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# Couple stress fluid flow due to slow steady oscillations of a permeable sphere

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**Abstract:** The study of oscillating flow of a Couple Stress fluid past a permeable sphere is considered. Analytical solution for the flow field in terms of stream function is obtained using modified Bessel functions. The formula for Drag acting on the sphere due external flow is evaluated. Pressure field for the flow region past and inside the sphere is obtained. Effects of physical parameters like couple stress parameter, permeability, frequency and geometric parameters on the drag due to internal and external flows are represented graphically. It is observed that the drag for viscous fluid flow will be less than the case of couple-stress fluid flow and hence couple stress fluids offer resistance for flow.

**Keywords:** couple stress fluid, oscillatory flow, permeable sphere, non-stick and hyper stick conditions

## Nomenclature

$a$ : radius of the sphere  
 $\vec{Q}$ : fluid velocity vector with dimensional and non-dimensional form.  
 $R, r$ : Non-dimensional and dimensional distances from origin to a generic point  
 $P, p$ : the non- dimensional and dimensional pressures  
 $\eta$ : couple stress viscosity coefficient  
 $\rho$ : density of couple stress fluid  
 $\omega$ : frequency parameter  
 $\Psi$ : the non-dimensional and dimensional stream func-

tions

$\lambda_1, \lambda_2$ : roots of the main equation for stream function

$\mu$ : material constants known as viscosity coefficient

$S$ : couple stress parameter

$\sigma$ : frequency parameter

$Re$ : reynolds number

$M$ : couple stress tensor

$U_\infty$ : velocity at infinity

$V_0$ : filtration velocity

$m$ : normal couple stress component

$\theta$ : angle between  $z$  axis and radius vector

$\Omega$ : angular velocity of the sphere

## 1 Introduction

By the heavy technical demand of industries, many researchers are using Non-Newtonian fluids extensively in the problems of extraction of petrol from porous wells, sedimentation, dilute polymers, suspensions and lubrications of journal bearings. The polar effects namely couple stresses and non-symmetric tensors are well explained by a simple model of couple stress fluids introduced by Stokes [1]. Stokes solved creeping flow of couple stress fluid across a sphere [2]. The study of the flow of couple stress fluid past axi-symmetric bodies was carried out [3]. Ramkissoon [4] has derived an elegant and useful formula for drag on an axi-symmetric body in terms of a limit on the stream function. Uniform flow of a Couple stress fluid past a permeable sphere was analyzed by Ramana Murthy and et al. [5]. Devakar et al. [6] studied analytical solutions of some fully developed flows of Couple Stress fluids between two cylinders with slip boundary conditions. Couple stress fluid flow past a porous spheroidal shell with solid core under Stokesian assumption was studied and analyzed by Iyengar and Radhika [7]. A study of a Couple acting on a couple- stress fluid for rotary flows across a permeable sphere was carried out [8]. Vandana Mishra and Ram Gupta [9] studied the concept of analytically uniform flow of steady axi-symmetric creeping flow of an incompressible micro-polar fluid around the permeable sphere. They considered non homogeneous boundary conditions for micro-rotation vector. Arbitrary oscilla-

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tory Stokes flow past a porous sphere for viscous fluid was studied by Prakesh et al. [10]. The slow and stationary flow of a viscous fluid was investigated by Leonov [11]. The concept of micro-polar fluids was first found by Eringen [12]. Gupta and Deo [13] examined Stokes flow of micro-polar fluid past a porous sphere with hyper-stick condition on micro-rotation vector. Recently, Choudhuri et al. [14] developed a method to find a solution to Stokes flow of a viscous and incompressible fluid flow across a sphere coated by a thin fluid of different viscosity. Ramkissoon [15] obtained a formula for drag coefficient of a micropolar fluid flow past a sphere. Recently, Vainshtein and Shapiro [16] have examined the forces acting on a porous sphere oscillating in a viscous fluid. Newtonian fluid flow inside and outside sphere is governed by Darcy–Brinkman equations of porous medium. Jai Prakash and Raja Sekhar [17] analyzed the arbitrary oscillatory Stokes flow past a porous sphere using Brinkman model. Crittenden et al. [18] studied the influence of oscillatory flow on axial dispersion in packed bed of spheres. They observed that the best reduction of axial dispersion coefficient (up to 50%) from the non-oscillation base value occurs when the column particle size is the smallest.

Many elastic properties of dilute polymers can be detected and measured conveniently by a suitable choice of oscillatory flows. The problems that are concerned with the effects of free stream oscillations are of physical significance. The problems of unsteady flows are initiated by Lighthill [19] by giving analytical solution of functions in stream function due to heat transfer. Fatter [20] has discussed the problems of oscillating sphere in an elastic viscous fluid. Latter many authors have studied the phenomena of oscillations of external flow over a non-zero mean velocity. Thomas and Walters [21] examined the flow due to the oscillatory motion of a sphere with convective terms present in a elastic viscous liquid using Laplace Transform technique. Lai and Fan [22] have considered the flow due to oscillating sphere in an elastico viscous fluid by neglecting the nonlinear terms. They also studied the flow past a sphere accelerating with aperiodic and arbitrary motion in the visco-elastic fluid using Fourier Transform technique and obtained expressions for drag experienced by the sphere. Entropy generation on non-Newtonian Eyring-Powell nanofluid has been analysed through a permeable stretching sheet by Bhatti et al. [23]. Variable viscosity and inclined magnetic field on the peristaltic motion of a non-Newtonian fluid in an inclined asymmetric channel was studied by Khan et al. [24]. New analytical method for the study of natural convection flow of a non-Newtonian fluid was studied by Rashidi et al. [25]. A steady flow of a sphere in a rotation motion in a micro-polar fluid was ana-

