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A second order numerical method for a class of parameterized singular perturbation problems on adaptive grid

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Abstract: A nonlinear singularly perturbed boundary value problem depending on a parameter is considered. First, we solve the problem using the backward Euler finite difference scheme on an adaptive grid. The adaptive grid is a special nonuniform mesh generated through equidistribution principle by a positive monitor function depending on the solution. The behavior of the solution, the stability and the error estimates are discussed. Then, the Richardson extrapolation technique is applied to improve the accuracy of the computed solution associated to the backward Euler scheme. The proofs of the uniform convergence for the backward Euler scheme and the Richardson extrapolation are carried out. Numerical experiments validate the theoretical estimates and indicates that the estimates are sharp.

Keywords: Singular perturbation, layer adapted mesh, parameterized problems, boundary layer, Richardson extrapolation

1 Introduction

The study of singularly perturbed problems (SPPs) is exceptionally useful because they describe many problems of practical interest. These types of problems are identified by the differential equations in which a small parameter (ε) known as perturbation parameter multiplies the highest order derivative. It is well known that the classical numerical methods for solving SPPs are unstable and fail to give accurate results when 'ε' is very small, $0 < \varepsilon \ll 1$. Therefore, special approaches are required for obtaining a good approximation. So, it is important to develop such methods, whose accuracy do not depend on ε. The layer-adapted meshes: Shishkin type meshes (standard Shishkin mesh (S-mesh), Bakhvalov-Shishkin mesh (B-S-mesh)) and newly developed adaptive grid can be used to obtain parameter uniform numerical methods. In recent years, development of parameter uniform numerical methods based on layer adapted meshes have witnessed substantial progress. For more details on the singular perturbation and layer adapted meshes, one can refer the books ([7, 13, 19]) and the references therein.

Consider the following singularly perturbed Boundary Value Problem(BVP) depending on a parameter:

$$\begin{cases} \varepsilon u'(x) + f(x,u,\lambda) = 0, \quad x \in \Omega = (0,1], \\ u(0) = s_0, \quad u(1) = s_1, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$ and λ known as the control parameter. Here $s_0,\ s_1$ are the given constants. The function $f(x,u,\lambda)$ is assumed to be sufficiently smooth and satisfies the following bounds:

$$\begin{cases} f(x,u,\lambda) \in C^3([0,1]\times R^2), \\ 0 < \alpha \le \dfrac{\partial f}{\partial u} \le \alpha^* < \infty \qquad (x,u,\lambda) \in [0,1]\times R^2, \\ 0 < m \le \left|\dfrac{\partial f}{\partial \lambda}\right| \le M < \infty \qquad (x,u,\lambda) \in [0,1]\times R^2. \end{cases} \tag{1.2}$$

With these assumptions, the BVP (1.1) possesses a unique solution having a boundary layer of width $O(\varepsilon)$ near $x = 0$ (refer [1, 17, 18]). The parameter λ has no connection with the eigenvalue of the nonlinear differential equation. Since there are two unknowns, two boundary conditions are given in (1.1) to determine it exactly.

Parameterized BVP have been considered for many years. The existence and uniqueness of the solution for the BVP(1.1) were first considered by Pomentale [17]. Jankowski [8] and Liu [11] constructed monotonic iterative methods for these problems. But the above-mentioned articles were concerned with the regular case. In recent years, many researchers considered the singular perturbation cases for these problems. Amiraliyev et. al. [1] gave a uniform finite difference method on a standard Shishkin mesh [7] for BVP(1.1) and shown that the method is first order convergent up to a logarithmic factor *i.e.* of the order $O(N^{-1}\ln N)$. A hybrid difference scheme which combines

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upwind scheme on the fine mesh with the midpoint upwind scheme on the coarse mesh was considered by Cen [3]. Xie et. al. [23] used the boundary layer correction technique to solve the parameterized problem. Turkyilmazoglu [21] constructed a methodology based on the homotopy analysis technique to approximate the analytic solution. A uniform difference scheme was developed by Amiraliyeva et. al. [2] for parameterized delay differential equation. Recently, Das [6] provide a comparison of *apriori* and *posteriori* meshes for BVP (1.1).

