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Comparative study of homotopy perturbation transformation with homotopy perturbation Elzaki transform method for solving nonlinear fractional PDE

https://doi.org/10.1515/nleng-2018-0136
Received September 14, 2018; revised December 4, 2018; accepted January 28, 2019.

Abstract: We apply homotopy perturbation transformation method (combination of homotopy perturbation method and Laplace transformation) and homotopy perturbation Elzaki transformation method on nonlinear fractional partial differential equation (fpde) to obtain a series solution of the equation. In this case, the fractional derivative is described in Caputo sense. To avow the adequacy and authenticity of the technique,we have applied both the techniques to Fractional Fisher's equation,time-fractional Fornberg-Whitham equation and time fractional Inviscid Burgers' equation. Finally, we compare the results obtained from homotopy perturbation transformation technique with homotopy perturbation Elzaki transformation.

Keywords: nonlinear fractional partial differential equation; HPTM; Caputo sense; He's polynomial; Elzaki transform; HPETM

1 Introduction

Most of the real world problems arising in the field of biology, fluid mechanics, ecology and thermodynamics etc. are modelled as nonlinear partial differential equation(PDE). The fractional calculus is important tool to refine the description of most of the natural phenomenon. Fractional differential equations have attracted considerable interest of many researcher because of their successive appearance in diverse fields of science and engineering. Many numerical and semi-analytical techniques are used to obtain the solution of linear and nonlinear partial differential equations.

Dr. Ji Huan He in 1999, proposed homotopy perturbation method (HPM) [11, 13] which is coupling of homotpoy method and classical perturbation technique, has been successfully implemented on linear and nonlinear problems like nonlinear wave equation[15, 16], fractional diffusion equation [3], fractional convection–diffusion equation [19], space–time fractional advection–dispersion equation [35], fractional Zakharov–Kuznetsov equations [34], fractional partial differential equations in fluid mechanics [33], fractional Schrödinger equation [32]. The significance of HPM is that it doesn't require a small parameter in the equation , so it overcome the impediments of classical perturbation technique. The other semi-analytical techniques such as HAM (Homotopy analysis method) [18], Laplace homotopy analysis method [20] Adomian decomposition method [4], HPTM(Homotopy perturbation transformation method) [10, 17, 24, 26, 28] and HPSTM(Homotopy perturbation Sumudu transform method) [23, 27], we can always obtain better result than the numerical one for partial differential equation. In recent years, many researchers have used numerical and analytical technique [1, 29–31] for the solution of fractional partial differential equation. In[9], author suggested a new form of fractional differentiation to model comlex physical problems. In this work, we apply HPTM [17, 24, 25] technique (which is combination of homotopy perturbation method and Laplace transformation) and HPETM [5–8, 21] (Homotopy perturbation method with Elzaki transform) to find the solution of Fractional Fisher's equation,time-fractional Fornberg-Whitham equation and time fractional Inviscid Burgers' equation and we get a power series solution in the form of a rapidly convergent series and only a few iterations lead to high accurate solutions. In these techniques, there is no need of algorithm like discritizing the problem, no linearization is required for nonlinear problem , only few iterations will lead to the solution which can be easily calculated. There are many sym-

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bolic computation software like Maple, Mathematica etc. with which we can easily calculate more terms very easily, hence it reduces the computational cost for solving such complex problem. Finally, we compare the result obtained by these methods.

2 Basic definitions and properties

Definition 2.1. A real function $g(t) \in C_\mu, t > 0, \mu \in \mathbb{R}$ if $\exists\ q \in \mathbb{R}; (q > \mu)$, such that $g(t) = t^q k(t)$, where $k(t) \in C[0, \infty)$ and $g(t) \in C_\mu^m$ if $g^{(m)} \in C_\mu, m \in \mathbb{N}$.

Definition 2.2. The Caputo fractional derivative of $g(\tau)$ [2, 22] is defined as

$$\frac{\partial^\alpha}{\partial \tau^\alpha} g(\tau) = J^{m-\alpha} \frac{\partial^m}{\partial \tau^m} g(\tau) = \frac{1}{\Gamma(m-\alpha)} \int_0^\tau (\tau - \eta)^{m-\alpha-1} g^m(\eta) d\eta,$$

$g \in C_{-1}^m, m - 1 < \alpha \le m, m \in \mathbb{N}, \tau > 0$. Here $\frac{\partial^\alpha}{\partial \tau^\alpha}$ and Γ denotes Caputo derivative operator and the Gamma function respectively.

Definition 2.3. The Elzaki transformation [5, 7] of $g(\tau)$ is defined as

$$E[g(\tau)] = v \int_0^\infty g(\tau) e^{\frac{-\tau}{v}} d\tau = F(v), \tau > 0. \tag{2.1}$$

2.1 Properties

Elzaki transform of the Caputo fractional derivative is

$$E\left\{ \frac{\partial^\alpha}{\partial \tau^\alpha} g(\tau) \right\} = \frac{E\{g(\tau)\}}{v^\alpha} - \sum_{k=0}^{n-1} v^{k-\alpha+2} g^{(k)}(0), n - 1 < \alpha \le n. \tag{2.2}$$

Definition 2.4. The Laplace transformation of $f(\tau)$ is defined as

$$\mathcal{L}[f(\tau)] = \int_0^\infty f(\tau) e^{-s\tau} d\tau = F(s), \tau > 0. \tag{2.3}$$

2.2 Properties

Laplace transform of the Caputo fractional derivative is given by [2]

$$\mathcal{L}\left\{ \frac{\partial^\alpha}{\partial \tau^\alpha} f(\tau) \right\} = s^\alpha \mathcal{L}\{f(\tau)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), n - 1 < \alpha \le n. \tag{2.4}$$

Definition 2.5. The Mittag-Leffler function of two parameter α and β is given by [12]

$$E_{\alpha,\beta}(\tau) = \sum_{n=0}^\infty \frac{\tau^n}{\Gamma(\alpha n + \beta)}, \alpha, \beta > 0 \tag{2.5}$$

