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Existence and Stability of Difference Equation in Imprecise Environment

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Abstract: In this paper, first order linear homogeneous difference equation is evaluated in fuzzy environment. Difference equations become more notable when it is studied in conjunction with fuzzy theory. Hence, here amelioration of these equations is demonstrated by three different tactics of incorporating fuzzy numbers. Subsequently, the existence and stability analysis of the attained solutions of fuzzy difference equations (FDEs) are then discussed under different cases of impreciseness. In addition, considering triangular and generalized triangular fuzzy numbers, numerical experiments are illustrated and efficient solutions are depicted, graphically and in tabular form.

Keywords: Fuzzy number; generalized fuzzy number; Fuzzy difference equation; Fuzzy stability

1 Introduction

The theory of difference equations has been developed greatly during last three decays. As it is well-known that it appears naturally as discrete analogous having many applications in population dynamics, sociology, physics, economics, engineering and many others. Recently, the study of the qualitative behavior of difference equation and system of difference equation is undergoing to a great extent [1–10]. In general, difference equations specify the change in a variable between two periods, therefore on using these equations one can study the concerning factors that cause a change in the value of functions in different time periods.

Virtuously, introduction of fuzzy theory by Chang et al. [11] have initiated a novel perception for different fields of science in modeling different real-world phenomena with a better adequacy. Since then, this theory has become the center of attention of many researchers and has been practiced, to cope the impreciseness of the models. Many experts have widely practiced difference equations with fuzzy concepts in order to make some problems under study more comprehensible. In this regard, there exists several research papers where difference equation is solved with fuzzy numbers. Khastan [12] illustrated the solvability of different first order linear fuzzy difference equations and studied the stability and boundedness of the multiple solutions of these equations. Allahviranloo et al. [13] presented an application of fuzzy difference equations in predicting a specific cardiovascular disturbance. Papaschinopoulos et al. [14] gave detailed description on the asymptotic behavior of the solution and boundness of some different type of fuzzy difference equations. Umekkan et al. [15] showed the application of fuzzy difference equation in finance. Stefanidou et al. [16] considered exponential type fuzzy difference equation and investigated the existence of positive solutions and nonnegative equilibrium points and many others [17–19].

In spite of the above-mentioned developments, few analyses are still there that are to be accomplished. In this manuscript, we have explored difference equation in fuzzy environment by considering initial condition and coefficient as fuzzy numbers, consecutively. The stability conditions and existence of solutions under each assumption for FDE is achieved here. The assessments are made for triangular and generalized triangular fuzzy numbers. Further, the remaining structure of sequel is as follows: Section 2 covers the preliminary concepts of fuzzy set theory. Difference equation and its stability conditions are elucidated in Section 3. In Section 4, FDE is modelled and its existence of solution is elaborated. Furthermore, some illustrative examples are carried out in Section 5 and an effective conclusion of graphical and tabular discussion is drawn as a final segment in Section 6.

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2 Preliminaries

In this section, we define some basic definitions of fuzzy numbers that are essential for the whole paper. These descriptions are found, in detail, in many research papers such as [20–22].

Definition 2.1

A function defined as $A : R \to [0, 1]$ is said to be a fuzzy number if it is upper semi-continuous, normal, fuzzy convex and compactly supported on **R**. The α-level set (interval of confidence at level α) of the fuzzy number A are closed intervals represented as $[A]_\alpha = [\underline{A_\alpha}, \bar{A_\alpha}]$, where $\underline{A_\alpha}$, and $\bar{A_\alpha}$ represent the lower and upper bound of interval, respectively, such that, lower bound is left-continuous non-decreasing and upper bound is left-continuous non-increasing over the interval $[0, 1]$.

Definition 2.2

A triangular fuzzy number defined as $\tilde{A} = (a_1, a_2, a_3; \omega)$, for $\omega \in (0, 1]$, is called generalized triangular fuzzy number with membership function

$$\mu_{\tilde{A}}(x) = \begin{cases} \omega \frac{x-a_1}{a_2-a_1} & \text{if } a_1 \le x \le a_2 \\ \omega \frac{a_3-x}{a_3-a_2} & \text{if } a_2 \le x \le a_3 \\ 0 & \text{otherwise} \end{cases}$$

That can also be illustratively defined as:

1. $\mu_{\tilde{A}}(x) : \mathbf{R} \to [0, 1]$
2. $\mu_{\tilde{A}}(x) = 0$ for $x \le a_1$
3. $\mu_{\tilde{A}}(x)$ is strictly increasing function for $a_1 \le x \le a_2$
4. $\mu_{\tilde{A}}(x)$ for $x = a_2$
5. $\mu_{\tilde{A}}(x)$ is strictly decreasing function for $a_2 \le x \le a_3$
6. $\mu_{\tilde{A}}(x) = 0$ for $a_3 \le x$

If $\omega = 1$ then $\tilde{A}$ becomes simple triangular fuzzy number. The α-level set of generalized triangular fuzzy number $\tilde{A}$ are closed intervals represented as, $[A]_\alpha \left[\left(a_1 + \frac{(a_2-a_1)\alpha}{\omega}\right), \left(a_3 - \frac{(a_3-a_2)\alpha}{\omega}\right)\right]$.