Here, we have considered two types of numerical schemes: the backward Euler scheme and the post-processing Richardson extrapolation technique on the adaptive grid generated via equidistribution principle. Richardson extrapolation is a technique where two computed solutions are approximated by an average to provide a better approximation. This technique has the advantage that it can be extended to problems in more than one dimensions. Vulanovic et. al. [22] used this technique for singularly perturbed reaction-diffusion problem using Bakhvalov mesh. Its application to the numerical solution of singularly perturbed convection-diffusion problem was examined in [16] by Natividad and Stynes. Using this idea, Mohapatra and Natesan [14] constructed a second order post processing technique for solving singularly perturbed delay differential equation. In this work, we have solved the BVP (1.1) by backward Euler difference scheme on an adaptive grid. Then, we have successfully applied the Richardson extrapolation technique on computed solution to enhance the accuracy from first order to second order.

This paper is organized as follows: In section 3, we describe the numerical scheme for the problem (1.1). In section 4, a brief description of the nonuniform mesh is presented. The error estimates for the approximate solution are obtained for the proposed schemes in section 5. Finally, in section 6, we present two numerical examples, which validate the theoretical estimates. Throughout this paper 'C' denotes a generic positive constant independent of both ε and N which can take different values at different places and subscripted C's are fixed constants. Here, we denote $g(x_i) = g_i$.

2 Analytical results

Lemma 2.1. *The BVP (1.1) has a unique solution* $\{u(x), \lambda\} \in C^1([0,1] \times \mathbb{R})$.

Proof. Following the proof of Theorem 2.1 in [23], one can prove the above Lemma. □

Lemma 2.2. *The solution $\{u(x), \lambda\}$ of (1.1) satisfies the following the inequalities:*

$$|\lambda| \le C, \quad |u^k(x)| \le C\left\{1 + \varepsilon^{-k} \exp\left(-\frac{\alpha x}{\varepsilon}\right)\right\},$$
$$x \in \overline{\Omega}, \quad k = 0, 1, 2, 3,$$

Proof. The proof for $k = 0, 1$ is given in Lemma 1 of [1], same argument can be used to prove for $k = 2, 3$. □

3 Numerical schemes

In this section, we describe a difference scheme for BVP (1.1). On a arbitrary nonuniform mesh $\Omega^N : 0 < x_0 < x_1 < \ldots < x_N$, define the operator:

$$D^- U_i^N = \frac{U_i^N - U_{i-1}^N}{h_i},$$

where $h_i = x_i - x_{i-1}$. The BVP (1.1) is discretized by the following difference scheme:

$$\begin{cases} T^N U_i^N \equiv \varepsilon D^- U_i^N + f(x_i, U_i^N, \lambda^N) = 0\,, \quad 1 \le i \le N-1. \\ U_0^N = s_0, \quad U_N^N = s_1, \end{cases} \tag{3.1}$$

In order to increase the accuracy of the difference scheme (3.1), we follow the idea described in [16]. We construct a new nonuniform mesh $\Omega^{2N} : 0 < \widetilde{x}_0 < \widetilde{x}_1 < \ldots < \widetilde{x}_{2N}$ and $\widetilde{h}_i = \widetilde{x}_{i-1} - \widetilde{x}_i$, which is generated by bisecting each interval of the original mesh Ω^N. Now, the difference scheme on the mesh Ω^{2N} is defined as:

$$\begin{cases} \widetilde{T}^{2N} \overline{U}_i^{2N} \equiv \varepsilon D^- \overline{U}_i^{2N} + f(x_i, \overline{U}_i^{2N}, \lambda^{2N}) = 0\,,\ 1 \le i \le 2N-1, \\ \widetilde{U}_0^{2N} = s_0, \quad \widetilde{U}_2^{2N} N = s_1. \end{cases} \tag{3.2}$$

We are interested in the extrapolated solution defined on the mesh Ω^N by $U_{extp,i}^N = 2\overline{U}_{2i}^{2N} - U_i^N$, for $i = 0, 1, \ldots, N$; which we expect will improve the accuracy of the approximation obtained at the points $x_i \in \Omega^N$. Also, It will enhance the order of the convergence of the scheme.

4 The nonuniform mesh

Numerical methods using the standard finite difference schemes on uniform meshes are inadequate for solving SPPs [7, 13]. Special approaches are required to dealt with these problems. Layer-adapted meshes are of great interest, not only because they capture the layer behaviour, but also because they can result in numerical approximation whose accuracy is guaranteed independent of the width of the layer. These meshes can be divided into two categories: a priori mesh, for which a prior information about the location and the width of the solution is required and a posteriori mesh, for which we do not need such information. In past few years, different approaches are developed to get uniform convergence where the mesh is chosen apriori. Very few literature are available for the posteriori meshes. One of such approach is the adaptive grid. In [10], an adaptive grid based numerical method is developed for solving a quasi-linear one-dimensional convection-diffusion problem. Mackenzie [12] constructed the uniform convergent upwind method for a convection-diffusion problem on an adaptive grid generated via equidistribution principle. Chen [4] provided uniform convergence of finite difference approximation for SPP on an adaptive grid. Mohapatra and Natesan [15] proved the uniform convergence of upwind scheme for approximating the global solution and the global normalized flux using the grid equidistribution. Here, our aim is to construct efficient numerical method on an adaptively generated posteriori mesh.