3 Homotopy perturbation method (HPM)

Consider the nonlinear partial differential equation

$$L(w) + N(w) = f(r), r \in \Omega \tag{3.1}$$

with boundary condition

$$B\left(w, \frac{\partial w}{\partial n}\right) = 0, r \in \Gamma \tag{3.2}$$

where L and N are linear and nonlinear differential operator and $f(r)$ is an analytic function. Ji-huan He [13], [14] construct a homotopy of eq. (3.1) as $H : \Omega \times [0, 1] \to \mathbb{R}$ which satisfies

$$H(v, p) = (1 - p)(L(v) - L(w_0)) + p(L(v) + N(v) - f(r)) = 0 \tag{3.3}$$

or

$$H(v, p) = L(v) - L(w_0) + pL(w_0) + p(N(v) - f(r)) \tag{3.4}$$

where $p \in [0, 1]$ is an embedding parameter and w_0 is an initial approximation which satisfies the boundary conditions. Clearly, from (3.3), we have

$$H(v, 0) = L(v) - L(w_0) = 0 \tag{3.5}$$

$$H(v, 1) = (L(v) + N(v) - f(r)) = 0 \tag{3.6}$$

The process of changing of p from zero to unity is that v varies from w_0 to $w(x, t)$. The basic assumption for this method is that the solution of (3.1) can be expressed as

$$v = v_0 + v_1 p + v_2 p^2 + v_3 p^3 + \dots$$

The solution of (3.1) is given by

$$w(x, t) = \lim_{p \to 1} (v_0 + v_1 p + v_2 p^2 + v_3 p^3 + \dots) \tag{3.7}$$

$$= v_0 + v_1 + v_2 + \dots \tag{3.8}$$

4 Homotopy perturbation Elzaki transform method (HPETM)

Consider the following general fractional nonlinear partial differential equation

$$\frac{\partial^\alpha}{\partial t^\alpha} w(x,t) + Lw(x,t) + Nw(x,t) = f(x,t), t > 0, x \in \mathbb{R}, n-1 < \alpha \le n, \tag{4.1}$$

here, $\frac{\partial^\alpha}{\partial t^\alpha}$, is the fractional Caputo derivative with respect to t, L and N are linear and non linear differential operators respectively which satisfy Lipschitz condition, $f(x,t)$ is the source term. Now applying Elzaki transform, we get

$$E\left\{\frac{\partial^\alpha}{\partial t^\alpha} w + Lw + Nw\right\} = E\{f(x,t)\}.$$

Using (2.2), we have

$$E\{w\} = \sum_{k=0}^{n-1} v^{k+2} w^{(k)}(x,0) + v^\alpha\left(E\left\{f(x,t) - Lw - Nw\right\}\right).$$

Applying the inverse Elzaki transform, we have

$$w = \sum_{k=0}^{n-1} \frac{t^k}{k!} w^{(k)}(x,0) + E^{-1}\left\{v^\alpha\left(E\left\{f(x,t) - Lw - Nw\right\}\right)\right\}. \tag{4.2}$$

By applying HPM, we get

$$0 = (1-p)\left(w(x,t) - w(x,0)\right) + p\left(w(x,t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} w^{(k)}(x,0) - E^{-1}\left\{v^\alpha E\left\{f(x,t) - Lw - Nw\right\}\right\}\right),$$

Let

$$\begin{aligned} w(x,t) &= \sum_{n=0}^{\infty} p^n w_n(x,t), \\ Nw(x,t) &= \sum_{n=0}^{\infty} p^n H_n(w(x,t)) \end{aligned} \tag{4.3}$$

where

$$H_n(w(x,t)) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(\sum_{i=0}^{\infty} p^i w_i\right)_{(p=0)}, n = 0, 1, 2, 3, \ldots \tag{4.4}$$

So, (4.2) becomes

$$\sum_{n=0}^{\infty} p^n w_n = w(x,0) + p\left(\sum_{k=1}^{n-1} \frac{t^k}{k!} w^{(k)}(x,0) + E^{-1}\left\{v^\alpha\left(E\left\{f(x,t) - L\sum_{n=0}^{\infty} p^n w_n - \sum_{n=0}^{\infty} p^n H_n(w)\right\}\right)\right\}\right).$$

Comparing the coefficients of like powers of p, we have

$$\begin{aligned} p^0 &: w_0 = w(x,0); \\ p^1 &: w_1 = \sum_{k=1}^{n-1} \frac{t^k}{k!} w^{(k)}(x,0) + E^{-1}\left\{v^\alpha\left(E\left\{f(x,t) - Lw_0(x,t) - H_0\right\}\right)\right\}; \\ p^2 &: w_2 = -E^{-1}\left\{v^\alpha\left(E\left\{Lw_1(x,t) + H_1\right\}\right)\right\}; \\ p^3 &: w_3 = -E^{-1}\left\{v^\alpha\left(E\left\{Lw_2(x,t) + H_2\right\}\right)\right\}, \\ &\vdots \end{aligned}$$

therefore, the HPETM series solution is obtained as $p \to 1$

$$w(x,t) = w_0 + w_1 + w_2 + w_3 + \ldots.$$

5 Convergence analysis

In this section, we emphasis on the condition of convergence of the proposed method for the series solution of eq. (4.1).

Theorem 5.1. *Let w and $w_n(x,t)$ be defined in Banach space, then the condition that series solution defined by eq. (4.3) converges to the solution of eq.(4.1) if $\exists\ \eta \in (0,1)$ such that $||w_{n+1}|| \le \eta\ ||w_n||$.The condition of convergence has been proved in [27, 28].*

Theorem 5.2. *The maximum absolute truncation error of the series solution eq. (4.3) of eq. (4.1) is given by*

$$|w(x,t) - \sum_{k=0}^{n} w_k(x,t)| \le \frac{\eta^{n+1}}{1-\eta}||w_0||$$

6 Application

Example 6.1. Consider the time fractional nonlinear Fisher's equation [21]

$$\frac{\partial^\alpha w}{\partial t^\alpha} = \frac{\partial^2 w}{\partial x^2} + 6w(1-w), t > 0, x \in \mathbb{R}, 0 < \alpha \le 1, \tag{6.1}$$

with initial condition $w(x,0) = \frac{1}{(1+e^x)^2}$.
By applying HPETM on (6.1), we have

$$\sum_{n=0}^{\infty} p^n w_n = \frac{1}{(1+e^x)^2} + p\left(E^{-1}\left\{v^\alpha\left(E\left\{\sum_{n=0}^{\infty}(p^n w_n)_{xx} + 6\left\{\sum_{n=0}^{\infty} p^n w_n - 6\sum_{n=0}^{\infty} p^n H_n(w)\right\}\right\}\right)\right\}\right), \tag{6.2}$$

where $H_n(w)$ represents He's polynomial . The first few components of He's polynomial are given by

$$\begin{aligned} H_0(w) &= w_0^2; \\ H_1(w) &= 2w_0w_1; \\ H_2(w) &= w_1^2 + 2w_0w_2, \\ &\vdots \end{aligned}$$