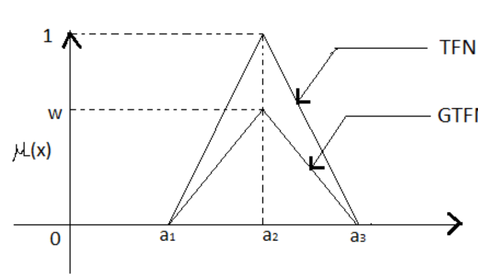

Fig. 1: Triangular and genaralized triangular fuzzy membership function

3 Difference Equation

Theorem 3.1: Let I be an interval of real numbers, and let $f : I \times I \to I$ be a continuous function then the difference equation can be expressed as:

$$x_{n+a} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots \tag{1}$$

where $x_{-1}, x_0 \in I$ are the initial values and f satisfies the following conditions:

1. There exist positive number a and b with $a < b$ such that $a \le f(x_n, x_{n-1})$ for all $x_n, x_{n-1} \in [a, b]$.
2. $f(x_n, x_{n-1})$ is an increasing function in $x_n \in [a, b]$ for each $x_{n-1} \in [a, b]$, and a decreasing function in $x_{n-1} \in [a, b]$ for each $x_n\ in[a, b]$.
3. Eq. (1) has no solutions of prime period two in $[a, b]$.

Then there exists exactly one equilibrium solution $\hat{x}$ of Eq. (1) which lies in $[a, b]$. Moreover, every solution of Eq. (1) with initial conditions $x_{-1}, x_0 \in [a, b]$ converges to $\hat{x}$.

Any qth order linear difference equation is a set of equations of the form

$$x_n-(a_{n-1}x_{n-1}+a_{n-2}x_{n-2}+\cdots+a_{n-q}x_{n-q}) = r_n \ n = q, q+1, \ldots \tag{2}$$

where r_n is the forcing factor. If $r_n = 0$, for all n, then Eq. (2) is said to be a homogeneous difference equation otherwise it is non homogeneous difference equation and if $a_i, i = 1, 2, \ldots, n$, do not depend on n then the equation said to have constant coefficients [1–10].

3.1 Stability Analysis

Next, we look over the stability analysis of difference equation and system of difference equations. The detailed stability analysis and existence of solutions for different types of difference equation is also found in [4–10].

3.1.1 Linear Difference Equation

Let us consider an autonomous linear discrete equation of the form

$$u_n = au_{n+1} + b \quad (a \neq 0) \tag{3}$$

If u^* be the equilibrium solution of the model, then $u_n = u_{n+1} = u^*$ (there is no change from $n - 1$ generation to n generation) i.e.

$$u^* = \frac{b}{1-a} \tag{4}$$

The equilibrium point u^* is said to be stable if all the solutions of the above difference equation approaches to $\frac{b}{1-a}$ as n becomes large. The equilibrium point u^* is unstable if all solutions diverges from u^* to $\pm\infty$. The stability of the equilibrium solution depends on a. It is stable if $|a| < 1$ and unstable if $|a| > 1$ and an ambiguous case if $a \pm 1$.

3.1.2 System of Linear Homogeneous Difference Equation

Let us consider the system of linear homogeneous difference equation

$$u_{n+1} = au_n + bv_n, \; v_{n+1} = cu_n + dv_n \tag{5}$$

That can also be expressed in matrix form as,

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (u_n v_n) \tag{6}$$

Clearly, $(0, 0)$ is the equilibrium point of the homogeneous system (5).

Theorem 3.1.

Let λ_1 and λ_2 be two real distinct eigenvalues of the coefficients matrix of system (5), then the equilibrium point $(0, 0)$ is

1. Stable if both $|\lambda_1|$ and $||lambda_2| < 1$.
2. Unstable if both $|\lambda_1| > 1$ and $|\lambda_2| > 1$.
3. Saddle if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ or $|\lambda_1| > 1$ and $|\lambda_2| < 1$.

Theorem 3.2.

Let $\lambda_1 = \lambda_2 = \lambda^*$ be real and equal eigenvalues of the coefficients matrix then the equilibrium point $(0, 0)$ is

1. Stable if $|\lambda^*| < 1$.
2. Unstable if $|\lambda^*| > 1$.

Theorem 3.3.

If $u + iv$ and $u - iv$ are the complex conjugate eigenvalues of the coefficients matrix then the equilibrium point $(0, 0)$ is

1. Stable if $|u \pm iv| < 1$.
2. Unstable if $|u \pm iv| > 1$.

4 Fuzzy difference equation

Consider the first order homogeneous FDE as

$$u_{n+1} + ku_n = 0, \quad n = 0, 1, 2 \ldots \tag{7}$$

with u_n as sequence of fuzzy numbers and having $u_{n=0} = u_0$ as initial condition. Now, let Eq. (7) be classified into three types under following paradigms,
Type I: When initial condition u_0 is taken as fuzzy number.
Type II: When coefficient k is taken as fuzzy number.
Type III: When initial condition u_0 and coefficient k are taken as fuzzy numbers.