Adaptive grid is one of the special kind of nonuniform mesh. A commonly-used technique to generate the adaptive grid is based on equidistribution of an arbitrary non-negative function $M(u(x), x)$ defined on $[0, 1]$ known as the monitor function. A grid Ω^N is said to be equidistributed if

$$\int_{x_{j-1}}^{x_j} M\big(u(s), s\big)ds = \int_{x_j}^{x_{j+1}} M\big(u(s), s\big)ds, \quad j = 1, 2, \ldots, N-1. \tag{4.1}$$

The monitor functions are usually depend on the gradient of the solution. The solution of (4.1) along with a discretized version of the BVP (1.1) produces the numerical approximation to the solution of the BVP (1.1).

We shall follow the similar mesh generation algorithm for solving the parameterized BVP (1.1). The following adaptive algorithm is used to generate the appropriate nonuniform mesh (One may refer [4, 10, 12] for more details on the adaptive algorithm).

Adaptive mesh generation algorithm

Step 1: Consider the initial mesh $\{0, 1/N, 2/N, \ldots 1\}$ as uniform.

Step 2: Solve the discrete problem for $k = 0, 1, \ldots$ on the mesh $\{x_i^k\}$ to compute the solution $\{U_i^{N,(k)}\}$.

Step 3: Define $D^2 = D^+D^-$, where $D^+U_i^N = \frac{U_{i+1}^N - U_i^N}{x_{i+1}-x_i}$ and $D^-U_i^N = \frac{U_i^N - U_{i-1}^N}{x_i - x_{i-1}}$. Find the discretized the monitor function as

$$M_i^k = [1 + |\overline{D}^2 U_i^{N,(k)}|^{1/2}] \text{ for } i = 1, \cdots, N,$$

by defining $\overline{D}^2 U_i^N = \left(D^2U_i^N + D^2U_{i-1}^N\right)/2$ with $\overline{D}^2 U_1^N = D^2U_1^N$ and $\overline{D}^2 U_N^N = D^2U_{N-1}^N$. Compute

$$l_j^k = \sum_{i=1}^{j} h_i^k M_i^k,$$

where $h_i = x_{i+1} - x_i$ for $i = 1, 2, \cdots, N-1$.

Step 4: Let C_0 be a user chosen constant with $C_0 > 1$. Now if, $\dfrac{\max_{i=1,\cdots,N} h_i^k M_i^k}{l_N^k} \le \dfrac{C_0}{N}$, then go to Step 6, otherwise continue to Step 5.

Step 5: Generate a new mesh by equidistributing the monitor function using the current computed solution from Step 2 and M_i^k from Step 3. Set $U_i^{N,(k)} = il_N^k/N$ for $i = 0, \cdots, N$. Now interpolate $(x_i^{k+1}, U^{N,(k)}(x_i^k))$ to (x_i^k, l_i^k) using the piecewise linear interpolation. Generate a new mesh $x_i^{(k+1)} = 0 = x_0^{(k+1)} < x_1^{(k+1)} < \ldots < x_N^{(k+1)} = 1$ and return to Step 2.

Step 6: Set $x_0, x_1, \ldots, x_N = x_i^{(k+1)}$ as the final desired nonuniform mesh and $U = U^{N,(k+1)}$. **Stop**.

5 Error estimates

In this section, we wish to establish the uniform convergence of the extrapolation scheme on the layer-adapted mesh. To investigate the error function of the method, the error function is denoted as $z_i^N = U_i^N - u_i,\ 0 \le i \le N,\ \mu^N = \lambda^N - \lambda$ which is the solution of the following discrete problem:

$$\begin{cases} \varepsilon \dfrac{z_i^N - z_{i-1}^N}{h_i} + f(x_i, U_i^N, \lambda^N) - f(x_i, u_i, \lambda) = \varepsilon(u_i' - D^-U_i^N), \\ z_0^N = z_N^N = 0. \end{cases} \tag{5.1}$$