Comparing the like powers of p on both sides of (6.2), we have

$$\begin{aligned} p^0 : w_0 &= \frac{1}{(1+e^x)^2}, \\ p^1 : w_1 &= 10\frac{e^x t^\alpha}{(1+e^x)^3\Gamma(\alpha+1)}, \\ p^2 : w_2 &= 50\frac{e^x(2e^x-1)t^{2\alpha}}{(1+e^x)^4\Gamma(2\alpha+1)}, \\ p^3 : w_3 &= \left(50\frac{e^x(-16e^{3x}-15e^{2x}+30e^x+5)}{(1+e^x)^6}\right. \\ &\quad \left. +600e^{2x}\frac{\Gamma(2\alpha+1)}{(1+e^x)^6\Gamma(\alpha+1)^2}\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}. \\ &\vdots \end{aligned}$$

Hence the solution is

$$\begin{aligned} w(x,t) &= \frac{1}{(1+e^x)^2} + 10\frac{e^x t^\alpha}{(1+e^x)^3\Gamma(\alpha+1)} \\ &+ 50\frac{e^x(2e^x-1)t^{2\alpha}}{(1+e^x)^4\Gamma(2\alpha+1)} \\ &+ \left(50\frac{e^x(-16e^{3x}-15e^{2x}+30e^x+5)}{(1+e^x)^6}\right. \\ &\left. +600e^{2x}\frac{\Gamma(2\alpha+1)}{(1+e^x)^6\Gamma(\alpha+1)^2}\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots. \end{aligned} \tag{6.3}$$

The above solution obtained is equivalent to the closed form solution when $\alpha = 1$ i.e $w(x,t) = \frac{1}{(1+e^{x-5t})^2}$ up to fourth order term approximation.

Example 6.2. Consider the time-fractional Fornberg-Whitham equation [26]

$$\frac{\partial^\alpha}{\partial t^\alpha}w(x,t) = \frac{\partial^3 w}{\partial x^2\partial t} - \frac{\partial w}{\partial x} + w\frac{\partial^3 w}{\partial x^3} - w\frac{\partial w}{\partial x} + 3\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2},$$
$$t > 0, x \in \mathbb{R}, 0 < \alpha \le 1, \tag{6.4}$$

with initial condition $w(x,0) = e^{\frac{x}{2}}$.
By applying HPETM on (6.4), we have

$$\begin{aligned} \sum_{n=0}^{\infty} p^n w_n &= e^{\frac{x}{2}} + p\left(E^{-1}\left\{v^\alpha\left(E\left\{\sum_{n=0}^{\infty} p^n (w_n)_{xxt}\right.\right.\right. \\ &\left.\left.\left. + \sum_{n=0}^{\infty} p^n(-w_n)_x + \left\{\sum_{n=0}^{\infty} p^n H_n(w)\right\}\right\}\right)\right\}\right), \end{aligned} \tag{6.5}$$

where $H_n(w)$ are He's polynomial represents nonlinear terms. The first few components of He's polynomial are given by

$$\begin{aligned} H_0(w) &= w_0w_{0xxx} - w_0w_{0x} + 3w_{0x}w_{0xx}; \\ H_1(w) &= w_0w_{1xxx} + w_1w_{0xxx} - w_0w_{1x} - w_1w_{0x} \\ &\quad +3w_{0x}w_{1xx} + 3w_{1x}w_{0xx}; \end{aligned}$$

$$\begin{aligned} H_2(w) = {} & w_0w_{2xxx} + w_1w_{1xxx} + w_2w_{0xxx} - w_0w_{2x} - w_1w_{1x} \\ & - w_2w_{0x} + 3w_{2x}w_{0xx} + 3w_{1x}w_{1xx} + 3w_{0x}w_{2xx}. \end{aligned}$$

Comparing the like powers of p on both sides of (6.5), we have

$$\begin{aligned} p^0 : w_0 &= e^{\frac{x}{2}}; \\ p^1 : w_1 &= \frac{-e^{\frac{x}{2}}}{2}\frac{t^\alpha}{\Gamma(\alpha+1)}; \\ p^2 : w_2 &= \frac{-e^{\frac{x}{2}}}{8}\frac{t^{2\alpha-1}}{\Gamma 2\alpha} + \frac{e^{\frac{x}{2}}}{4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}; \\ p^3 : w_3 &= e^{\frac{x}{2}}\left(\frac{-1}{32}\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} + \frac{1}{8}\frac{t^{3\alpha-1}}{\Gamma(3\alpha)} - \frac{1}{8}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right), \\ &\vdots \end{aligned}$$

Hence, the solution is

$$\begin{aligned} w(x,t) &= e^{\frac{x}{2}} - \frac{e^{\frac{x}{2}}}{2}\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{e^{\frac{x}{2}}}{8}\frac{t^{2\alpha-1}}{\Gamma 2\alpha} + \frac{e^{\frac{x}{2}}}{4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &+ e^{\frac{x}{2}}\left(\frac{-1}{32}\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} + \frac{1}{8}\frac{t^{3\alpha-1}}{\Gamma(3\alpha)} - \frac{1}{8}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right) + \ldots. \end{aligned} \tag{6.6}$$

From the above solution, it is clear that the approximate solution(up to fourth order approximation) obtained from above said technique is very closed to the exact solution i.e. $w(x,t) = e^{\frac{1}{2}(x-\frac{4t}{3})}$ for $\alpha = 1$.