4.1 Existence of Solution to Type I: When initial condition is fuzzy number

Consider the Eq. (7) with fuzzy initial condition $[u_{n=0}]_\alpha = \lfloor \underline{u}_0(\alpha), \bar{u_0}(\alpha) \rfloor$, for all $\alpha \in [0, 1]$. In this situation, two cases will take place as:

4.1.1 When coefficient is positive i.e., $k > 0$

Taking the α-levels of Eq. (7),

$$\lfloor \underline{u}_{n+1}(\alpha), \bar{u}_{n+1}(\alpha) \rfloor + k \lfloor \underline{u}_n(\alpha), \bar{u}_n(\alpha) \rfloor = 0 \tag{8}$$

we obtain the solutions in α-levels as follows, for all $\alpha \in [0, 1]$,

$$\underline{u}_n(\alpha) = (-k)^n \underline{u}_0(\alpha), \; \bar{u}_n(\alpha) = (-k)^n \bar{u}_0(\alpha) \tag{9}$$

where the coefficient matrix can be constructed as:

$$\begin{pmatrix} -k & 0 \\ 0 & -k \end{pmatrix} \tag{10}$$

with the eigenvalues $\lambda = -k, -k$. Hence, the equilibrium point $(0, 0)$ is

1. Stable if $|-k| < 1$ i.e. $|k| < 1$.
2. Unstable if $|-k| > 1$ i.e. $|k| > 1$.

Thus, the solution is attained by means of the coefficient k.

4.1.2 When coefficient is negative i.e. $k < 0$

Let $k = -m$ in Eq. (7), then on taking its α-levels we have

$$\lfloor \underline{u}_{n+1}(\alpha), \bar{u}_{n+1}(\alpha) \rfloor - m \lfloor \underline{u}_n(\alpha), \bar{u}_n(\alpha) \rfloor = 0 \tag{11}$$

which changes to system of equations as,

$$\underline{u}_{n+1}(\alpha) - m\bar{u}_n(\alpha) = 0, \ \bar{u}_{n+1}(\alpha) - m\underline{u}_n(\alpha) = 0. \quad (12)$$

Hence, the solutions in the form of α-levels are formed as, for all $\alpha \in [0, 1]$,

$$\underline{u}_n(\alpha) = \frac{\underline{u}_0(\alpha) + \bar{u}_0(\alpha)}{2}(m)^n + \frac{\underline{u}_0(\alpha) - \bar{u}_0(\alpha)}{2}(-m)^n, \quad (13)$$

$$\bar{u}_n(\alpha) = \frac{\underline{u}_0(\alpha) + \bar{u}_0(\alpha)}{2}(m)^n - \frac{\underline{u}_0(\alpha) - \bar{u}_0(\alpha)}{2}(-m)^n. \quad (14)$$

with the coefficient matrix,

$$\begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \quad (15)$$

and eigenvalues $\lambda = m, -m$. Therefore, the equilibrium point $(0, 0)$ is

1. Stable if both $|m| < 1$ and $|-m| < 1$.
2. Unstable if both $|m| > 1$ and $|-m| > 1$.

4.2 Existence of Solution to Type II: When coefficient is fuzzy number

Consider Eq. (7) with fuzzy coefficient $[k]_\alpha = \lfloor \underline{k}(\alpha), \bar{k}(\alpha) \rfloor$, for all $\alpha \in [0, 1]$ and crisp initial condition $u_{n=0} = u_0$, then we have the following cases.

4.2.1 When coefficient is positive i.e. $k > 0$ for all $\alpha \in [0, 1]$

Taking Eq. (7) in its α-levels,

$$\lfloor \underline{u}_{n+1}(\alpha), \bar{u}_{n+1}(\alpha) \rfloor + \lfloor \underline{k}(\alpha), \bar{k}(\alpha) \rfloor \lfloor \underline{u}_n(\alpha), \bar{u}_n(\alpha) \rfloor = 0 \quad (16)$$

as $\underline{k}(\alpha) < \bar{k}(\alpha)$ and $\underline{u}_n(\alpha) < \bar{u}_n(\alpha)$, Eq. (16) takes the following form

$$\underline{u}_{n+1}(\alpha) + \underline{k}(\alpha)\underline{u}_n(\alpha) = 0, \ \bar{u}_{n+1}(\alpha) + \bar{k}(\alpha)\bar{u}_n = 0. \quad (17)$$

we obtain the solutions as, for all $\alpha \in [0, 1]$,

$$\bar{u}_n(\alpha) = u_0(-\bar{k}(\alpha))^n, \bar{u}_n(\alpha) = u_0(-\bar{k}(\alpha))^n. \quad (18)$$

with the coefficient matrix,

$$\begin{pmatrix} -\underline{k}(\alpha) & 0 \\ 0 & -k\bar{k}(\alpha) \end{pmatrix} \quad (19)$$

and eigenvalues $\lambda = -\underline{k}(\alpha), -\bar{k}(\alpha)$. Therefore, the equilibrium point $(0, 0)$ is

1. Stable if $|-\underline{k}(\alpha)| < 1$ and $|-\bar{k}(\alpha)| < 1$.
2. Unstable if both $|-\underline{k}(\alpha)| > 1$ and $|-\bar{k}(\alpha)| > 1$.
3. Saddle if $|-\underline{k}(\alpha)| < 1$ and $|-\bar{k}(\alpha)| > 1$ or, $|-\underline{k}(\alpha)| > 1$ and $|-\bar{k}(\alpha)| < 1$.