For, $i = 1, 2, \ldots, N$, we use the Taylor's expansion for f around (x_i, u_i, λ) which gives,

$$L^N z_i^N \equiv \varepsilon \frac{z_i^N - z_{i-1}^N}{h_i} + a_i z_i^N = b_i \mu^N + R_i, \tag{5.2}$$

where,

$$\begin{cases} a_i = \dfrac{\partial}{\partial u} f(x_i, u_i + \sigma z_i, \lambda + \sigma\mu^N), \\ b_i = \dfrac{\partial}{\partial \lambda} f(x_i, u_i + \sigma z_i, \lambda + \sigma\mu^N), \quad 0 < \sigma < 1, \\ R_i = \varepsilon(u_i' - D^-U_i^N). \end{cases} \tag{5.3}$$

It is easy to verify that the matrix associated with L^N is an M-matrix. Hence, we can use the discrete maximum principle, which states "If $\{v_i\}_0^N$ and $\{w_i\}_0^N$ are the mesh functions that satisfy $v_0 \geq w_0$ and $v_N \geq w_N$ and $L^N v_i \geq L^N w_i$ for $i = 0, \ldots, N-1$ then $v_i \geq w_i$ for all i." Here we use the discrete norms $\|v\|_{\infty,\Omega} = \max_i |v_i|$ and $\|v\|_{\star,\Omega} = \max_{i=1,\ldots,N} |\sum_{j=i}^N h_j v_j|$.

Lemma 5.1. *For the pair $\{z_i^N, \mu^N\}$, the following estimates hold:*

$$\begin{cases} |\mu^N| \leq m^{-1}\|R\|_{\infty,\Omega}, \\ \|z^N\|_{\infty,\Omega} \leq \alpha^{-1}(1 + m^{-1}M)\|R\|_{\infty,\Omega}. \end{cases} \tag{5.4}$$

Proof. One can refer Lemma 4.2 of [1] for the proof. □

The next theorem gives the error estimate of the solution before extrapolation.

Theorem 5.2. *Let $\{u(x), \lambda\}$ and $\{U_i^N, \lambda^N\}$ be the exact solution and discrete solution on adaptive grid respectively. Then, there exists a constant C independent of N and ε such that*

$$\|u - U^N\|_{\infty,\Omega} < CN^{-1}. \tag{5.5}$$

Proof. From (5.3) for R_i, we use Taylor's series expansion for $u(x)$ around x_i to obtain

$$\begin{aligned} |R_i| &\leq \varepsilon|u_i' - D^- U_i^N| \leq \varepsilon \int_{x_{i-1}}^{x_i} |u''(x)|dx, \\ &\leq \int_{x_{i-1}}^{x_i} (\varepsilon + \varepsilon^{-1}\exp(-\alpha x/\varepsilon))dx, \\ &\leq C\left(\varepsilon h_i + \varepsilon^{-1}\int_{x_{i-1}}^{x_i} \exp(-\alpha x/\varepsilon)dx\right), \\ &\leq C\left(\varepsilon N^{-1} + \int_{x_{i-1}}^{x_i} (1 + |u''(x)|^{1/2})dx\right), \\ &\leq C\left(\varepsilon N^{-1} + \int_{x_{i-1}}^{x_i} M(u(x), x)dx\right), \\ &\leq C\left(\varepsilon N^{-1} + \frac{1}{N}\int_0^1 M(u(x), x)dx\right) \leq CN^{-1}. \end{aligned}$$

Combining the above with (5.4), we get the desired bound. □

Now, we will prove the estimate on the adaptive grid after extrapolation. Equation (1.1) can be written as:

$$\mathfrak{L}u = \varepsilon u' + a(x)u = b(x)\lambda + F(x), \tag{5.6}$$

where,

$$\begin{cases} a(x) = \dfrac{\partial f}{\partial u}(x, u, \lambda), \\ b(x) = -\dfrac{\partial f}{\partial \lambda}(x, u, \lambda), \\ F(x) = f(x, 0, 0). \end{cases} \tag{5.7}$$

So, the backward Euler scheme for (5.6) is given as:

$$L^N U_i^N = \varepsilon D^- U_i^N + a_i U_i^N = b_i \lambda^N + F_i. \tag{5.8}$$

The operator L^N enjoys the following stability property:

$$\|v\|_{\infty,\Omega} \leq C\|L^N v\|_{\star,\Omega} \text{ for all } v \in R_0^{N+1}. \tag{5.9}$$