lyzed by the author [26]. Webster [27] has considered non-Newtonian and turbulent fluid models. He developed a finite difference numerical technique to solve incompressible fluid flow problems. Casanellas and Ortin [28] studied the laminar oscillatory flow of Maxwell and Oldroyd-B fluids [28]. Jayalakshamma et al. [29] studied numerically the steady flow of an incompressible micropolar fluid past an impervious sphere. Mishra and Gupta [30] studied creeping flow of micro polar fluid past composite sphere. Numerical and analytical study of flow past porous sphere embedded in micropolar fluid by Ramalakshmi and Pankaj Shukla [31]. Ashmawy [32] developed a simple formula for drag acting on a sphere for couple stress fluid. The problem of rotary oscillation of a rigid sphere in an incompressible couple stress fluid is investigated by Shehadeh and Ashmawy [33]. Ashmawy [34] studied unsteady Stokes flow of a couple stress fluid around a rotating sphere with slip condition on the boundary. Jaiswal and Gupta [35] have considered the flow over composite sphere: liquid core with permeable shell. Jaiwal [36] studied analytically, Stokes flow over Reiner-Rivlin liquid sphere embedded in a porous medium filled with micropolar fluid using Brinkman's model. Nagaraju and Mahesh [37] studied the analytical investigation of two-dimensional heat transfer behavior of an axisymmetric incompressible dissipative viscous fluid flow in a circular pipe.

The oscillatory flow of incompressible couple stress fluid flow past a permeable sphere is considered in the present study due to its practical importance. The velocity and pressure field on the sphere are obtained. The drag experienced by the sphere is evaluated. Effects of couple stress parameter, permeability parameter, frequency parameter and geometric parameter on the drag due to internal and external flows are found numerically and are shown graphically.

## 2 Fundamental equations and formulation of the problem

Here we consider an oscillating flow of the form  $U_{\infty} e^{i\omega t} \bar{k}$  of incompressible couple stress fluid, the direction of the oscillation being along  $\bar{k}$ . A spherical membrane of radius  $a$  with porous surface is introduced into the flow and held fixed at the origin. Since the sphere is having a porous membrane, the couple stress fluid flows across a fixed permeable sphere and divides the entire region into flow region-I external to the sphere and region-II internal to the sphere.

The basic equations governing the flow of an incompressible couple stress fluid as proposed by V.K. Stokes are

$$\nabla \cdot \bar{Q} = 0 \quad (1)$$

$$\rho \frac{d\bar{Q}}{dt} = -\nabla P - \mu \nabla \times \nabla \times \bar{Q} - \eta \nabla \times \nabla \times \nabla \times \bar{Q} \quad (2)$$

Neglecting convective terms, on the basis of Stokes assumption that flow is very slow and Reynolds number  $Re$  is very small ( $Re \ll 1$ ), equation (2) reduces to

$$\rho \frac{\partial \bar{Q}}{\partial t} = -\nabla P - \mu \nabla \times \nabla \times \bar{Q} - \eta \nabla \times \nabla \times \nabla \times \bar{Q} \quad (3)$$

Spherical coordinate system with origin at the center of the sphere and  $Z$  axis along the flow direction is considered. Velocity field and pressure suitable for this oscillating flow are of the form,

$$\bar{Q} = \nabla \times \left( \frac{\Psi \bar{e}_\phi}{h_3} \right) e^{i\omega t'} = \left( \frac{1}{R^2 \sin \theta} \frac{\partial \Psi}{\partial \theta} - \frac{1}{R \sin \theta} \frac{\partial \Psi}{\partial R} \bar{e}_\theta \right) e^{i\omega t'}$$

$$\text{and } P = P_0 e^{i\omega t'} \quad (4)$$

where the scale factors for spherical coordinate system are  $h_1=1$ ,  $h_2=R$  and  $h_3=R\sin\theta$  and  $\Psi$  the stream function is taken to satisfy equation (1).

By the choice of equation for velocity in (4), we note that

$$\nabla \times \bar{Q} = -\frac{E_0^2 \Psi}{h_3} \bar{e}_\phi \quad \text{and} \quad \nabla \times \nabla \times \nabla \times \bar{Q} = \frac{E_0^4 \Psi}{h_3} \bar{e}_\phi e^{i\omega t'} \quad (5)$$

By taking curl to equation (3), the pressure is eliminated and we get,

$$\rho i\omega \nabla \times \bar{Q} = -\mu \nabla \times \nabla \times \nabla \times \bar{Q} - \eta \nabla \times \nabla \times \nabla \times \nabla \times \bar{Q}. \quad (6)$$