Example 6.3. Consider the nonlinear nonhomogeneous time fractional Inviscid Burgers' equation [36]

$$D_t^\alpha w + ww_x = 1 + x + t, w(x,0) = x, 0 < \alpha \le 1 \tag{6.7}$$

By applying HPETM on eq.(6.7), we have

$$\sum_{n=0}^{\infty} p^n w_n = x + p\left(E^{-1}\left\{v^\alpha E\left\{1+x+t\right\} - \frac{1}{s^\alpha}E\left\{\sum_{n=0}^{\infty} p^n H_n(w)\right\}\right\}\right) \tag{6.8}$$

where

$$ww_x = \sum_{n=0}^{\infty} p^n H_n(w)$$

i.e.

$$\left(\sum_{n=0}^{\infty} p^n w_n\right)\left(\sum_{n=0}^{\infty} p^n w_n\right)_x = \sum_{n=0}^{\infty} p^n H_n(w)$$

The first few components of He's polynomial i.e. $H_n(w)$ are given as

$$\begin{aligned} H_0 &= w_0 w_{0x};\\ H_1 &= w_0 w_{1x} + w_1 w_{0x};\\ H_2 &= w_0 w_{2x} + w_1 w_{1x} + w_2 w_{0x},\\ &\vdots \end{aligned}$$

On comparing the like powers of p on both sides of (6.8), we have

$$\begin{aligned} p^0 &: w_o = x;\\ p^1 &: w_1 = (1+x)\frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - E^{-1}\left\{v^\alpha E\{H_0\}\right\}\\ &= \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)};\\ p^2 &: w_2 = -E^{-1}\left\{v^\alpha E\{H_1\}\right\}\\ &= -\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right);\\ p^3 &: w_3 = -E^{-1}\left\{v^\alpha E\{H_2\}\right\}\\ &= \left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right),\\ &\vdots \end{aligned}$$

Hence the solution of (6.7) is

$$w(x,t) = x + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right) + \left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right) + \cdots$$

or

$$w(x,t) = x + \left(\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right) + \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \cdots\right)$$

or

$$w(x,t) = x - \sum_{n=1}^{\infty}\frac{(-1)^n t^{n\alpha}}{\Gamma(n\alpha+1)} - t\sum_{n=1}^{\infty}\frac{(-1)^n t^{n\alpha}}{\Gamma(n\alpha+2)}$$

or

$$w(x,t) = x + 1 + t - E_{\alpha,1}(-t^\alpha) - tE_{\alpha,2}(-t^\alpha) \tag{6.9}$$

where $E_{\alpha,2}(-t^\alpha)$ in eq. (6.9) is Mittag-Leffler function defined in (2.5). When $\alpha = 1$, the exact solution of (6.7) is $w(x,t) = x + t$.

7 Homotopy perturbation transformation method (HPTM)

Now we present the solution of (4.1) using Laplace transformation,

$$\mathcal{L}\left\{\frac{\partial^\alpha}{\partial t^\alpha}w + Lw + Nw\right\} = \mathcal{L}\{f(x,t)\}.$$

Using (2.4), we have

$$\mathcal{L}\{w\} = \frac{1}{s^\alpha}\left(\sum_{k=0}^{n-1} s^{\alpha-k-1}w^{(k)}(x,0)\right) + \frac{1}{s^\alpha}\mathcal{L}\left\{f(x,t) - Lw - Nw\right\}.$$

$$\mathcal{L}\{w\} = \sum_{k=0}^{n-1} s^{-k-1}w^{(k)}(x,0) + \frac{1}{s^\alpha}\mathcal{L}\left\{f(x,t) - Lw - Nw\right\}.$$

Operating inverse Laplace transform , we get

$$w(x,t) = \sum_{k=0}^{n-1}\frac{t^k}{k!}w^{(k)}(x,0) + \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{f(x,t) - Lw - Nw\right\}\right\},$$

By applying HPM, we get

$$\begin{aligned} 0 =& (1-p)\left(w(x,t) - w(x,0)\right) + p\left(w(x,t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}w^{(k)}(x,0)\right.\\ &\left. - \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{f(x,t) - Lw - Nw\right\}\right\}\right), \end{aligned}$$

$$\begin{aligned} w(x,t) &= w(x,0)\\ &+ p\left(\sum_{k=1}^{n-1}\frac{t^k}{k!}w^{(k)}(x,0) + \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{f(x,t) - Lw - Nw\right\}\right\}\right) \end{aligned} \tag{7.1}$$

Let

$$\begin{aligned} w(x,t) &= \sum_{n=0}^{\infty} p^n w_n(x,t),\\ Nw(x,t) &= \sum_{n=0}^{\infty} p^n H_n(w(x,t)) \text{ and } w(x,0) = w_0(x,t) \end{aligned} \tag{7.2}$$

where

$$H_n(w(x,t)) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(\sum_{i=0}^{\infty} p^i w_i\right) \tag{7.3}$$

Substituting (7.2) and (7.3) in (7.1), we get

$$\begin{aligned} \sum_{n=0}^{\infty} p^n w_n(x,t) = w_0(x,t) + p\left(\sum_{k=1}^{n-1}\frac{t^k}{k!}w^{(k)}(x,0) + \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\right.\right.\\ \left.\left.\left\{f(x,t) - L\left(\sum_{n=0}^{\infty} p^n w_n(x,t)\right) - \sum_{n=0}^{\infty} p^n H_n(w(x,t))\right\}\right\}\right) \end{aligned} \tag{7.4}$$

On comparing the coefficients of the like powers of p, we get

$$\begin{aligned}
p^0 : w_0 =& w(x,0);\\
p^1 : w_1 =& \sum_{k=1}^{n-1} \frac{t^k}{k!} w^{(k)}(x,0)\\
&+ \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{f(x,t) - Lw_0 - H_0(w)\right\}\right\};\\
p^2 : w_2 =& -\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{Lw_1 + H_1(w)\right\}\right\};\\
p^3 : w_3 =& -\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{Lw_2 + H_2(w)\right\}\right\},\\
&\vdots
\end{aligned}$$

hence, the approximate solution is obtained as $p \to 1$

$$w(x,t) = w_0 + w_1 + w_2 + \dots.$$

8 Application

Example 8.1. Consider the time fractional nonlinear Fisher's equation[21]

$$\frac{\partial^\alpha w}{\partial t^\alpha} = \frac{\partial^2 w}{\partial x^2} + 6w(1-w),\ t > 0,\ x \in \mathbb{R},\ 0 < \alpha \le 1, \quad (8.1)$$

with initial condition $w(x,0) = \frac{1}{(1+e^x)^2}$.
By applying HPTM on (8.1), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} p^n w_n =& \frac{1}{(1+e^x)^2} + p\left(\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\left(\mathcal{L}\left\{\sum_{n=0}^{\infty}(p^n w_n)_{xx}\right.\right.\right.\\
&\left.\left.\left. + 6\left\{\sum_{n=0}^{\infty} p^n w_n - \sum_{n=0}^{\infty} p^n H_n(w)\right\}\right\}\right)\right\}\right) \quad (8.2)
\end{aligned}$$

where $H_n(w)$ are He's polynomial represents nonlinear terms. The first few components of He's polynomial are given by

$$\begin{aligned}
H_0(w) &= w_0^2;\\
H_1(w) &= 2w_0w_1;\\
H_2(w) &= w_1^2 + 2w_0w_2,\\
&\vdots
\end{aligned}$$