Lemma 5.3. *Let ζ be the solution of the following parameterized BVP:*

$$\mathfrak{L}\xi = b(x)\lambda + F(x) \text{ where } \zeta' = b(x)\lambda + F(x), \ x \in (0,1),$$
$$\xi(0) = 0, \ \xi_1 = 0, \tag{5.10}$$

where ζ is piecewise continuously differentiable, then

$$\sum_{k=i}^{N-1} h_k[L^N\xi]_k = \sum_{k=i}^{N-1} h_k(a\xi)_k - \int_{x_i}^1 (a\xi)dx + \int_{x_i}^1 (F(x) + \lambda b(x))dx \tag{5.11}$$

Proof. We know

$$\sum_{k=i}^{N-1}[L^N\xi]_k = \varepsilon[\xi]_N - \varepsilon[\xi]_i + \sum_{k=i}^{k=N} h_k(a\xi)_k. \tag{5.12}$$

Integrating $\mathfrak{L}\xi = b(x)\lambda + F(x)$ over (x_i, x_N), we get

$$\varepsilon([\xi]_N - [\xi]_i) = \int_{x_i}^1 (F(x) + \lambda b(x))dx - \int_{x_i}^1 (a\xi)dx. \tag{5.13}$$

Combining (5.12) and (5.13), we get (5.11). □

From Lemma 5.3, the error $L^N(u - U^N)$ of the scheme is given by:

$$\sum_{k=i}^{k=N-1} h_k[L^N(u - U^N)]_k \leq \sum_{k=i}^{k=N-1} h_k g_k - \int_{x_i}^1 g(x)dx + C_1|\lambda - \lambda^N|, \tag{5.14}$$

where $g(x) = (F(x) - a(x)u(x))$. Thus,

$$\begin{aligned} \|L^N(u - U^N)\|_{\infty,\Omega} \leq& \varepsilon \max_{i=1,\ldots N} |[D^- U^N]_i - u_i'| \\ &+ \max_{i=0,\ldots,N}\left|\int_{x_i}^1 g(x)dx - \sum_{k=i}^N h_k g_k\right|. \end{aligned} \tag{5.15}$$

Both terms in RHS of (5.15) are of first order.

Let ψ, the leading term of the error expansion be the solution of the following problem:

$$\mathfrak{L}\psi(x) = \Psi', \ \ \psi(0) = \psi(1) = 0,$$

$$\Psi(x) = \varepsilon\frac{h(x)}{2}u''(x) - \int_x^1 h(s)g(s)ds. \qquad (5.16)$$

Then by Lemma 5.3, we have

$$\sum_{k=i}^{k=N-1}[L^N(u-\psi-U^N)]_k = \varepsilon([D^-U^N]_i - u_i' + \frac{h_i}{2}u_i'') - \varepsilon\frac{h_N}{2}u_N''$$
$$+\int_{x_i}^1 (g(x)-h(x)g'(x))dx - \sum_{k=i}^{N-1}h_k g_k. \qquad (5.17)$$

Now, consider the first term in RHS of (5.17). Differentiating (5.6) twice, we get $\varepsilon u'' + (a(x)u)' = b'(x)\lambda + F'(x)$ and $\varepsilon u''' = b''(x)\lambda + F''(x) - (a(x)u)''$ which implies that $|\varepsilon u'''| \le C(1+|u''|)$. So, we have

$$\begin{aligned}\varepsilon([D^-U^N])_i - u_i' + \frac{h_i}{2}u_i'' &= \frac{\varepsilon}{2h_i}\left|\int_{x_{i-1}}^{x_i}(x-x_{i-1})^2u'''(x)dx\right|,\\ &\le \frac{C}{2h_i}\int_{x_{i-1}}^{x_i}(x-x_{i-1})^2(1+|u''(x)|)dx,\\ &\le C\max_{[x_{i-1}x_i]} h_i^2[1+|u''(x)|],\\ &\le C\max_i\max_{[x_{i-1}x_i]} h_i^2[1+|u''(x)|^{1/2}]^2.\end{aligned} \qquad (5.18)$$

From Taylor's expansion, we obtain

$$g(s) = g_{i-1} + (s-x_{i-1})g'(s) - \int_{x_{i-1}}^{s}(x-x_{i-1})g''(x)dx. \qquad (5.19)$$

So, we can deduce

$$\begin{aligned}\int_{x_i}^{x_{i-1}}(g(x)-(x-x_{i-1})g'(x))dx - h_i g_{i-1} &= \int_{x_{i-1}}^{x_i}\int_{x_{i-1}}^{s}g''(x)dxds\\ &\le Ch_i^3.\end{aligned} \qquad (5.20)$$