Using (5) in (6), the equation for stream function  $\Psi$  is obtained as is

$$E_0^2 \left( E_0^2 - \frac{\lambda_1^2}{a^2} \right) \left( E_0^2 - \frac{\lambda_2^2}{a^2} \right) \Psi = 0 \quad (7)$$

where

$$\lambda_1^2 + \lambda_2^2 = \frac{\mu a^2}{\eta} = S \quad \text{and} \quad \lambda_1^2 \lambda_2^2 = \frac{i\rho\omega a^4}{\eta} = i.Re.S.\sigma \quad (8)$$

and  $E_0^2 = \frac{E^2}{a^2}$  and  $E^2 = \frac{\partial^2}{\partial r^2} - \frac{\cot \theta}{R^2} \frac{\partial}{\partial \theta} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2}$  = Stokes stream function operator

We introduce the non-dimensional scheme and the non-dimensional parameters like  $\sigma$  the frequency parameter, “ $Re$ ” Reynolds number and  $S$  the couple stress parameter as follows;

$$R = ra, \quad \Psi = I_\infty a^2 \psi, \quad P_0 = p\rho I_\infty^2, \quad t' = t/\omega, \quad \sigma = \frac{\omega a}{U_\infty},$$

$$Re = \frac{\rho U_\infty a}{\mu}, \quad S = \frac{\mu a^2}{\eta} \quad (9)$$

In equation (9), the small letters on RHS indicate non-dimensional quantities and the capital letters on LHS indicate dimensional quantities.

By this non-dimensional scheme the equation (7) reduces to

$$E^2(E^2 - \lambda_1^2)(E^2 - \lambda_2^2)\psi = 0 \quad (10)$$

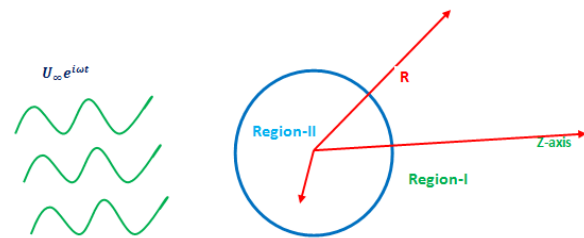


Figure 1: Flow geometry

Let

$$\psi = \psi_e, p = p_e \quad \text{for region I for } r \geq 1 \quad (11)$$

$$\psi = \psi_i, p = p_i \quad \text{for region II for } r \leq 1 \quad (12)$$

Now we find the solution of the equation (10) for  $\psi$  under the following conditions:

Region-I	Region-II
$E^2(E^2 - \lambda_1^2)(E^2 - \lambda_2^2)\psi_e = 0$	$E^2(E^2 - \lambda_1^2)(E^2 - \lambda_2^2)\psi_i = 0$
(i) $Lt_{r \rightarrow \infty} \psi_e = \frac{1}{2} r^2 \sin^2 \theta$	$Lt_{r \rightarrow \infty} \psi_i = \text{finite}$
(ii) $\frac{\partial \psi_e}{\partial r} = 0$ on $r = 1$	$\frac{\partial \psi_i}{\partial r} = 0$ on $r = 1$
(iii) $m_{r\phi} = 0$ on $r = 1$	$m_{r\phi} = 0$ on $r = 1$ type A (13.i-iv)
Or $E^2 \psi_e = 0$ on $r = 1$	$E^2 \psi_i = 0$ on $r = 1$ type B
(iv) $\psi_e = \psi_i = V_0$	on $r = 1$

Condition (i) of (13) represents the uniform flow condition (after removing oscillation term  $e^{i\omega t}$ ) far away from the sphere and finite velocity at the origin (centre of the sphere).

Condition (ii) of (13) represents no slip tangential velocity on the surface of sphere.

Condition (iii) represents vanishing of couple stresses on the surface (this is called type A condition) or represents hyper-stick condition which means vanishing of micro-rotations (this condition is called type B condition). Here either type A or type B condition is taken. Both conditions are not valid simultaneously. Long chain fluids satisfy type A condition and suspension like fluids satisfy type B condition.

Condition (iv) represents continuity condition for normal velocity which is equal to suction velocity  $V_0$  on the surface.

### 3 Solution for the problem undertaken

The solution for (10) is sought in the form,

$$\psi = \psi_0 + \psi_1 + \psi_2 = f(r) \sin^2 \theta = \{f_0 + b_1 f_1(r) + c_1 f_2(r)\} \sin^2 \theta \quad \text{where}$$

for  $r \geq 1$

$$\psi = \{f_0 + b_2 f_1(r) + c_2 f_2(r)\} \sin^2 \theta$$

for  $r \leq 1$

For external flow

$$f_0(r) = r^2 + a_1/r; f_1(r) = \sqrt{r} K_{3/2}(\lambda_1 r) \text{ and } f_2(r) = \sqrt{r} K_{3/2}(\lambda_2 r)$$

For internal flow

$$f_0(r) = a_2 r^2; f_1(r) = \sqrt{r} I_{3/2}(\lambda_1 r) \text{ and } f_2(r) = \sqrt{r} I_{3/2}(\lambda_2 r)$$

