to 1$ i.e.

$$v(x,t) = v_0 + v_1 + v_2 + v_3 + \dots$$

$$\rho(x,t) = \rho_0 + \rho_1 + \rho_2 + \rho_3 + \dots$$

$$v(x,t) = me^{-x^2}\left(1 + a\left(-2+4x^2\right)\frac{t}{1} + a^2\left(12 - 48x^2 + 16x^4\right)\frac{t^2}{2}\right) + me^{-x^2}\left(a^3\left(-120 + 720x^2 - 480x^4 + 64x^6\right)\frac{t^3}{6}\right) + \dots.$$

$$\rho(x,t) = ne^{-x^2} + \frac{te^{-x^2}}{1}\left((cm-dn) - 2nb\left(2x^2-1\right)\right) + \frac{t^2e^{-x^2}}{2}\left(d(-cm+dn) + 2acm\left(-1+2x^2\right)\right)$$

$$+\frac{t^3e^{-x^2}}{6}\left(\begin{array}{l} d^2(cm-dn) + b\left(6 - 24x^2 + 8x^4\right) + 2acm\left(d - 2dx^2\right) + 4b^2(cm-3dn)\left(3 - 12x^2 + 4x^4\right) \\ +8b^3n\left(-15 + 90x^2 - 60x^4 + 8x^6\right) + 4a^2cm\left(3 - 12x^2 + 4x^4\right) + 2bd(-2cm+3dn)\left(-1+2x^2\right) \end{array}\right) + \dots.$$

Case-II: Consider the Keller Siegel equation with sensitivity function $\chi(\rho) = \rho$. Then

$$\frac{\partial}{\partial x}\left(v\frac{\partial \chi(\rho)}{\partial x}\right) = \frac{\partial v}{\partial x}\frac{\partial \rho}{\partial x} + v\frac{\partial^2 \rho}{\partial x^2};$$

Hence Keller-segel equation (14) reduces to

$$\frac{\partial v}{\partial t} = a\frac{\partial^2 v}{\partial x^2} - \left(\frac{\partial v}{\partial x}\frac{\partial \rho}{\partial x} + v\frac{\partial^2 \rho}{\partial x^2}\right),$$

$$\frac{\partial \rho}{\partial t} = b\frac{\partial^2 \rho}{\partial x^2} + cv - d\rho \tag{39}$$

Now, for the solution of Eq. (19), we apply HPTM on Eq. (19), we have

$$\sum_{n=0}^{\infty} p^n v_n = v(x,0) + pL^{-1}\left[\frac{1}{s}L\left\{a\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n U_n\right) - \sum_{n=0}^{\infty} p^n H_n\right\}\right] \tag{40}$$

$$\sum_{n=0}^{\infty} p^n \rho_n = \rho(x,0) + pL^{-1}\left[\frac{1}{s}L\left\{b\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n \rho_n\right)\right\}\right] + pL^{-1}\left[\frac{1}{s}L\left\{c\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n \rho_n\right) - d\left(\sum_{n=0}^{\infty} p^n \rho_n\right)\right\}\right] \tag{41}$$

Where

$$\sum_{n=0}^{\infty} p^n H_n(x,t) = \left(\frac{\partial v}{\partial x}\frac{\partial \rho}{\partial x} + v\frac{\partial^2 \rho}{\partial x^2}\right)$$

An initial couple of terms of He's polynomial i.e. $H_n(x,t)$ are given below:

$$H_o(x,t) = v_{0x}\rho_{0x} + v_0\rho_{0xx};$$

$$H_1(x,t) = v_{0x}\rho_{1x} + v_{1x}\rho_{0x} + v_0\rho_{1xx} + v_1\rho_{0xx};$$

$$H_2(x,t) = v_{0x}\rho_{2x} + v_{1x}\rho_{1x} + v_{2x}\rho_{0x} + v_0\rho_{2xx} + v_1\rho_{1xx} + v_2\rho_{0xx};$$

On looking at the like terms of p of Eq. (20) & (21) and using Eq. (15) and He's polynomial, we get

$p^0 : v_0(x,t) = me^{-x^2}$;

$p^0 : \rho_0(x,t) = ne^{-x^2}$;