4.2.2 When coefficient is negative i.e. $k < 0$for all $\alpha \in [0, 1]$

Let $k = -m$ in Eq. (7), i.e. $[m]_\alpha = \lfloor \underline{m}(\alpha), \bar{m}(\alpha) \rfloor$, and expressing it in form of α-levels we have,

$$\underline{u}_{n+1}(\alpha) - \bar{m}(\alpha)\bar{u}_n(\alpha) = 0, \bar{u}_{n+1}(\alpha) - \underline{m}(\alpha)\underline{u}_n(\alpha) = 0 \quad (20)$$

that changes to system of equations as,

$$\underline{u}_{n+1}(\alpha) - \bar{m}(\alpha)\bar{u}_n(\alpha) = 0, \bar{u}_{n+1}(\alpha) - \underline{m}(\alpha)\underline{u}_n(\alpha) = 0, \quad (21)$$

with initial conditions $u_{n=0} = u_0$, then its solution is obtained as

$$\underline{u}_n(\alpha) = u_0 \left(\sqrt{\underline{m}(\alpha)\bar{m}(\alpha)}\right)^n + u_0 \left(-\sqrt{\underline{m}(\alpha)\bar{m}(\alpha)}\right)^n$$

$$\bar{u}_n(\alpha) = u_0 \left(\sqrt{\underline{m}(\alpha)\bar{m}(\alpha)}\right)^n - u_0 \left(-\sqrt{\underline{m}(\alpha)\bar{m}(\alpha)}\right)^n \quad (22)$$

and consequently, we obtain the coefficient matrix,

$$\begin{pmatrix} 0 & \underline{m}(\alpha) \\ \bar{m}(\alpha) & 0 \end{pmatrix} \quad (23)$$

with the eigenvalues $\lambda = \pm\sqrt{\underline{m}(\alpha)\bar{m}(\alpha)}$. So, the equilibrium point $(0, 0)$ is

1. Stable if both $|\sqrt{\underline{m}(\alpha)\bar{m}(\alpha)}| < 1$ and $|-\sqrt{\underline{m}(\alpha)\bar{m}(\alpha)}| < 1$.
2. Unstable if both $|\sqrt{\underline{m}(\alpha)\bar{m}(\alpha)}| > 1$ and $|-\sqrt{\underline{m}(\alpha)\bar{m}(\alpha)}| > 1$.
3. Saddle if $|\sqrt{\underline{m}(\alpha)\bar{m}(\alpha)}| < 1$ and $|-\sqrt{\underline{m}(\alpha)\bar{m}(\alpha)}| > 1$ or, $|\underline{m}(\alpha)\bar{m}(\alpha)| > 1$ and $|-\sqrt{\underline{m}(\alpha)\bar{m}(\alpha)}| < 1$.

Thus, the solution depends upon the fuzzy number $[k]_\alpha = \lfloor \underline{k}(\alpha), \bar{k}(\alpha) \rfloor$ at the different values of $\alpha \in [0, 1]$.

4.3 Existence of Solution to Type III: Initial condition and coefficient are fuzzy numbers

Now taking Eq. (7) with fuzzy coefficient $[k]_\alpha = \lfloor \underline{k}(\alpha), \bar{k}(\alpha) \rfloor$ and fuzzy initial condition $[u_{n=0}]_\alpha = \lfloor \underline{u}_0(\alpha), \bar{u}_0(\alpha) \rfloor$, for all $\alpha \in [0, 1]$, Then the following cases are constructed.

4.3.1 When coefficient is positive i.e. $k > 0$ for all $\alpha \in [0, 1]$

Writing Eq. (7) in its α-levels,

$$\lfloor \underline{u}_{n+1}(\alpha), \bar{u}_{n+1}(\alpha) \rfloor + \lfloor \underline{k}(\alpha), \bar{k}(\alpha) \rfloor \lfloor \underline{u}_n(\alpha), \bar{u}_n(\alpha) \rfloor = 0, \quad (24)$$

then its solutions are given by

$$\underline{u}_n(\alpha) = \underline{u}_0(\alpha)(-\underline{k}(\alpha))^n, \bar{u}_n(\alpha) = \bar{u}_0(\alpha)(-\bar{k}(\alpha))^n, \text{ for all } \alpha \in [0, 1]. \quad (25)$$