The suitable solutions for regularity condition are given by

$$\psi_e = \{r^2 + \frac{a_1}{r} + b_1 \sqrt{r} K_{3/2}(\lambda_1 r) + c_1 \sqrt{r} K_{3/2}(\lambda_2 r)\} I_2(x) \quad (14)$$

$$\psi_i = \{a_2 r^2 + b_2 \sqrt{r} I_{3/2}(\lambda_1 r) + c_2 \sqrt{r} I_{3/2}(\lambda_2 r)\} I_2(x) \quad (15)$$

Now from the condition (i) – (iv) we obtain the equations as given below:

$$2 - a_1 - b_1 K_{3/2}(\lambda_1) \Delta_1(\lambda_1) - c_1 K_{3/2}(\lambda_2) \Delta_1(\lambda_2) = 0, \quad (16a)$$

$$2a_2 - b_2 I_{3/2}(\lambda_1) \Delta_2(\lambda_1) - c_2 I_{3/2}(\lambda_2) \Delta_2(\lambda_2) = 0, \quad (16b)$$

$$\begin{aligned} 1 + a_1 + b_1 K_{3/2}(\lambda_1) + c_1 K_{3/2}(\lambda_2) \\ = a_2 + b_2 I_{3/2}(\lambda_1) + c_2 I_{3/2}(\lambda_2) = V_0 \end{aligned} \quad (16c)$$

$$\begin{aligned} \lambda_1^2 b_1 K_{3/2}(\lambda_1) + \lambda_2^2 c_1 K_{3/2}(\lambda_2) = 0 \\ \text{and } \lambda_1^2 b_2 I_{3/2}(\lambda_1) + \lambda_2^2 c_2 I_{3/2}(\lambda_2) = 0 \end{aligned} \quad (16d)$$

$b'_1, b'_2, c'_1$  and  $c'_2$  are defined as given below:

$$b'_1 = b_1 K_{3/2}(\lambda_1) \text{ and } b'_2 = b_2 I_{3/2}(\lambda_1)$$

$$c'_1 = c_1 K_{3/2}(\lambda_2) \text{ and } c'_2 = c_2 I_{3/2}(\lambda_2)$$

Now from (16.e), we get the constants as,

$$b'_1 = -\epsilon c'_1, \quad b'_2 = -\epsilon c'_2$$

where

$$\epsilon = \frac{\lambda_2^2}{\lambda_1^2}.$$

By solving the equations (16.a – d), in the following form, we get the constants

$$\begin{bmatrix} 1 & \Delta_3 & 0 \\ 0 & 0 & \Delta_4 \\ 1 & 1 - \epsilon & \epsilon - 1 \end{bmatrix} \begin{bmatrix} a_1 \\ c'_1 \\ c'_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2a_2 \\ a_2 - 1 \end{bmatrix}$$

$$c'_1 = \frac{-3\Delta_4 + (\Delta_4 - 2\epsilon + 2)\tau}{\Delta_4 \Delta_5}; \quad c'_2 = \frac{2\tau}{\Delta_4} \quad \text{and} \quad a_1 = 2 - \Delta_3 c'_1 \quad (17)$$

$$\Delta_1(x) = 1 + \frac{x K_{1/2}(x)}{K_{3/2}(x)}$$

and

$$\Delta_2(x) = 1 - \frac{x I_{1/2}(x)}{I_{3/2}(x)},$$

$$\Delta_3 = \Delta_1(\lambda_2) - \epsilon \Delta_1(\lambda_1),$$

$$\Delta_4 = \Delta_2(\lambda_2) - \epsilon \Delta_2(\lambda_1),$$

$$\Delta_5 = (1 - \epsilon - \Delta_3) \text{ and } I_2(x) = \frac{1}{2}(1 - x^2)$$

The arbitrary constants  $a_1, a_2, b_1, b_2, c_1, c_2$  in (16) are expressed as  $a_1, a_2, b'_1, b'_2, c'_1, c'_2$  in (17). The arbitrary constants are six (6) in number, but the number of equations are five (5).  $b'_1, b'_2$ , are expressed in terms of  $c'_1, c'_2$ . Hence 4 arbitrary constants are expressed in 3 equations as in (17). Hence one of the constants is arbitrary.

$\tau$  is arbitrary which is taken in the place of  $a_2$ . Hence  $\tau$  need not take real values and need not start from zero value. Now  $\tau$  is defined as permeability parameter.

### 4 Pressure distribution

From equation (3) pressure is given by

$$\nabla P = -i\rho\omega\bar{Q} - \mu\nabla \times \nabla \times \bar{Q} - \eta\nabla \times \nabla \times \nabla \times \bar{Q}$$

The equations in non-dimensional form along radial and transverse directions are given by

$$Re S \frac{\partial p}{\partial r} = \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (-i Re . S \sigma \psi + S E^2 \psi - E^4 \psi)$$

$$Re S \frac{\partial p}{\partial \theta} = \frac{1}{\sin \theta} \frac{\partial}{\partial r} (i Re . S \sigma \psi - S E^2 + E^4 \psi)$$

From these pressure can be obtained as

$$\begin{aligned} Re . S . p &= -\frac{d}{dr} \left( D^4 f - S . D^2 f + i Re . S . \sigma f \right) . \cos \theta \\ &= -\lambda_1^2 \lambda_2^2 f'_0(r) \cos \theta \end{aligned}$$