Thus,

$$\begin{aligned}&\left|\int_{x_i}^{1}(g(x)-h(x)g'(x))dx - \sum_{k=i+1}^{N}h_k g_{k-1}\right|\\ &\le \sum_{k=i+1}^{N}\left|\int_{x_{k-1}}^{x_k}(g(x)-(x-x_{k-1})g'(x)dx\right|,\\ &\le C\sum_{k=i+1}^{N}h_k^3 \le C\max_k h_k^2\sum_{k=i+1}^{N}h_k \le C\max_i h_i^2,\\ &\le C\max_i\max_{[x_{i-1}x_i]} h_i^2[1+u''(x)^{1/2}]^2.\end{aligned} \qquad (5.21)$$

Combining the above estimates, we have

$$\|L^N(u-\psi-U^N)\|_{\star,\Omega^N} \le C\max_i\max_{[x_{i-1}x_i]} h_i^2[1+u''(x)^{1/2}]^2. \qquad (5.22)$$

Thus, from the stability property,

$$\|(u-\psi-U^N)\|_{\infty,\Omega^N} \le C\max_i\max_{[x_{i-1}x_i]} h_i^2[1+|u''(x)|^{1/2}]^2. \qquad (5.23)$$

In order to provide idea of extrapolation, consider the discrete problem for $i = 1, 2, \ldots 2N-1$:

$$\overline{L}^N\overline{U}^{2N} = \varepsilon D^-\overline{U}_i^{2N} + \overline{a}_i\overline{U}_i^{2N} = \overline{b}_i\lambda^{2N} + \overline{F}_i \qquad (5.24)$$

Following the similar procedure as we follow for (5.6) we can prove that

$$\|(u-\frac{\psi}{2}-\overline{U}^{2N})\|_{\infty,\Omega^N} \le C\max_i\max_{[x_{i-1}x_i]} h_i^2[1+|u''(x)|^{1/2}]^2. \qquad (5.25)$$

Then using the triangle inequality and combining (5.23) with (5.25), we get

$$||u-U_{extp}^N||_{\infty,\Omega^N} \le C\max_i\max_{[x_{i-1}x_i]} h_i^2[1+|u''(x)|^{1/2}]^2. \qquad (5.26)$$

Finally on the adaptive grid, we can state the following convergence result for the Richardson extrapolation technique.

Theorem 5.4. *Let $\{u(x),\lambda\}$ and $\{U_{extp}^N,\lambda^N\}$ be the exact solution and discrete solution obtained by the Richardson extrapolation technique on the adaptive grid (defined in Section 4) respectively. Then, there exists a constant C such that*

$$\|u-U_{extp}^N\|_{\infty,\Omega^N} \le CN^{-2}.$$

Proof. From (5.26) we have,

$$\|u-U_{extp}^N\|_{\infty,\Omega^N} \le C\max_i\max_{[x_{i-1}x_i]} h_i^2[1+|u''(x)|^{1/2}]^2.$$

From [5] we know that if the mesh is generated by the adaptive algorithm, then $h_i \le CN^{-1}$. Applying the bound of the derivatives of the solution in (5.26), we can prove the desired estimates. □

6 Numerical Results

We consider two test problems to show the applicability and efficiency of the proposed methods.

Example 6.1. *Consider the singularly perturbed problem*

$$\begin{cases}\varepsilon u'(x) + 2u - \exp(-u) + \lambda = 0, & x \in \Omega = (0,1),\\ u(0) = 0, \quad u(1) = 1.\end{cases} \qquad (6.1)$$

Table 1: $E^N_{\varepsilon,u}$ and the corresponding $r^N_{\varepsilon,u}$ for Example 6.1 on the adaptive grid.

ε		Number of intervals N						
		16	32	64	128	256	512	1024
$1e-4$	before	2.577e-2	1.238e-2	6.093e-3	3.020e-3	1.502e-3	7.503e-4	3.749e-4
	rate	1.059	1.023	1.013	1.008	1.001	1.001	
	after	2.370e-3	5.182e-4	1.230e-4	2.998e-5	7.405e-6	1.974e-6	5.921e-7
	rate	2.193	2.075	2.037	2.018	1.908	1.737	
$1e-8$	before	2.796e-2	1.332e-2	6.188e-3	3.033e-3	1.504e-3	7.506e-4	3.742e-4
	rate	1.069	1.106	1.029	1.012	1.003	1.004	
	after	2.776e-3	5.849e-4	1.269e-4	3.023e-5	7.678e-6	1.937e-6	4.947e-7
	rate	2.246	2.204	2.070	1.977	1.987	1.969	