$p^1 : v_1(x,t) = 2mte^{-2x^2}\left\{n - 4nx^2 + ae^{x^2}\left(-1+2x^2\right)\right\}$;

$p^1 : \rho_1(x,t) = te^{-x^2}\left[2bn\left(2x^2-1\right) + (cm-nd)\right]$;

$p^2 : v_2(x,t) = mt^2\left(-cme^{-2x^2}\left(-1+4x^2\right) + 2a^2e^{-x^2}\left(3 - 12x^2 + 4x^4\right)\right)$

$- mt^2\left(2ane^{-2x^2}\left(7 - 58x^2 + 40x^4\right) + dne^{-2x^2}\left(-1+4x^2\right)\right) - mt^2(2be^{-2x^2}\left(3 - 18x^2 + 8x^4\right)$

$+ 2ne^{-3x^2}\left(1 - 18x^2 + 24x^4\right))$;

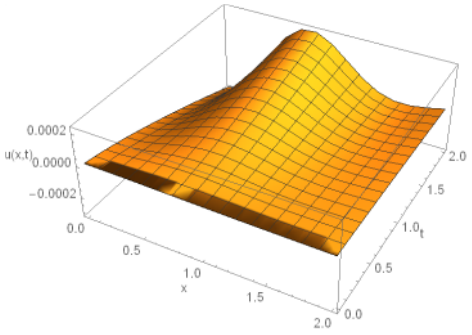
$$p^2 : \rho_2(x,y) = \tfrac{1}{2}e^{-2x^2}t^2\left(-cdme^{x^2} + d^2ne^{x^2}\right)$$
$$+ \tfrac{1}{2}e^{-2x^2}t^2\left(2cmn - 8cmnx^2 + 2acme^{x^2}\left(-1+2x^2\right)\right)$$
$$+ \tfrac{1}{2}e^{-2x^2}t^2\left(2be^{x^2}(cm-2dn)\left(-1+2x^2\right)\right)$$
$$+ \tfrac{1}{2}e^{-2x^2}t^2(4b^2e^{x^2}n\left(3-12x^2+4x^4\right)) \vdots$$

The solution of Eq. (19) obtained as $p \to 1$, i.e.

$$v(x,t) = v_0 + v_1 + v_2 + \ldots$$
$$\rho(x,t) = \rho_0 + \rho_1 + \rho_2 + \ldots$$

$$v(x,t) = me^{-x^2} + 2mte^{-2x^2}\left(n - 4nx^2 + ae^{x^2}\left(-1+2x^2\right)\right)$$
$$- mt^2(2ae^{-2x^2}n\left(7-58x^2+40x^4\right) + n\left(de^{-2x^2}\left(-1+4x^2\right)\right)$$
$$+ \ldots$$

$$\rho(x,t) = ne^{-x^2} + te^{-x^2}\left(cm - \left(nd + b\left(2-4x^2\right)\right)\right)$$
$$+ \frac{1}{2}t^2e^{-2x^2}\left(-cde^{x^2}m + d^2e^{x^2}n\right)$$
$$+ \frac{1}{2}t^2e^{-2x^2}\left(2cmn - 8cmnx^2 + 2ace^{x^2}m\left(-1+2x^2\right)\right)$$
$$+ \frac{1}{2}t^2e^{-2x^2}\left(2be^{x^2}(cm-2dn)\left(-1+2x^2\right)\right)$$
$$+ \frac{1}{2}t^2e^{-2x^2}\left(4b^2e^{x^2}n\left(3-12x^2+4x^4\right)\right) + \ldots.$$



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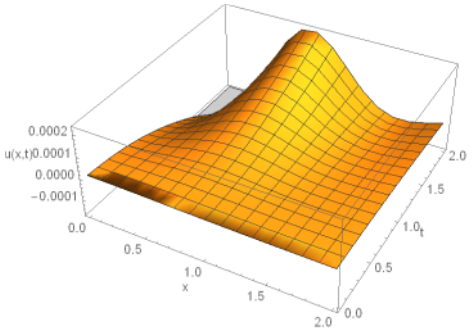

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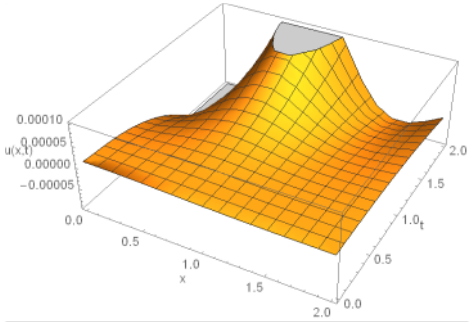