For external and internal flows this reduces to,

$$P_e = i\rho U_\infty a \omega \left(2r - \frac{a_1}{r^2}\right) \cos \theta = i\rho U_\infty^2 \sigma \left(2r - \frac{a_1}{r^2}\right) \cos \theta,$$

i.e.

$$p_e = i\sigma \left(2r - \frac{a_1}{r^2}\right) \cos \theta$$

$$P_i = 2i\rho a U_\infty \omega \tau r \cos \theta = i2\rho U_\infty^2 \sigma \tau r \cos \theta,$$

i.e.

$$p_i = 2i\sigma \tau r \cos \theta$$

## 5 Bounds for permeability parameter $\tau$

On the surface the filtration velocity  $V_0 = 0$  gives  $\tau = 0$

And  $\Delta P = P_e - P_i = 0$  gives  $\tau = \frac{3\Delta_3\Delta_4}{3\Delta_3\Delta_4 + 2(1-\epsilon)(\Delta_4 - \Delta_3)}$

Hence bounds for  $\tau$  are as follows:

$$0 \leq |\tau| \leq \left| \frac{3\Delta_3\Delta_4}{3\Delta_3\Delta_4 + 2\Delta_3(\epsilon_2 + 1) - 2\Delta_4(\epsilon_1 + 1)} \right| \quad (18)$$

It is to be observed that  $\tau$  takes complex values. For the sake of calculations, for  $\tau$  any real value between 0 and the maximum bound can be taken.

## 6 Drag on the sphere

The drag  $D$  on the sphere is given by the formula

$$\text{Drag} = D_r = \int_0^\pi (T_{rr} \cos \theta - T_{r\theta} \sin \theta) 2\pi R^2 \sin \theta d\theta$$

on  $r = 1$  (19)

Stress tensor for couple stress fluid is given by the constitutive equation

$$\mathbf{T} = -P\mathbf{I} + \lambda \nabla \cdot \mathbf{Q} + 2\mu \mathbf{E} + \frac{1}{2} I \times \nabla \cdot \mathbf{M} \quad (20)$$

The constitutive equation for Couple stress tensor  $\mathbf{M}$  is given by

$$\mathbf{M} = m\mathbf{I} + 2\eta \nabla (\nabla \times \mathbf{Q}) + 2\eta' [\nabla (\nabla \times \mathbf{Q})]^T \quad (21)$$

The strain tensor  $\mathbf{E}$  in (20) is calculated by

$$\mathbf{E} = \frac{\partial U}{\partial R} e_r e_r + \frac{1}{2} \left( \frac{\partial V}{\partial R} + \frac{1}{R} \cdot \frac{\partial U}{\partial \theta} - \frac{V}{R} \right) (e_r e_\theta + e_\theta e_r) + \frac{1}{R} \left( \frac{\partial V}{\partial \theta} + U \right) e_\theta e_\theta + \frac{1}{R} (U + \cot \theta V) e_\phi e_\phi$$

This strain in non-dimensional form in terms of stream function is given by

$$\mathbf{E} = \frac{U_\infty}{a} \left[ \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial^2 \psi}{\partial r \partial \theta} - \frac{2}{r} \frac{\partial \psi}{\partial \theta} \right\} e_r e_r + \frac{1}{2r \sin \theta} \left\{ E^2 \psi - 2 \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \frac{\partial \psi}{\partial r} \right\} (e_r e_\theta + e_\theta e_r) + \frac{1}{r^2 \sin \theta} \left\{ \frac{1}{r} \frac{\partial \psi}{\partial \theta} + \cot \theta \frac{\partial \psi}{\partial r} - \frac{\partial^2 \psi}{\partial r \partial \theta} \right\} e_\theta e_\theta + \frac{1}{r^2 \sin \theta} \left\{ \frac{1}{r} \frac{\partial \psi}{\partial \theta} - \cot \theta \frac{\partial \psi}{\partial r} \right\} e_\phi e_\phi \right]$$

The couple stress tensor  $\mathbf{M}$  in (21) is given by

$$\mathbf{M} = \frac{\rho U_\infty^2}{a} m\mathbf{I} + \frac{\rho \eta U_\infty^2}{a} \left[ \frac{1}{\sin \theta} \left\{ \frac{\epsilon E^2 \psi}{r^2} - \frac{\partial}{\partial r} \left( \frac{E^2 \psi}{r} \right) \right\} e_r e_\phi + \frac{1}{\sin \theta} \left\{ \frac{E^2 \psi}{r^2} - \frac{\epsilon \partial}{\partial r} \left( \frac{E^2 \psi}{r} \right) \right\} e_\phi e_r + \frac{1}{r} \left\{ \frac{\epsilon E^2 \psi}{r \sin \theta} - \frac{\partial}{\partial \theta} \left( \frac{E^2 \psi}{r \sin \theta} \right) \right\} e_\theta e_\phi + \frac{1}{r} \left\{ \frac{E^2 \psi}{r \sin \theta} - \epsilon \frac{\partial}{\partial \theta} \left( \frac{E^2 \psi}{r \sin \theta} \right) \right\} e_\phi e_\theta \right]$$

where  $\epsilon = \eta'/\eta$ .