Comparing the like powers of p on both sides of (8.2), we have

$$\begin{aligned}
p^0 : w_0 =& \frac{1}{(1+e^x)^2};\\
p^1 : w_1 =& 10\frac{e^x t^\alpha}{(1+e^x)^3\Gamma(\alpha+1)};\\
p^2 : w_2 =& 50\frac{e^x(2e^x-1)t^{2\alpha}}{(1+e^x)^4\Gamma(2\alpha+1)};\\
p^3 : w_3 =& \left(50\frac{e^x(-16e^{3x}-15e^{2x}+30e^x+5)}{(1+e^x)^6}\right.\\
&\left.+600e^{2x}\frac{\Gamma(2\alpha+1)}{(1+e^x)^6\Gamma(\alpha+1)^2}\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\\
&\vdots
\end{aligned}$$

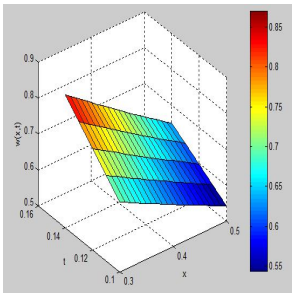

Fig. 1: Surface graph of $w(x,t)$ of eq. (8.1), when $\alpha = 0.6$

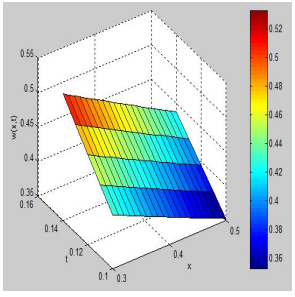

Fig. 2: Surface graph of $w(x,t)$ of eq. (8.1), when $\alpha = 0.8$

Hence, the solution is given as

$$\begin{aligned}
w(x,t) =& \frac{1}{(1+e^x)^2} + 10\frac{e^x t^\alpha}{(1+e^x)^3\Gamma(\alpha+1)}\\
&+ 50\frac{e^x(2e^x-1)t^{2\alpha}}{(1+e^x)^4\Gamma(2\alpha+1)}\\
&+ \left(50\frac{e^x(-16e^{3x}-15e^{2x}+30e^x+5)}{(1+e^x)^6}\right.\\
&\left.+600e^{2x}\frac{\Gamma(2\alpha+1)}{(1+e^x)^6\Gamma(\alpha+1)^2}\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots. \quad (8.3)
\end{aligned}$$

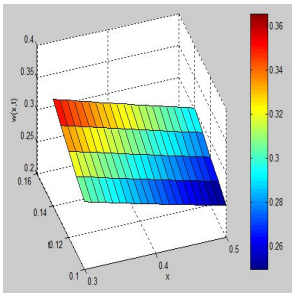

Fig. 3: Surface graph of $w(x, t)$ of eq. (8.1), when $\alpha = 1$

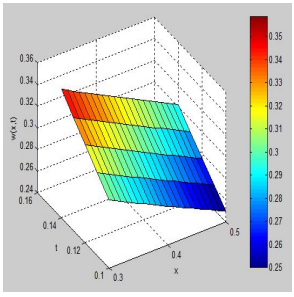

Fig. 4: Surface graph of $w(x, t)$ of eq. (8.1), when $\alpha = 1$(exact solution)

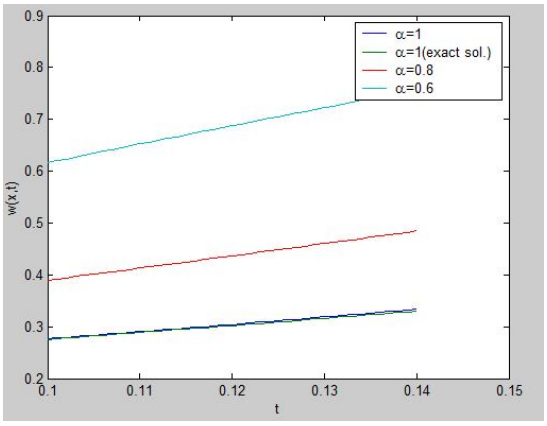

Fig. 5: Plot of of $w(x, t)$ of eq. (8.1) when $x = 0.4$ and $\alpha = 0.6, 0.8$ and 1

Example 8.2. Consider the time-fractional Fornberg-Whitham equation [26]

$$\frac{\partial^\alpha}{\partial t^\alpha} w(x,t) = \frac{\partial^3 w}{\partial x^2 \partial t} - \frac{\partial w}{\partial x} + w\frac{\partial^3 w}{\partial x^3} - w\frac{\partial w}{\partial x} + 3\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2},$$
$$t > 0, x \in \mathbb{R}, 0 < \alpha \le 1, \tag{8.4}$$

with initial condition $w(x, 0) = e^{\frac{x}{2}}$.
By applying HPTM on (8.4), we have

$$\sum_{n=0}^{\infty} p^n w_n = e^{\frac{x}{2}} + p\left(\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\left(\mathcal{L}\left\{\sum_{n=0}^{\infty} p^n (w_n)_{xxt} + \sum_{n=0}^{\infty} p^n (-w_n)_x + \left\{\sum_{n=0}^{\infty} p^n H_n(w)\right\}\right\}\right)\right\}\right) \tag{8.5}$$

where $H_n(w)$ are He's polynomial represents nonlinear terms. The first few components of He's polynomial are given by

$$\begin{aligned} H_0(w) &= w_0 w_{0xxx} - w_0 w_{0x} + 3w_{0x} w_{0xx}; \\ H_1(w) &= w_0 w_{1xxx} + w_1 w_{0xxx} - w_0 w_{1x} - w_1 w_{0x} \\ &\quad + 3w_{0x} w_{1xx} + 3w_{1x} w_{0xx}; \end{aligned}$$

$$H_2(w) = w_0 w_{2xxx} + w_1 w_{1xxx} + w_2 w_{0xxx} - w_0 w_{2x} - w_1 w_{1x} - w_2 w_{0x} + 3w_{2x} w_{0xx} + 3w_{1x} w_{1xx} + 3w_{0x} w_{2xx}.$$