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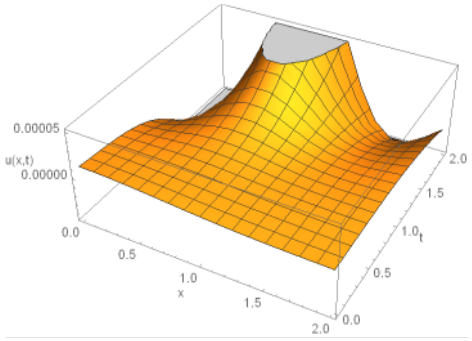
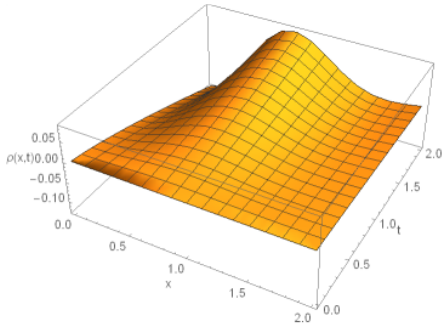


(d)

Figure 1: The surface graph of approximate solution $v(x,t)$ for case-I: **(a)** $v(x,t)$ for $\beta = 0.4$**(b)** $v(x,t)$ for $\beta = 0.6$ **(c)** $v(x,t)$ for $\beta = 0.8$ **(d)** $v(x,t)$ for $\beta = 1$



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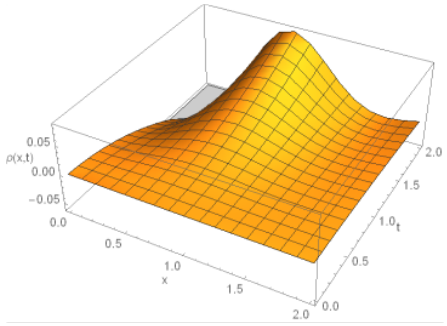

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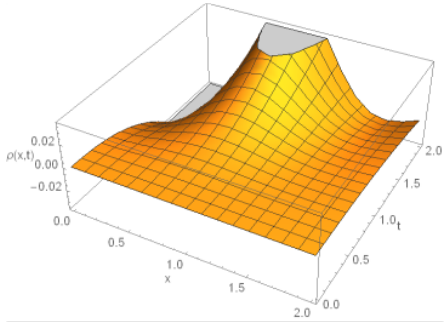

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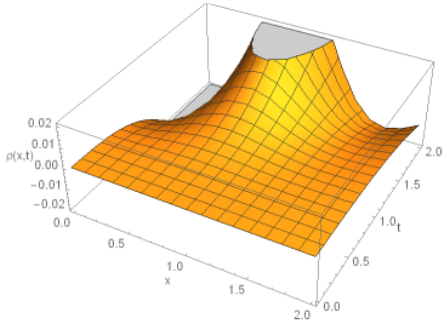
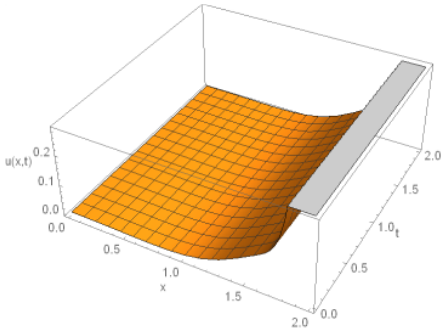


(d)

Figure 2: The surface graph of approximate solution $\rho\,(x,\,t)$ for case-I: **(a)** $\rho\,(x,\,t)$ for $\beta\,=\,0.4$ **(b)** $\rho\,(x,\,t)$ for $\beta\,=\,0.6$ **(c)** $\rho\,(x,\,t)$ for $\beta\,=\,0.8$ **(d)** $\rho\,(x,\,t)$ for $\beta\,=\,1$