Now the following quantities can be evaluated.

$$\nabla \cdot \mathbf{M} = -\frac{\rho \eta U_\infty^2}{a^2} 2E^4 \psi e_\phi$$

$$\text{and } I \times \nabla \cdot \mathbf{M} = -\frac{\rho \eta U_\infty^2}{a^2} 2E^4 \psi (e_r e_\theta - e_\theta e_r)$$

$$T_{rr} = \frac{\mu U_\infty}{a} \left\{ i\sigma \cdot \text{Re} f'_0 + \frac{2}{r^2} \left( f' - \frac{2f}{r} \right) \right\} \cos \theta$$

$$T_{r\theta} = \frac{\mu U_\infty}{a r} \left\{ \left( D^2 f - 2f'' + \frac{2}{r^2} f' \right) + \frac{1}{S} D^4 f \right\} \sin \theta$$

From this we get that the non-dimensional drag  $D'_{ex}$  due to external flow and  $D'_{in}$ .

Drag due to external flow  $D_{ex}$  is given by,

$$D_{ex} = 2\pi \mu U_\infty a \cdot e^{i\sigma t} \int_0^\pi \left[ \left\{ i\sigma \text{Re} f'_0 + 2 \left( f' - \frac{2f}{r} \right) \right\} \cos^2 \theta - \left\{ \frac{D^4 f}{S} - D^2 f - 2f + 2f' \right\} \sin^2 \theta \right] \sin \theta \cdot d\theta$$

By taking  $f' = 0$  and  $f = V_0$  on  $r = 1$ , the drag simplifies to

$$D_{ex} = \frac{8\pi \mu U_\infty a}{3} \cdot i\sigma \text{Re} \left( V_0 - \frac{3a_1}{2} \right) e^{i\sigma t} \quad (22)$$

Similarly Drag due to internal flow=

$$D_{in} = \frac{8\pi \mu U_\infty a}{3} \cdot i\sigma \text{Re} V_0 e^{i\sigma t} \quad (23)$$

The non-dimensional drag  $D^*$  is obtained by comparing the drag with Stokes drag.

$$D_{ex}^* = \frac{D_{ex}}{6\pi\mu U_\infty a} = \frac{4}{9}i\sigma Re \left( V_0 - \frac{3a_1}{2} \right) e^{i\sigma t}$$

$$\text{and } D_{in}^* = \frac{4}{9}i\sigma Re V_0 e^{i\sigma t} \quad (24)$$

## 7 Results and discussion

The geometric parameters  $\lambda_1^2$  and  $\lambda_2^2$  of equation (8) are computed by solving the quadratic equation

$$\lambda^4 - S\lambda^2 + i\sigma \cdot Re \cdot S = 0. \quad (25)$$

Then, the constants in the stream function  $\psi$  in (14) and (15) for internal flow and external flow are obtained by using the equations (17).

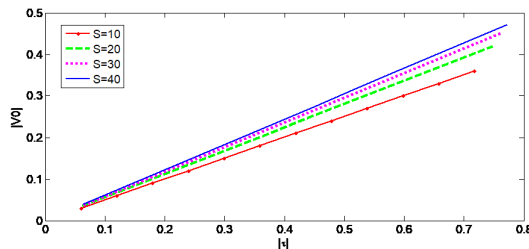
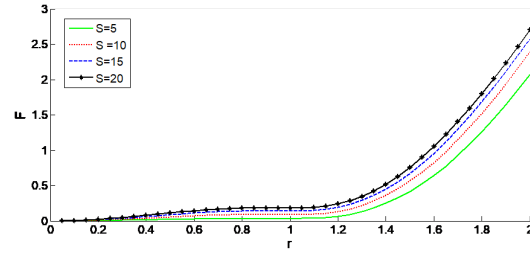


Figure 2: Filtration velocity  $V_0$  vs. permeability parameter  $\tau$

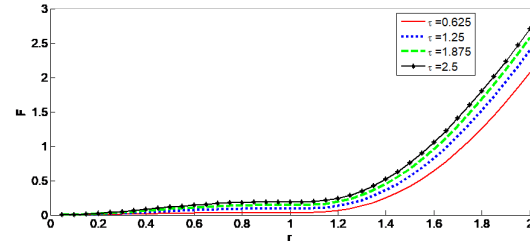
**Filtration velocity  $V_0$  on the surface:** Then the permeability parameter  $\tau$  is fixed by choosing a value within the bounds given in (18). It is to be noted that  $\tau$  is not real. For choosing a value for  $\tau$  we can fix a real value which is less than the maximum bound in (18). Now filtration velocity  $V_0$  in (16.4) can be computed. This filtration velocity  $V_0$  is presented in figure2. With an increase of  $S$  there is an increase in the filtration velocity also. But for any value of  $S$ , filtration velocity is less than 50% of the velocity at infinity.

**Radial velocity:** The stream function  $\psi$  in terms of radial function  $f(r)$  is shown in Figure 3 at different values of  $\tau$  and  $S$ . As  $\tau$  or  $S$  increases,  $f$  (radial velocity) increases. i.e. as couple stresses increase, they increase the radial velocity. This means in the case of viscous flow, the radial velocity will be less than that in couple stress fluid flow (since as  $S \rightarrow \infty$ , the flow reduces to viscous flow).