Stability of the solutions are same as defined in Section 4.2.1

4.3.2 When coefficient is negative i.e. $k < 0$ for all $\alpha \in [0, 1]$

Let $k = -m$ in Eq. (7) with α-levels,

$$\lfloor \underline{u}_{n+1}(\alpha), \bar{u}_{n+1}(\alpha) \rfloor - \lfloor \underline{m}(\alpha), \bar{m}(\alpha) \rfloor \lfloor \bar{u}_n(\alpha), \underline{u}_n(\alpha) \rfloor = 0, \quad (26)$$

then its solutions are established as,

$$\underline{u}_n(\alpha) = \frac{\underline{u}_0(\alpha) + \bar{u}_0(\alpha)}{2}(\sqrt{\underline{m}(\alpha)\bar{m}(\alpha)})^2 + \frac{\underline{u}_0(\alpha) - \bar{u}_0(\alpha)}{2}(-\sqrt{\underline{m}(\alpha)\bar{m}(\alpha)})^n, \quad (27)$$

$$\bar{u}_n(\alpha) = \frac{\underline{u}_0(\alpha) + \bar{u}_0(\alpha)}{2}(\sqrt{\underline{m}(\alpha)\bar{m}(\alpha)})^2 - \frac{\underline{u}_0(\alpha) - \bar{u}_0(\alpha)}{2}(-\sqrt{\underline{m}(\alpha)\bar{m}(\alpha)})^n \quad (28)$$

Stability cases are same as defined in Section 4.2.2.

5 Numerical example

In this section, some illustrative examples are carried out to exemplify the analysis demonstrated in previous section.

Example 5.1:

Let the fuzzy difference equation

$$u_{n+1} + au_n = 0, \ n = 0, 1, \ldots \quad (29)$$

with initial condition $u_{n=0} = u_0$ be taken under the following cases, i.e.

Type I: $a = 4$ and $u_0 = (10, 12, 15)$
Type II: $a = (3, 4, 6)$ and $u_0 = 12$
Type III: $a = (3, 4, 6)$ and $u_0 = (10, 12, 15)$

Solution in case of Type I:

Since initial condition is a fuzzy number that can be written in α-levels as:

$$\lfloor \underline{u}_0(\alpha), \bar{u}_0(\alpha) \rfloor = [(2\alpha + 10), (15 - 3\alpha)] \quad (30)$$

and $a > 0$ so the solution is obtained according to the case I of type I as elaborated in Section 4.1.1, i.e.,

$$\underline{u}_n(\alpha) = (10 + 2\alpha)(-4)^n, \ \bar{u}_n(\alpha) = (15 - 3\alpha)(-4)^n. \quad (31)$$

Table 1 shows some of the numerical values of Eq. (31) for $n = 2$ and $n = 6$, whereas Fig. 2 represents graphical view of above compact solution for $n = 4$. Clearly, from the Table 1 and Fig. 2 we see that $\underline{u}_n(\alpha)$ is increasing and $\bar{u}_n(\alpha)$ is decreasing as α moves from 0 to 1, which concludes that the solution is also a fuzzy solution.

Table 1: Solution of $u_n(\alpha) = \lfloor \underline{u}_n(\alpha), \bar{u}_n(\alpha) \rfloor$ of Example 5.1 for Type I

α	$n = 2$	$n = 6$
0	[160.0,240.0]	[40960.0,61440.0]
0.2	[166.4,230.4]	[42598.4,58982.4]
0.4	[172.8,220.8]	[44236.8,56524.8]
0.6	[179.2,211.2]	[45875.2,54067.2]
0.8	[185.6,201.6]	[47513.6,51609.6]
1	[192.0,192.0]	[49152.0,49152.0]

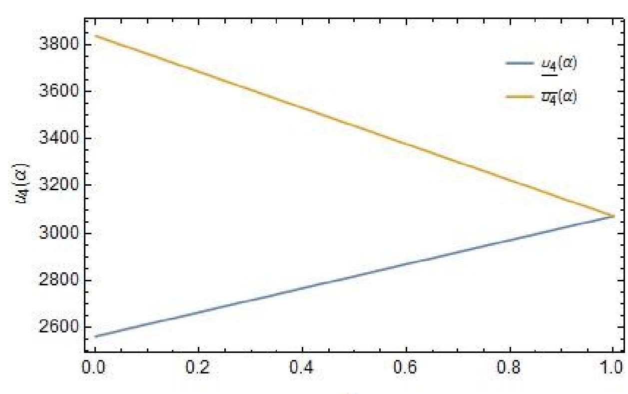

Fig. 2: Solution of $u_n(\alpha) = \lfloor \underline{u}_n(\alpha), \bar{u}_n(\alpha) \rfloor$ for $n = 4$ of Example 5.1 for Type I

Solution in case of Type II:

Here coefficient is a fuzzy number that can be written in α-levels as:

$$\lfloor \underline{a}(\alpha), \bar{a}(\alpha) \rfloor = [(\alpha + 3), (6 - 2\alpha)] \quad (32)$$

and $a > 0$ for all $\alpha \in [0, 1]$, therefore, the solution is obtained according to the case I of type II as elaborated in