The exact solution is not available for Example 6.1. In order to calculate the maximum point-wise error $E^N_{\varepsilon,u}$ and the rate of convergence $r^N_{\varepsilon,u}$, we use interpolation. Define $\widetilde{U}^{2N}_i$ and $\widetilde{U}^{2N}_{extp,\,i}$ as the piecewise linear interpolation to U^N_i and $U^N_{extp,\,i}$ respectively in Ω^N. For any value of N, maximum pointwise error $E^N_{\varepsilon,u}$ with respect to the variable u of the numerical solution before and after extrapolation will be calculated by $\max_i |U^N_i - \widetilde{U}^{2N}_i|$ and $\max_i |U^N_{extp,\,i} - \widetilde{U}^{2N}_{extp,\,i}|$ respectively.

Example 6.2. *Consider the following singularly perturbed problem:*

$$\begin{cases} \varepsilon u'(x) + u - \exp(-u) + (\lambda + x)\exp(-1/\varepsilon) \\ \quad + \exp(x\exp(-1/\varepsilon) - \exp(-x/\varepsilon)) + \exp(\lambda) + \lambda - 1 = 0, \\ \qquad x \in \Omega = (0,1), \\ u(0) = 1, \quad u(1) = 0. \end{cases} \tag{6.2}$$

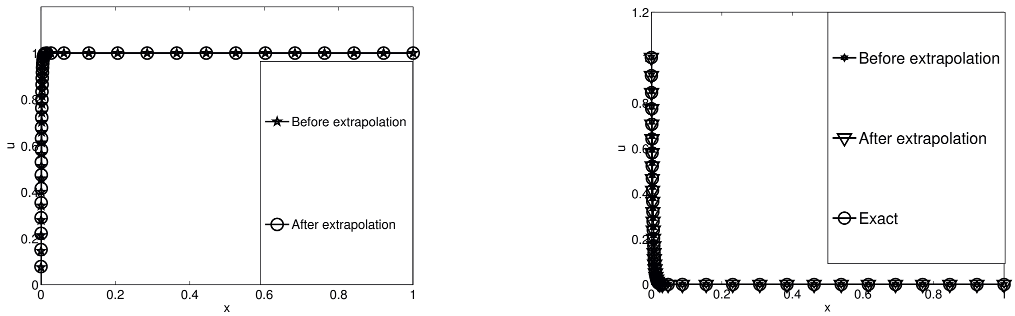
The above problem has exact solution $u(x) = \exp(-x/\varepsilon) - x\exp(-1/\varepsilon)$ whenever λ satisfies $(\lambda-\varepsilon)\exp(-1/\varepsilon)+\exp(\lambda)-1 = 0$

Let $u(x)$ be the exact solution. Let the numerical solution before and after extrapolation be U^N_i and $U^N_{extp,\,i}$ respectively, then the $E^N_{\varepsilon,u}$ with respect to the variable u of the numerical solution before and after extrapolation is calculated as: $\max_i |u_i - U^N_i|$ and $\max_i |u_i - U^N_{extp,i}|$ respectively. The corresponding rate of convergence is calculated as $r^N_{\varepsilon,u} = \log_2\left(\frac{E^N_{\varepsilon,u}}{E^{2N}_{\varepsilon,u}}\right)$.

Figure 1(a) and Figure 1(b) represent the numerical solution before and after extrapolation for Example 6.1 and Example 6.2 respectively for $N = 40$ and $\varepsilon = 2^{-8}$. The error behaviour of the scheme for Example 6.1 and 6.2 for $N = 20$ and $\varepsilon = 10^{-2}$ are displayed in the Figure 2. One can clearly observe that the error in the layer region as well as in the outer region is less after extrapolation. In Figure 3(a) and Figure 3(b), the maximum pointwise error versus the number of mesh interval is plotted in the logarithmic scale for $\varepsilon = 10^{-8}$. These figures show the effectiveness and accuracy of the extrapolation scheme. In Table 1, we represent the maximum pointwise error and the corresponding rate of convergence for Example 6.1 with $\varepsilon = 10^{-4}$, 10^{-8} which is enough to show the singularly perturbed nature of the problem. The rapid decrease in the error after extrapolation can be observed from these results. Moreover, improvement in the rate of convergence from first order to second order is also evident as claimed by the theoretical finding. The proposed improvement is compared with results on Shishkin type meshes in Table 2 for Example 6.2 with $\varepsilon = 10^{-8}$. The numerical approximation on the B-S-mesh also results in first-order convergence before extrapolation and second order convergence after extrapolation. But the advantage of the adaptive grid is that it does not need any *apriori* information about the location and width of the layer. The numerical results are the clear illustration of the error estimates.