**Drag:** Drag on the sphere because of the flow of couple stress fluid without the time factor  $e^{i\sigma t}$  is computed in Figure 4. Couple stress parameter  $S$  is not involved directly in the formula for drag. But it is found that with the increase in parameter  $S$ , there is a decrease in drag and tends to



(a)



(b)

Figure 3: Stream function  $F$  for various values of a) permeability parameter  $\tau$ ; b) couple stress parameter  $S$

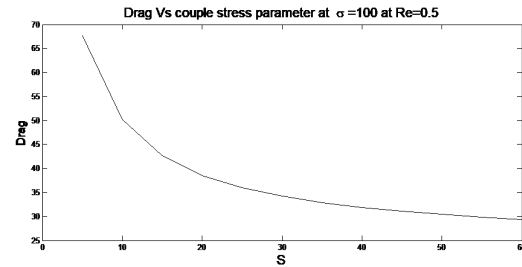


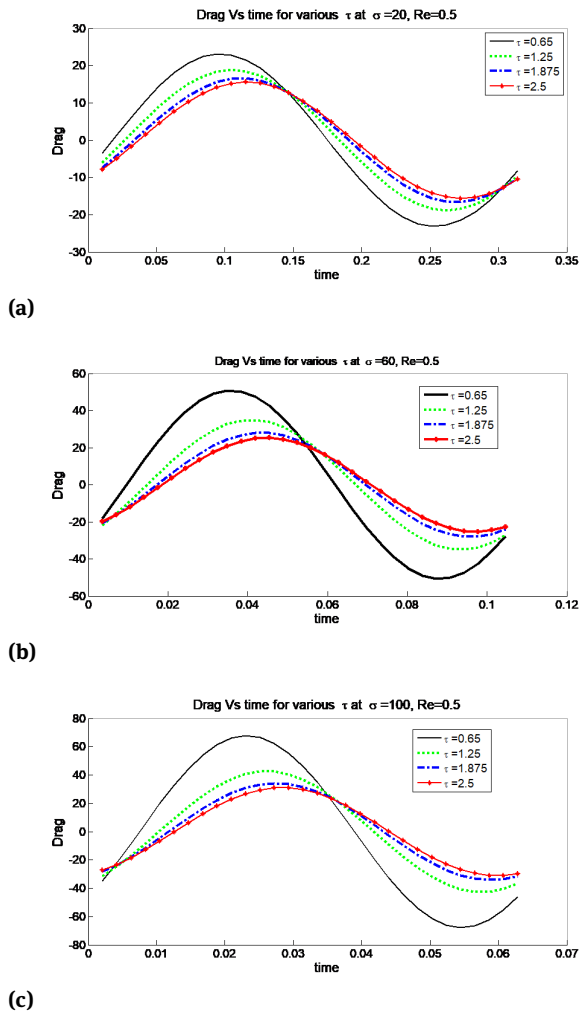
Figure 4: Drag vs. couple stress parameter

a fixed quantity. (in the figure near to 45 at  $\sigma = 100$  and  $Re = 0.5$ ). This indicates that the drag for viscous fluid flow will be less than the case of couple-stress fluids.

In Figure 5, drag is shown by including the time factor  $e^{i\sigma t}$ . Drag is drawn for a time period  $2\pi/\sigma$ . We notice that as  $\sigma$  increases drag increases and as  $\tau$  increases, magnitude of drag decreases. This can be expected. As frequency of oscillations increase, it is natural to expect high drag on the body.

**Stream function:** The stream function without time factor  $e^{i\sigma t}$  is shown in Figure 6. Three stream lines  $\psi=0.01$ , 0.05 and 0.12 are shown at different permeability parameter  $\tau$ . The flow is as perceived by an observer travelling with the flow. It is to be noted that all stream lines are with positive sign only. The stream line  $\psi=0$  passes through center of the sphere. As the value of  $\psi$  increases, the stream lines move away from the sphere and take uniform flow far away from the sphere. Three stream lines