Comparing the like powers of p on both sides of (8.5), we have

$$\begin{aligned} p^0 &: w_0 = e^{\frac{x}{2}}; \\ p^1 &: w_1 = \frac{-e^{\frac{x}{2}}}{2}\frac{t^\alpha}{\Gamma(\alpha+1)}; \\ p^2 &: w_2 = \frac{-e^{\frac{x}{2}}}{8}\frac{t^{2\alpha-1}}{\Gamma 2\alpha} + \frac{e^{\frac{x}{2}}}{4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}; \\ p^3 &: w_3 = e^{\frac{x}{2}}\left(\frac{-1}{32}\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} + \frac{1}{8}\frac{t^{3\alpha-1}}{\Gamma(3\alpha)} - \frac{1}{8}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right), \\ &\vdots \end{aligned}$$

Hence, the solution is given as

$$w(x,t) = e^{\frac{x}{2}} - \frac{e^{\frac{x}{2}}}{2}\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{e^{\frac{x}{2}}}{8}\frac{t^{2\alpha-1}}{\Gamma 2\alpha} + \frac{e^{\frac{x}{2}}}{4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + e^{\frac{x}{2}}\left(\frac{-1}{32}\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} + \frac{1}{8}\frac{t^{3\alpha-1}}{\Gamma(3\alpha)} - \frac{1}{8}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right) + \dots. \tag{8.6}$$

Example 8.3. Consider the nonlinear nonhomogeneous time fractional Inviscid Burgers' equation[36]

$$D_t^\alpha w + ww_x = 1 + x + t, w(x, 0) = x, 0 < \alpha \le 1 \tag{8.7}$$

By applying HPTM on eq.(8.7), we have

$$\sum_{n=0}^{\infty} p^n w_n = x + p\left(\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{1+x+t\right\} - \frac{1}{s^\alpha}\mathcal{L}\left\{\sum_{n=0}^{\infty} p^n H_n(w)\right\}\right\}\right) \tag{8.8}$$

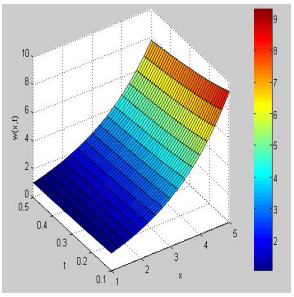

Fig. 6: Surface graph of $w(x, t)$ of eq. (8.4), when $\alpha = 0.6$

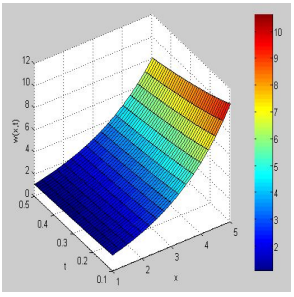

Fig. 7: Surface graph of $w(x, t)$ of eq. (8.4), when $\alpha = 0.8$

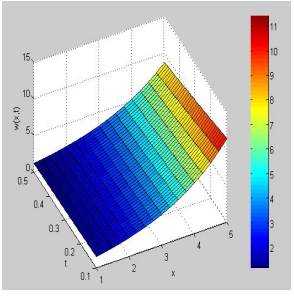

Fig. 8: Surface graph of $w(x, t)$ of eq. (8.4), when $\alpha = 1$

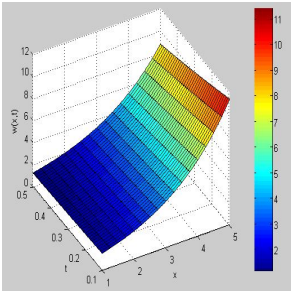

Fig. 9: Surface graph of $w(x, t)$ of eq. (8.4), when $\alpha = 1$(exact solution)

where

$$ww_x = \sum_{n=0}^{\infty} p^n H_n(w)$$

i.e.

$$\left(\sum_{n=0}^{\infty} p^n w_n\right)\left(\sum_{n=0}^{\infty} p^n w_n\right)_x = \sum_{n=0}^{\infty} p^n H_n(w)$$

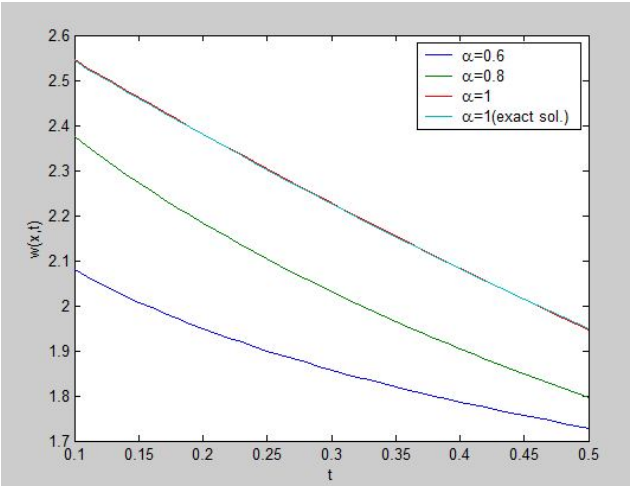

Fig. 10: Plot of of $w(x, t)$ of eq. (8.4) when $x = 2$ and $\alpha = 0.6, 0.8$ and 1

The first few components of He's polynomial i.e. $H_n(w)$ are given as

$$H_0 = w_0 w_{0x};$$
$$H_1 = w_0 w_{1x} + w_1 w_{0x};$$
$$H_2 = w_0 w_{2x} + w_1 w_{1x} + w_2 w_{0x},$$
$$\vdots$$