Section 4.2.1, i.e.,

$$\underline{u}_n(\alpha) = 12(-(3+\alpha))^n, \bar{u}_n(\alpha) = 12(-(6-2\alpha))^n. \quad (33)$$

Table 2 presents some of the numerical values of Eq. (33) for $n = 2$ and $n = 6$, whereas Fig. 3 displays pictorial view of above compact solution for $n = 4$. From the Table 2 and Fig. 3 it can be seen that $\underline{u}_n(\alpha)$ is increasing and $\bar{u}_n(\alpha)$ is decreasing as α moves from 0 to 1, which shows that the solution obtained in case when the coefficient is a fuzzy number is also a fuzzy solution.

Table 2: Solution of $u_n(\alpha) = \lfloor \underline{u}_n(\alpha), \bar{u}_n(\alpha) \rfloor$ of Example 5.1 for Type II

α	$n = 2$	$n = 6$
0	[108.00,432.00]	[8748.00,559872.00]
0.2	[122.88,376.32]	[12884.90,370092.00]
0.4	[138.72,324.48]	[18537.70,237247.00]
0.6	[155.52,276.48]	[26121.40,146767.00]
0.8	[173.28,232.32]	[36131.2,87075.80]
1	[192.00,192.00]	[49152.00, 49152.00]

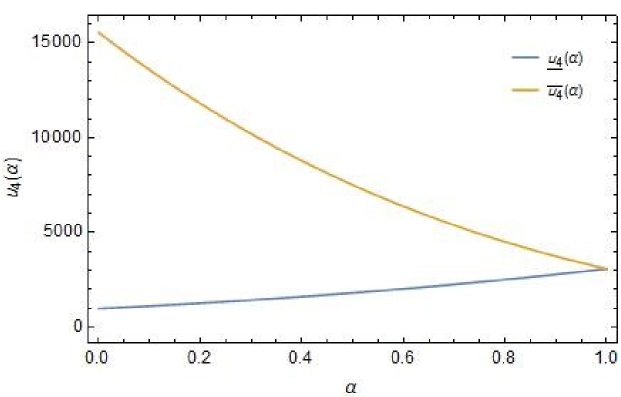

Fig. 3: Solution of $u_n(\alpha) = \lfloor \underline{u}_n(\alpha), \bar{u}_n(\alpha) \rfloor$ for $n = 4$ of Example 5.1 for Type II

Solution in case of Type III:

Here, coefficient and initial condition are given as fuzzy numbers, therefore on considering Eq. (30) and (32), the solution is constructed according to analysis mentioned in the case I of type III in Section 4.3.1, as:

$$\underline{u}_n(\alpha) = (-(3+\alpha))^n(10+2\alpha), \bar{u}_n(\alpha) = (-(6-2\alpha))^n(-3+15\alpha) \quad (34)$$

Table 3 demonstrates some of the numerical values of Eq. (34) for $n = 2$ and $n = 4$, while Fig. 4 exhibits pictorial view of above close solution for $n = 4$. From the Table 3 and Fig. 4 it can be seen that $\underline{u}_n(\alpha)$ is increasing and $\bar{u}_n(\alpha)$ is decreasing as α moves from 0 to 1, which shows that the solution obtained in case when the coefficient and initial condition are fuzzy numbers is also a fuzzy solution.

Table 3: Solution of $u_n(\alpha) = \lfloor \underline{u}_n(\alpha), \bar{u}_n(\alpha) \rfloor$ of Example 5.1 for Type III

α	$n = 2$	$n = 6$
0	[90.00,-108.00]	[7290.00,-139968.00]
0.2	[106.49,0.00]	[11166.90,0.00]
0.4	[124.85,81.12]	[16683.90,59311.80]
0.6	[145.15,138.24]	[24380.00,73383.50]
0.8	[167.50,174.24]	[34926.90,65306.80]
1	[192.00, 192.00]	[49152.00, 49152.00]

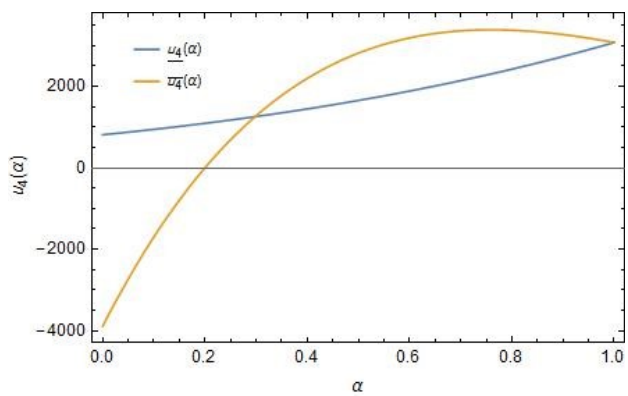

Fig. 4: Solution of $u_n(\alpha) = \lfloor \underline{u}_n(\alpha), \bar{u}_n(\alpha) \rfloor$ for $n = 4$ of Example 5.1 for Type III

Example 5.2:

Let the difference equation

$$u_{n+1} + a u_n = 0, \quad n = 0, 1, \ldots \quad (35)$$

with initial condition $u_{n=0} = u_0$ be taken under the following cases, i.e.