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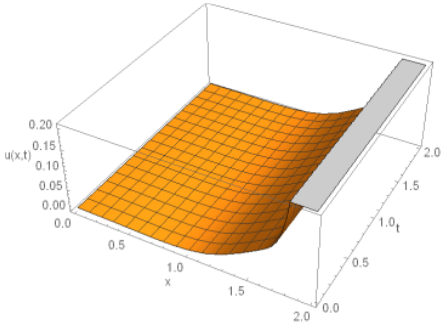

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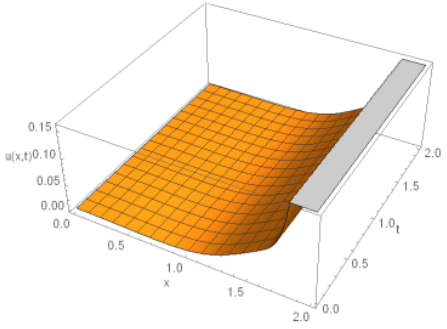

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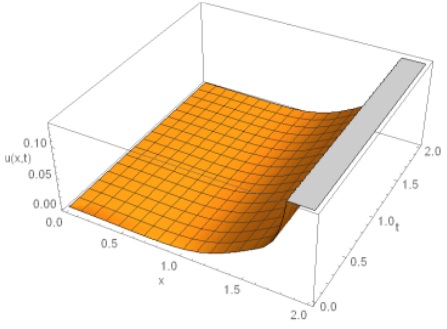
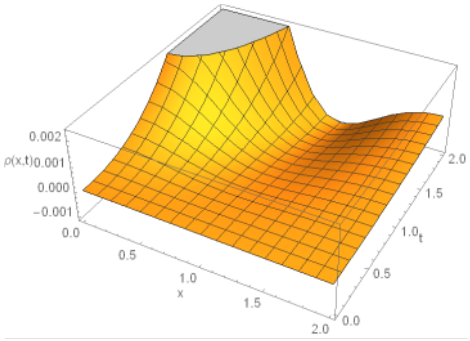


(d)

Figure 3: The surface graph of approximate solution $v\,(x,\,t)$ for case-II: **(a)** $v\,(x,\,t)$ for $\beta\,=\,1$**(b)** $v\,(x,\,t)$ for $\beta\,=\,0.8$ **(c)** $v\,(x,\,t)$ for $\beta\,=\,0.6$ **(d)** $v\,(x,\,t)$ for $\beta\,=\,0.4$



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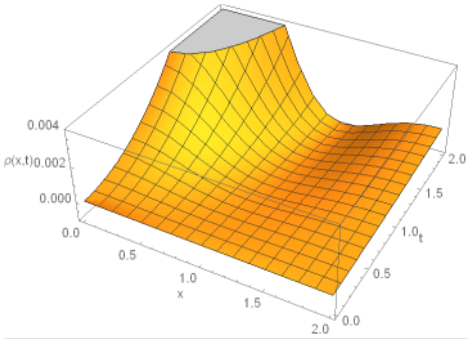

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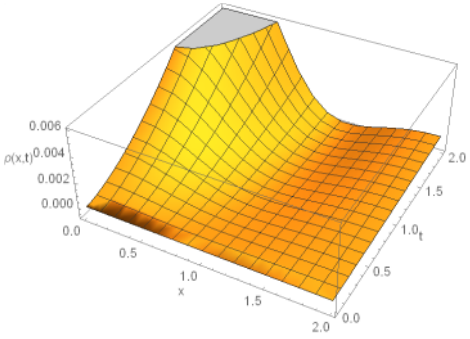

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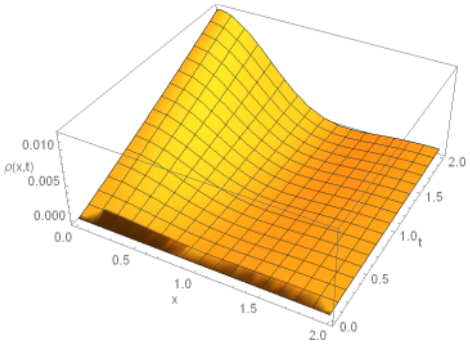
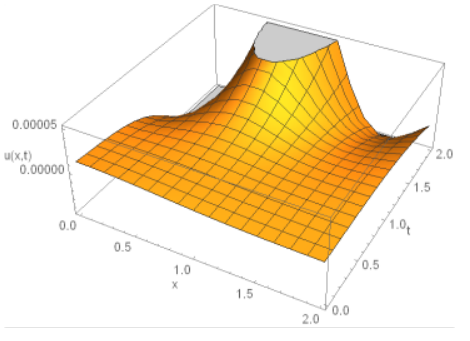