On comparing the like powers of p on both sides of (8.8), we have

$$p^0 : w_o = x;$$

$$p^1 : w_1 = (1+x)\frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\{H_0\}\right\}$$
$$= \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)};$$

$$p^2 : w_2 = -\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\{H_1\}\right\} = -\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right);$$

$$p^3 : w_3 = \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\{H_2\}\right\} = \left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right),$$

$$\vdots$$

Hence the solution of (8.7) is

$$w(x,t) = x + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right) + \left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right) + \dots$$

or

$$w(x,t) = x + \left(\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots\right) + \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \dots\right)$$

or

$$w(x,t) = x - \sum_{n=1}^{\infty} \frac{(-1)^n t^{n\alpha}}{\Gamma(n\alpha+1)} - t\sum_{n=1}^{\infty} \frac{(-1)^n t^{n\alpha}}{\Gamma(n\alpha+2)}.$$

or

$$w(x,t) = x + 1 + t - E_{\alpha,1}(-t^{\alpha}) - tE_{\alpha,2}(-t^{\alpha}).$$

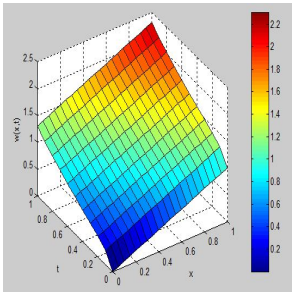

Fig. 11: Surface graph of $w(x,t)$ of eq. (8.7), when $\alpha = 0.6$

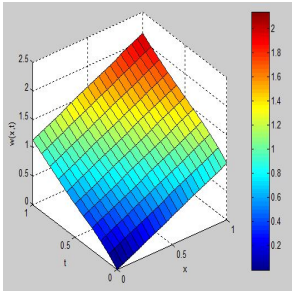

Fig. 12: Surface graph of $w(x,t)$ of eq. (8.7), when $\alpha = 0.8$

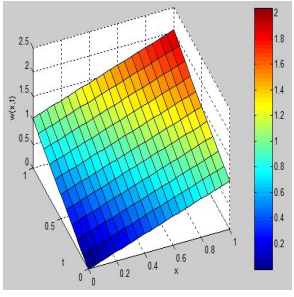

Fig. 13: Surface graph of $w(x,t)$ of eq. (8.7), when $\alpha = 1$

When $\alpha = 1$, the exact solution of (8.7) is $w(x,t) = x+t$.

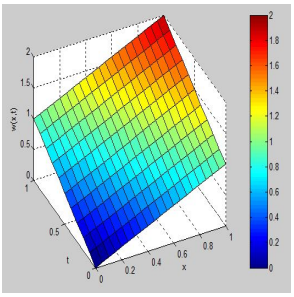

Fig. 14: Surface graph of $w(x,t)$ of eq. (8.7), when $\alpha = 1$(exact solution)

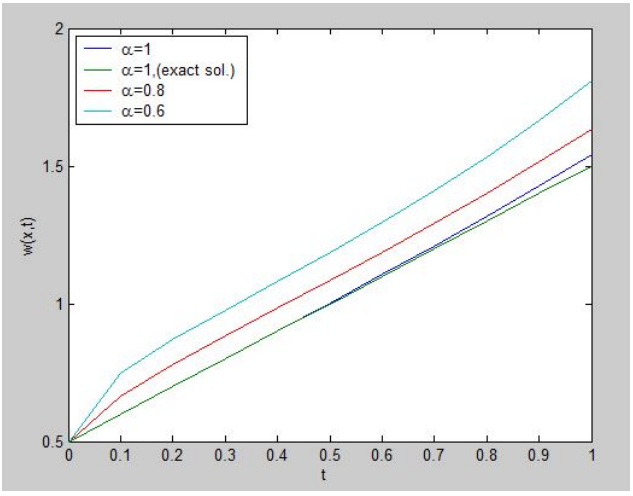

Fig. 15: Plot of of $w(x,t)$ of eq. (8.7) when $x = 0.5$ and $\alpha = 0.6, 0.8$ and 1

9 Analysis and conclusion

We Know that

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) = \bar{f}(s) \tag{9.1}$$

Also from (2.1) and (9.1), we have

$$E\{f(t)\} = F(v) = v\bar{f}\left(\frac{1}{v}\right) \tag{9.2}$$

or

$$\bar{f}\left(\frac{1}{v}\right) = \frac{1}{v}F(v)$$
$$\Rightarrow \bar{f}(s) = \frac{1}{v}F(v), \text{ where } v = \frac{1}{s}$$

Hence

$$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}. \tag{9.3}$$
$$\Rightarrow E\{t^n\} = v^{n+2}\Gamma(n+1), \text{ using(9.2).}$$

Table 1: Approximate solution of Fisher's equation (6.1) and (8.1) up to fourth order(when $\alpha = 1$)

x	t	w_{HPETM} (approx.)	w_{HPTM} (approx.)	w (exact sol.)	abs.error	$\|\|w_1\|\|$	$\|\|w_2\|\|$	$\|\|w_3\|\|$
	0.1	0.304691131	0.304691131	0.302317425	0.002373706	0.104031064	1.88E-02	7.49E-04
	0.11	0.319292625	0.319292625	0.316042418	0.003250207	0.11443417	2.28E-02	9.97E-04
0.3	0.12	0.334319781	0.334319781	0.329984205	0.004335576	0.124837277	2.71E-02	1.29E-03
	0.13	0.34977709	0.34977709	0.344120184	0.005656906	0.135240383	3.18E-02	1.65E-03
	0.14	0.365669045	0.365669045	0.358426914	0.007242131	0.145643489	3.69E-02	2.05E-03
	0.1	0.276611064	0.276611064	0.275603147	0.001007917	9.64E-02	1.92E-02	4.91507E-05
	0.11	0.29026645	0.29026645	0.288830839	0.001435611	0.106061562	2.32E-02	6.54196E-05
0.4	0.12	0.304302372	0.304302372	0.302317425	0.001984947	0.115703523	2.76E-02	8.49324E-05
	0.13	0.318718535	0.318718535	0.316042418	0.002676117	0.125345483	3.24E-02	0.000107984
	0.14	0.333514645	0.333514645	0.329984205	0.00353044	0.134987443	3.76E-02	0.00013487
	0.1	0.249765515	0.249765515	0.25	0.000234485	8.87E-02	1.92E-02	0.000734094
	0.11	0.262435106	0.262435106	0.262653581	0.000218475	9.76E-02	2.33E-02	0.00097708
0.5	0.12	0.275441031	0.275441031	0.275603147	0.000162116	0.10646815	2.77E-02	0.001268515
	0.13	0.288778885	0.288778885	0.288830839	5.19537E-05	0.115340496	3.25E-02	0.001612806
	0.14	0.302444264	0.302444264	0.302317425	0.000126839	0.124212842	3.77E-02	0.002014355