Type I: $a = 2$ and $u_0 = \left(\frac{1}{8}, \frac{1}{4}, \frac{1}{2}; 0.8\right)$
Type II: $a = (1, 2, 4; 0.7)$ and $u_0 = \frac{1}{4}$
Type III: $a = (1, 2, 4; 0.7)$ and $u_0 = \left(\frac{1}{8}, \frac{1}{4}, \frac{1}{2}; 0.8\right)$

Solution in case of Type I:

Since initial condition is a fuzzy number that can be written in α-levels as:

$$[\underline{u}_0(\alpha), \bar{u}_0(\alpha)] = \left[\left(\frac{4+5\alpha}{32}\right), \left(\frac{8-5\alpha}{16}\right)\right] \quad (36)$$

and $a > 0$ so the solution is obtained according to the case I of type I as elaborated in Section 4.1.1, i.e.,

$$\underline{u}_n(\alpha) = \left(\frac{4+5\alpha}{32}\right)(-2)^n, \bar{u}_n(\alpha) = \left(\frac{8-5\alpha}{16}\right)(-2)^n. \quad (37)$$

Table 4 presents some of the numerical values of Eq. (37) for $n = 2$ and $n = 10$, whereas Fig. 5 displays pictorial view of above compact solution for $n = 6$. Clearly from the Table 4 and Fig. 5 it is depicted that $\underline{u}_n(\alpha)$ is increasing and $\bar{u}_n(\alpha)$ is decreasing as α moves from 0 to 1, which concludes that the solution is also a fuzzy solution.

Table 4: Solution of $u_n(\alpha) = \lfloor \underline{u}_n(\alpha), \bar{u}_n(\alpha) \rfloor$ of Example 5.2 for Type I

α	$n = 2$	$n = 10$
0	[0.50,2.00]	[128.00,512.00]
0.2	[0.62,1.75]	[160.00,448.00]
0.4	[0.75,1.50]	[192.00,384.00]
0.6	[0.87,1.25]	[224.00,320.00]
0.8	[1.00,1.00]	[256.00,256.00]

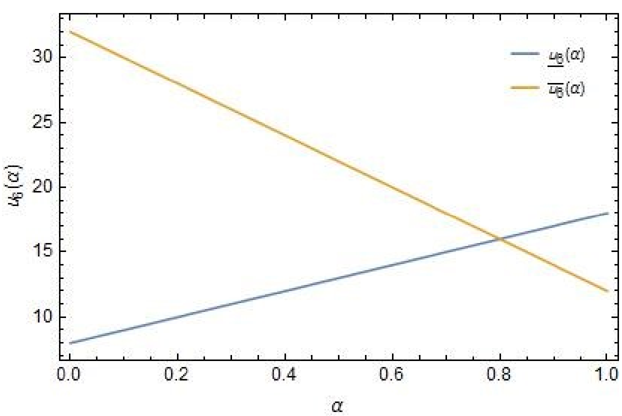

Fig. 5: Solution of $u_n(\alpha) = \lfloor \underline{u}_n(\alpha), \bar{u}_n(\alpha) \rfloor$ for $n = 6$ of Example 5.2 for Type I

Solution in case of Type II:

Here coefficient is a fuzzy number that can be written in α-levels as:

$$[\underline{a}(\alpha), \bar{a}(\alpha)] = \left[\left(\frac{7+10\alpha}{7}\right), \left(\frac{28-20\alpha}{7}\right)\right] \quad (38)$$

and $a > 0$ for all $\alpha \in [0, 1]$, therefore, the solution is obtained according to the case I of type II as elaborated in Section 4.2.1, i.e.,

$$\underline{u}_n(\alpha) = \frac{1}{4}\left(-\left(\frac{7+10\alpha}{7}\right)\right)^n,$$
$$\bar{u}_n(\alpha) = \frac{1}{4}\left(-\left(\frac{28-20\alpha}{7}\right)\right)^n. \quad (39)$$

Table 5 demonstrates some of the numerical values of Eq. (39) for $n = 2$ and $n = 10$, while Fig. 6 exhibits pictorial view of above close solution for $n = 6$. From the Table 5 and Fig. 6 it can be seen that $\underline{u}_n(\alpha)$ is increasing and $\bar{u}_n(\alpha)$ is decreasing as α moves from 0 to 1, which shows that the solution obtained in case when the coefficient is a fuzzy number is also a fuzzy solution.