(d)

Figure 4: The surface graph of approximate solution $\rho\,(x,\,t)$ for case-II: **(a)** $\rho\,(x,\,t)$ for $\beta = 1$ **(b)** $\rho\,(x,\,t)$ for $\beta = 0.8$ **(c)** $\rho\,(x,\,t)$ for $\beta = 0.6$ **(d)** $\rho\,(x,\,t)$ for $\beta = 0.4$



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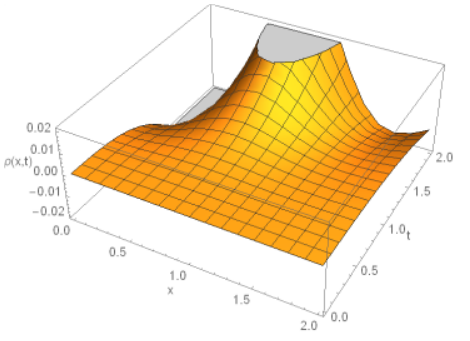

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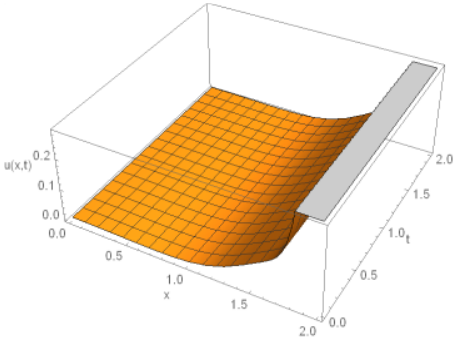

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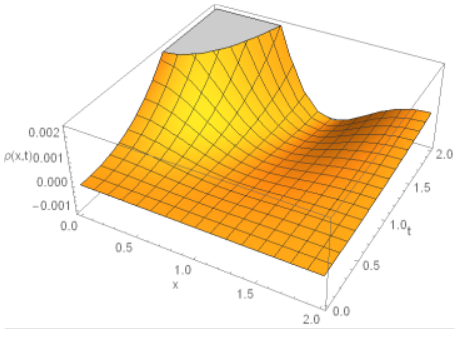

(d)

Figure 5: The surface graph of approximate solution $v(x,\,t)$ and$\rho(x,\,t)$ for $\beta\ =\ 1$: **(a)** $v(x,\,t)$ for case-I **(b)** $\rho(x,\,t)$ for case-I **(c)** $v(x,\,t)$ for case-II **(d)** $\rho(x,\,t)$ for case-II

4 Results and discussion

In this section, the numerical solution of examples obtained by HPSTM and HPTM through a graphical representation are studied. The surface graphs of Keller-Segel equation for respective cases (I & II) at different values of β are represented in Figures 1-4. For graphical representation of solution we take $m = 0.000012$, $n = 0.000016$, $a = 0.5$, $b = 3$, $c = 1$, $d = 2$. Figure 1 represents solution $v(x, t)$ at $\beta = 0.4$, $\beta = 0.6$, $\beta = 0.8$, $\beta = 1$, respectively, whereas Figure 2 indicates $\rho(x, t)$ corresponding to different values of β for Case-I.

Figures 3 and 4 show surface graphs of solution $v(x, t)$ and $\rho(x, t)$ for Case-II at different values of β. Figure 5 represents solution $v(x, t)$ and $\rho(x, t)$ obtained from HPTM for both cases. It is clear from the graphs that results of HPSTM and HPTM are in good harmony with each other for $\beta = 1$.

5 Conclusion

In this work, homotopy perturbation transform method (HPTM) combined with sumudu transform has been successfully applied to approximate solution for a system of nonlinear partial differential equations derived from an attractor for a one-dimensional Keller-Segel dynamics system. On comparing the results of this method with HPTM, it is observed HPSTM is extremely simple, straightforward and easy to handle the nonlinear terms. Maple 13 package is used to calculate series obtained from iteration. Further, the method needs much less computational work which shows fast convergent for solving nonlinear system of partial differential equations.

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