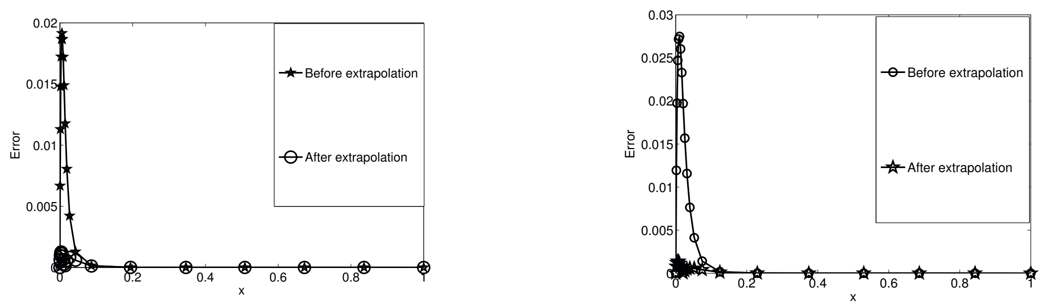
7 Concluding remarks

In this work, a higher order accurate method is used to compute the solution of a class of parameterized nonlinear singularly perturbed differential equation. First, we use the backward Euler difference scheme on the adaptive grid, then Richardson extrapolation technique is used on the computed solution. Clearly, after extrapolation the error of the numerical solution is considerably decreased. Theoretical estimates are supported with the help of numerical results. The advantage of the adaptive grid over Shishkin type meshes is evident from the theoretical es-



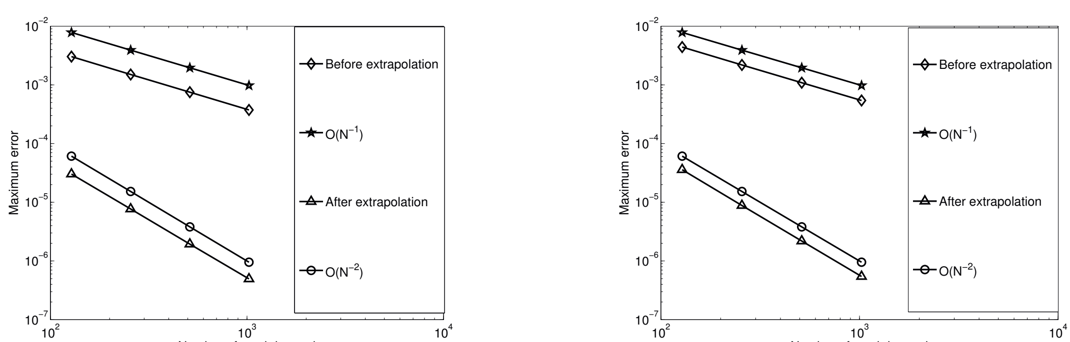
(a) For Example 1. **(b)** For Example 2.

Fig. 1: Numerical solution with $N = 40$ and $\varepsilon = 2^{-8}$.



(a) For Example 1. **(b)** For Example 2.

Fig. 2: Error behaviour with $N = 20$ and $\varepsilon = 10^{-2}$.



(a) For Example 1. **(b)** For Example 2.

Fig. 3: Loglog plot of maximum point-wise errors with $\varepsilon = 10^{-8}$.

timates obtained as well as from the numerical results shown.

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Table 2: $E^N_{\varepsilon,u}$ and the corresponding $r^N_{\varepsilon,u}$ for Example 6.2.

N		S-mesh	B-S-mesh	Adaptive grid
	before	1.488e-2	1.545e-2	1.859e-2
32	rate	0.705	0.932	0.959
	after	4.643e-4	3.850e-4	6.346e-4
	rate	1.403	1.769	1.998
	before	5.399e-3	4.141e-3	4.411e-3
128	rate	0.794	0.983	1.014
	after	6.154e-5	4.647e-5	3.590e-5
	rate	1.586	1.968	2.028
	before	1.759e-3	1.053e-3	1.087e-3
512	rate	0.843	0.995	1.003
	after	6.557e-6	4.181e-6	2.185e-6
	rate	1.686	1.986	2.007

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