Table 5: Solution of $u_n(\alpha) = \lfloor \underline{u}_n(\alpha), \bar{u}_n(\alpha) \rfloor$ of Example 5.2 for Type II

α	$n = 2$	$n = 10$
0	[0.25,4.00]	[0.25,262144.00]
0.1	[0.33,3.45]	[0.95,124938.00]
0.3	[0.510,2.47]	[8.85,23506.40]
0.5	[0.73,1.65]	[54.79,3159.98]
0.7	[1.00,1.00]	[256.00,256.00]

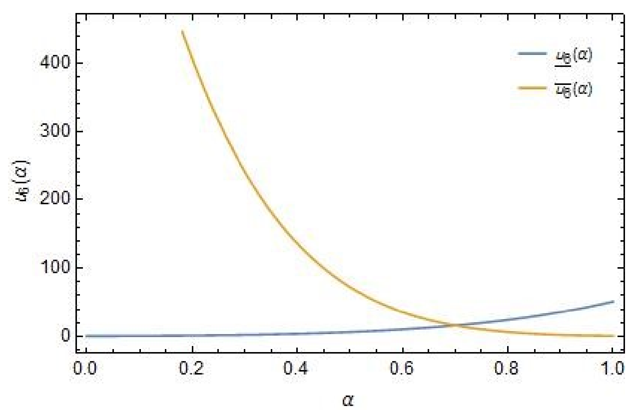

Fig. 6: Solution of $u_n(\alpha) = \lfloor \underline{u}_n(\alpha), \bar{u}_n(\alpha) \rfloor$ for $n = 6$ of Example 5.2 for Type II

Solution in case of Type III:

Here, coefficient and initial condition are given as fuzzy numbers, therefore on considering Eq. (36) and (38), the solution is constructed according to analysis mentioned in the case I of type III in Section 4.3.1, as:

$$\underline{u}_n(\alpha) = \left(\frac{4+5\alpha}{32}\right)\left(-\left(\frac{7+10\alpha}{7}\right)\right)^n,$$
$$\bar{u}_n(\alpha) = \left(\frac{8-5\alpha}{16}\right)\left(-\left(\frac{28-20\alpha}{7}\right)\right)^n. \quad (40)$$

Table 6 shows some of the numerical values of Eq. (40) for $n = 2$ and $n = 10$, whereas Fig. 7 represents graphical view of above compact solution for $n = 6$. From the Table 6 and Fig. 7 it can be seen that $\underline{u}_n(\alpha)$ is increasing and $\bar{u}_n(\alpha)$ is decreasing as α moves from 0 to 1, which shows that the solution obtained in case when the coefficient and initial condition are fuzzy numbers is also a fuzzy solution.

Table 6: Solution of $u_n(\alpha) = \lfloor \underline{u}_n(\alpha), \bar{u}_n(\alpha) \rfloor$ of Example 5.2 for Type III

α	$n = 2$	$n = 20$
0	[0.12,8.00]	[0.12,524288.00]
0.1	[0.19,6.47]	[0.53,234258.00]
0.3	[0.35,4.01]	[6.08,38197.90]
0.5	[0.59,2.27]	[44.52,4344.98]
0.7	[0.94,1.12]	[240.00,288.00]

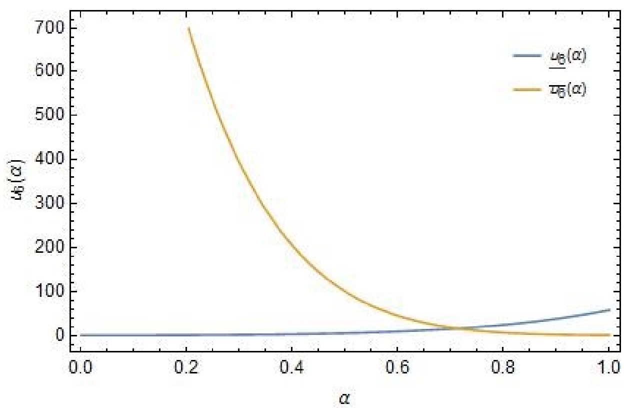

Fig. 7: Solution of $u_n(\alpha) = \lfloor \underline{u}_n(\alpha), \bar{u}_n(\alpha) \rfloor$ for $n = 6$ of Example 5.2 for Type III

6 Conclusion

In this paper, we exercised existence and stability analysis of linear difference equation in fuzzy environment. The stability of solutions of governing problems were discussed by encompassing initial condition and coefficient as fuzzy numbers. The explanation was examined on different numerical examples with triangular and generalized triangular fuzzy numbers, respectively. Thus, we turn up with the following findings:

- Modelling difference equation by incorporating fuzzy numbers, in the form of its initial condition or coefficient, represented the possibility to tackle the imprecision that occur either in initial condition or as coefficient, while modelling a real-world phenomenon.
- The stability analysis yields the existence of solution of fuzzy difference equations under all the cases, where the solutions are achieved by means of the coefficients of fuzzy difference equations.
- Stability deliberation further illustrated that the nature of fuzzy difference equation varies for different sign of coefficient, i.e., for negative sign the fuzzy difference equation converts into the related system of two difference equations and so solutions are attained by solving the system of difference equations.

The constructive theory and effective results are surely very important and beneficial for the concerned researchers working in this field.

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