Table 2: Approximate solution of Fornberg-Whitham equation (6.4) and (8.4) up to fourth order (when $\alpha = 1$)

t	x	w_{HPETM} (approx.)	w_{HPTM} (approx.)	w (exact sol.)	abs.error	$\|\|w_1\|\|$	$\|\|w_2\|\|$	$\|\|w_3\|\|$
	1	1.543580941	1.543580941	1.542390265	1.19E-03	0.082436064	0.018548114	0.004156152
	2	2.544934731	2.544934731	2.542971638	1.96E-03	0.135914091	0.030580671	0.006852335
0.1	3	4.195888024	4.195888024	4.19265143	3.24E-03	0.224084454	0.050419002	0.011297591
	4	6.917849834	6.917849834	6.912513593	5.34E-03	0.369452805	0.083126881	0.018626579
	5	11.40560617	11.40560617	11.3968082	8.80E-03	0.609124698	0.137053057	0.030710037
	1	1.351024036	1.351024036	1.349858808	1.17E-03	0.247308191	0.043278933	0.00711011
	2	2.227462066	2.227462066	2.225540928	1.92E-03	0.407742274	0.071354898	0.01172259
0.3	3	3.672464088	3.672464088	3.669296668	3.17E-03	0.672253361	0.117644338	0.019327284
	4	6.054869657	6.054869657	6.049647464	5.22E-03	1.108358415	0.193962723	0.031865304
	5	9.982792395	9.982792395	9.974182455	8.61E-03	1.827374094	0.319790467	0.052537005
	1	1.180724868	1.180724868	1.181360413	6.36E-04	0.412180318	0.05152254	0.004293545
	2	1.946686205	1.946686205	1.947734041	1.05E-03	0.679570457	0.084946307	0.007078859
0.5	3	3.209542954	3.209542954	3.211270543	1.73E-03	1.120422268	0.140052783	0.011671065
	4	5.291641738	5.291641738	5.29449005	2.85E-03	1.847264025	0.230908003	0.019242334
	5	8.72444229	8.72444229	8.729138364	4.70E-03	3.04562349	0.380702936	0.031725245

Also using (9.2) in (2.4), we have

$$\mathcal{L}\left\{\frac{\partial^\alpha}{\partial t^\alpha}f(t)\right\} = s^\alpha \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0),\ n-1 < \alpha \le n,$$

$$\Rightarrow \frac{1}{v}E\{f^\alpha(t)\} = \frac{1}{v^\alpha}\frac{F(v)}{v} - \sum_{k=0}^{n-1}\left(\frac{1}{v}\right)^{\alpha-k-1} f^{(k)}(0),\ n-1 < \alpha \le n,$$

$$\Rightarrow E\{f^\alpha(t)\} = \frac{F(v)}{v^\alpha} - \sum_{k=0}^{n-1} v^{k-\alpha+2} f^{(k)}(0),\ n-1 < \alpha \le n.$$

In this work, we intend to study two semi-analytical techniques to solve nonlinear fractional partial differential equations: homotopy perturbation with Elzaki transform and homotopy perturbation transformation method.So, from above analysis, we conclude that the Elzaki transformation and its properties could be derive from Laplace transformation. This is the reason that either we use HPTM or HPETM, we come out with same series solution of nonlinear PDE or fractional PDE. Fig. 1-4, represent the surface graph of approximate solution of (8.1) for various estimations of α and the exact solution for $\alpha = 1$ and we find that approximate solution up to order 4 converges to exact solution for $\alpha = 1$, in Table 1, the condition of convergence is verified i.e. we analyse that $||w_1|| < ||w_2|| < ||w_3||$. Moreover, from Fig.5, we conclude that with the decrease in the value of α, the value of $w(x, t)$ increases. On the other

Table 3: Approximate solution of Inviscid Burger's Equation (6.7) and (8.7) up to fourth order($\alpha = 1$)

x	t	w_{HPETM} (approx.)	w_{HPTM} (approx.)	w (exact sol.)	abs.error	$\|\|w_1\|\|$	$\|\|w_2\|\|$	$\|\|w_3\|\|$
	0.25	0.50016276	0.50016276	0.5	0.00016276	0.28125	0.033854167	0.002766927
0.25	0.5	0.752604167	0.752604167	0.75	0.002604167	0.625	0.145833333	0.0234375
	0.75	1.013183594	1.013183594	1	0.013183594	1.03125	0.3515625	0.083496094
	1	1.291666667	1.291666667	1.25	0.041666667	1.5	0.666666667	0.208333333
	0.25	0.75016276	0.75016276	0.75	0.00016276	0.28125	0.033854167	0.002766927
0.5	0.5	1.002604167	1.002604167	1	0.002604167	0.625	0.145833333	0.0234375
	0.75	1.263183594	1.263183594	1.25	0.013183594	1.03125	0.3515625	0.083496094
	1	1.541666667	1.541666667	1.5	0.041666667	1.5	0.666666667	0.208333333
	0.25	1.00016276	1.00016276	1	0.00016276	0.28125	0.033854167	0.002766927
0.75	0.5	1.252604167	1.252604167	1.25	0.002604167	0.625	0.145833333	0.0234375
	0.75	1.513183594	1.513183594	1.5	0.013183594	1.03125	0.3515625	0.083496094
	1	1.791666667	1.791666667	1.75	0.041666667	1.5	0.666666667	0.208333333

hand, Fig. 6-9 and Fig. 11-14 represents the surface graph of (8.4) and (8.7) for various estimations of α and the exact solution for $\alpha = 1$, the approximate solution of $w(x, t)$ converges to exact solution when $\alpha = 1$, but by slightly decreasing the value of α, the value of $w(x, t)$ also decreases which is shown in the Fig. 10 and Fig.15 . We have applied both the techniques (i.e HPTM and HPETM) on nonlinear homogeneous and non homogenous fractional PDE and the outcome exhibit the efficiency, simplicity and high rate of accuracy of the suggested methodologies to solve this type of complex equation.

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