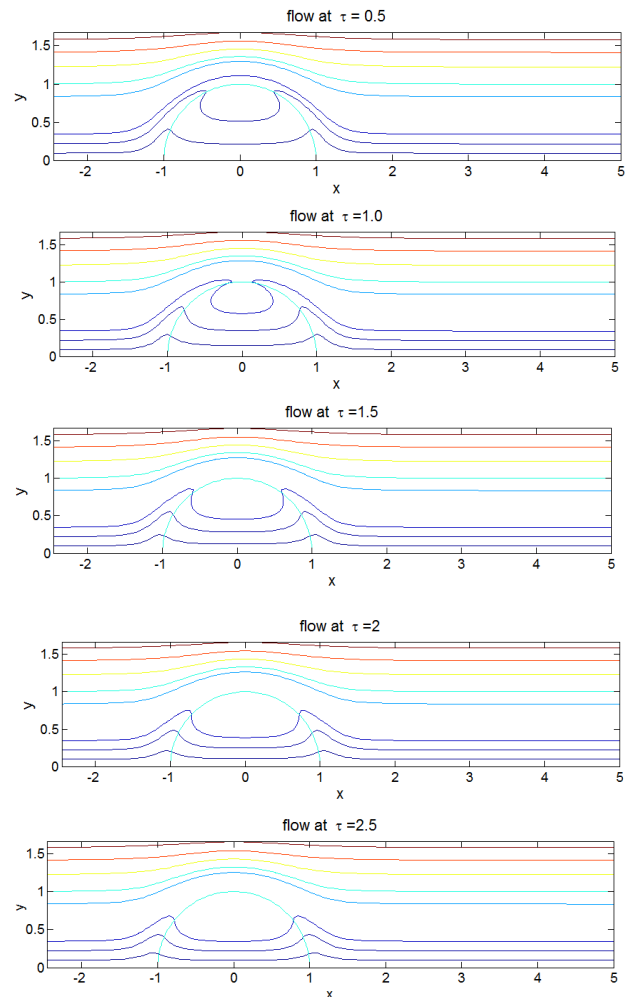




**Figure 5:** Drag vs. time for time period  $2\pi/\sigma$  at a)  $\sigma = 20$ ; b)  $\sigma = 50$ ; c)  $\sigma = 100$

$\psi=0.01, 0.05$  and  $0.12$  are coming near to the axis of sphere as  $\tau$  increases, which indicates that as  $\tau$  increases more number of stream lines are flowing through the sphere. It is observed that when  $\tau$  is small, below the top pole near to it there is small circulation. When  $\tau$  increases, this circulation disappears.

In Figure 7, the stream lines with time factor  $e^{i\sigma t}$  are shown. It is interesting to note that pattern of stream lines with time factor including and excluding differ completely. Now the stream lines take both negative and positive values. The flow is as per the observations of an observer fixed in space. It is exciting to note that near the sphere there is another fluid spherical region in which flow circulations take place. Within this fluid sphere  $\psi$  takes negative values. Outside this fluid sphere  $\psi$  is positive and flow is same as that of flow past an impermeable sphere. As  $\tau$  increases,



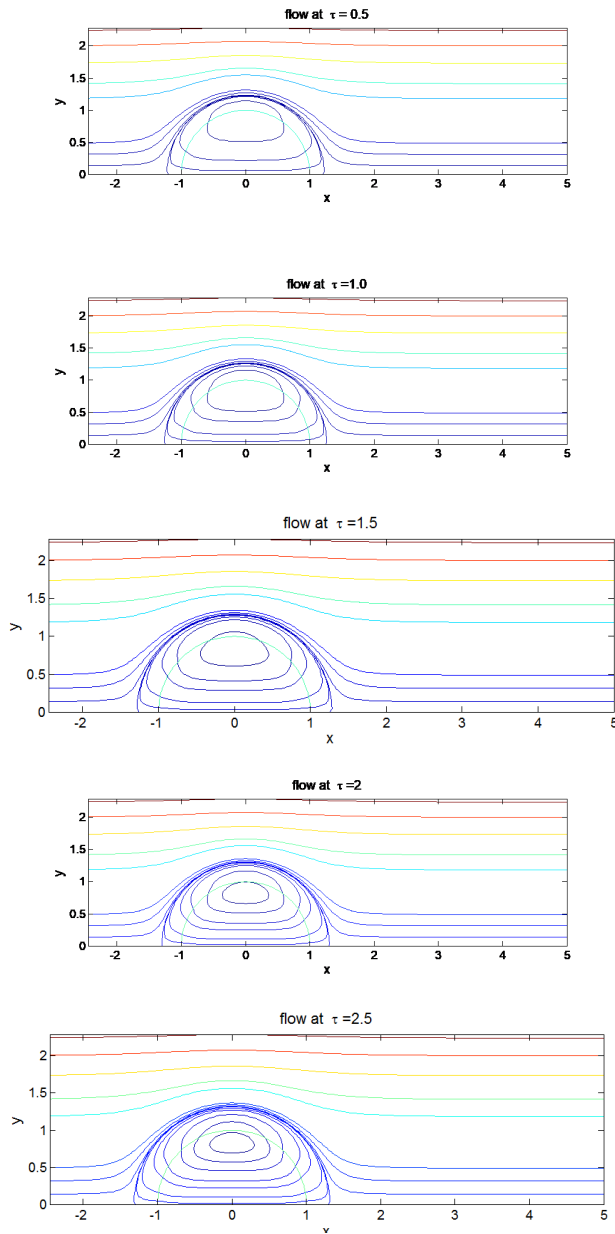
**Figure 6:** Stream lines without time factor  $e^{i\sigma t}$  at different permeabilities;  $\psi=0.01, \psi=0.05$  and  $\psi=0.12$  are passing through the sphere

the circulation is more prominent and the centre of the circulation is below the top pole near to it.

## 8 Conclusions

The following observations are made in the study of the oscillating flow of couple stress fluid past a permeable sphere.

1. As permeability parameter  $\tau$  increases, filtration velocity increases.
2. As couple-stress parameter increases, there is an increase in filtration velocity.
3. The observer with the flow (flow pattern excluding  $e^{i\sigma t}$ ), observes a small circulation near the pole at small permeabilities.



**Figure 7:** Stream lines with time factor  $e^{i\omega t}$ . Stream lines with circulations are negative.

4. The observer fixed in space (flow pattern including  $e^{i\omega t}$ ) observes a circulation of fluid within a fluid sphere which passes through the permeable sphere. The center of circulation is below the pole of the sphere.

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