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An analytical method with Padé technique for solving of variational problems

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Abstract: In this paper, the homotopy analysis method (HAM) is employed to solve a class of variational problems (VPs). By using the so-called $\hbar$-curves, we determine the convergence parameter $\hbar$, which plays key role to control convergence of solution series. Also we use Pade' approximant to improve accuracy of the method. Two test example are given to clarify the applicability and efficiency of the proposed method.

Keywords: Variational problems; Euler-Lagrange; Homotopy analysis method; Padé

MSC: 41A58; 39A10; 34K28; 41A10

1 Introduction

1.1 Motivation of paper

The investigation of exact solutions to fractional VPs plays an important role in the study of nonlinear physical phenomena.

1.2 Some history

The Homotopy analysis method (HAM) has been presented by Liao [1–3], to obtain the analytical solutions for various nonlinear problems. There are many letters that deal with Homotopy analysis method such as, Abbasbandy et al. [4] applied the Newton-homotopy analysis method to solve nonlinear algebraic equations, Allan [5] constructed the analytical solutions to Lorenz system by the HAM, Bataineh et al. [6, 7] proposed a new reliable modification of the HAM, Alomari and et al. introduced the solution of delay differential equation by means of homotopy analysis method , Chen and Liu [9] applied the HAM to increase the ciatonvergent region of the harmonic balance method, Liao [10] proposed the HAM to study nonlinear problems and others [11, 12].

1.3 Structure of paper

In this paper, we extend the application of HAM for solving variational problems. The structure of this paper is organized as follows.

In Section 2, we present our main results concerning to our method. In Section 3, we present a test example. Finally, we end the paper with few concluding remarks in Section 4.

2 VPs with Caputo fractional derivative

Consider the following functional

$$J\,[y] = \int_a^b F(x.y\,(x)\,, y^{'}(x))dx \tag{1}$$

defined on the set of continuous curves $y\,:\,[0,1] \to R$, where F has continuous derivatives with respect to the second, third and fourth variable. Consider the problem of finding the extremum of the functional (1) with boundary conditions $y\,(a) = A$ and $y\,(b) = B$. Let us denote this problem by (P). A necessary condition for problem (P) is given by the next theorem.

Theorem1: (see [12]) If y is a local minimizer to problem (P), then $y$satisfies the Euler-Lagrange equation

$$\begin{cases} \dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0, \\ y\,(a) = A\ ,\ y\,(b) = B \end{cases} \tag{2}$$

The Euler-Lagrange equation (2) is in general a nonlinear differential equation of fractional order, which does not always have an analytic solution.

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3 Solution guidelines for variational problems

First, we rewrite the Euler-Lagrange equation in the following operator form

$$\begin{cases} L[y(x)] + N[y(x)] - g(x) = 0, \\ y(a) = A\ ,\ y(b) = B. \end{cases} \tag{3}$$

where L is a linear operator, N is a nonlinear operator which contains differential operators less than 2 and $g(x)$ is a given function.

Let $\hbar$ denote a nonzero auxiliary parameter. By means of generalizing the traditional homotopy method, we construct the so-called zero th-order deformation equation

$$(1-p)\,L[Y(x;p) - Y_0(x)] = p\hbar[L[y(x)] + N[y(x)] - g(x)]), \tag{4}$$

Where $p\epsilon[0,1]$ is the so-called homotopy parameter, $(x;p) : [a,b] \times [0,1] \to R$, and Y_0 defines the initial approximation of the solution of (3). Assume the solution of (3) to be in the form

$$Y = Y_0 + pY_1 + p^2Y_2 + p^3Y_3 + \ldots \tag{5}$$

In order to determine the functions $Y_j,\ j = 1,2,3,\ldots$ substituting (5) into (4) and collecting terms of the same power of p gives

$$L[Y_1] = \hbar\ [L[Y_0] + N_0[Y_0] - g(x)]\,,$$

$$L[Y_2] = L[Y_1] + \hbar[L[Y_1] + N_1[Y_0, Y_1]]$$

$$L[Y_3] = L[Y_2] + \hbar[L[Y_2] + N_0[Y_0, Y_1, Y_2]]\,,$$

$$\vdots$$

where

$$N\left[Y_0 + pY_1 + p^2Y_2 + p^3Y_3 + \ldots\right] = N_0[Y_0] + pN_1[Y_0, Y_1] + P^2N_2[Y_0, Y_1, Y_2]\,,$$

and

$$\begin{pmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \\ N_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & Y_1 & 0 & 0 & 0 & \ldots \\ 0 & Y_2 & \frac{1}{2!}Y_1^2 & 0 & 0 & \ldots \\ 0 & Y_3 & Y_1Y_2 & \frac{1}{3!} & 0 & \ldots \\ 0 & Y_4 & \frac{1}{2!}Y_2^2 + Y_1Y_3 & \frac{1}{2!}Y_1^2Y_2 & \frac{1}{4!}Y_1^4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \end{pmatrix} \begin{pmatrix} \tilde{N}[Y_0] \\ \tilde{N'}[Y_0] \\ \tilde{N''}[Y_0] \\ N^{(\tilde{'''})}[Y_0] \\ N^{\tilde{(4)}}[Y_0] \\ \vdots \end{pmatrix}$$

where

$$N^{\tilde{(n)}}[Y_0] = N^{(n)}\left[Y_0 + \sum_{m=1}^{\infty} Y_m p^m\right]_{|_{P=0}}\ ,\quad n = 0,1,2,\ldots$$

Then, the solution of (2) has the form

$$y(x) = \lim_{n\to\infty} \sum_{j=0}^{n} Y_j(x)\,. \tag{6}$$

If it is difficult to determine the sum of series (6), then as an approximate solution of the equation, we approximate the solution $y(x)$ by the truncated series

$$y^n(x) = \sum_{j=0}^{n} Y_j(x)\,. \tag{7}$$

3.1 Convergence theorem

Theorem 1. (see [1]) If the solution series defined by (6) is convergent, then it must be the solution of the equation (2).

Theorem 2. (see [13]) Let $A(\hbar)$ be a continuous function on $[a_1, b_1]$. Further, suppose that $y^n(x, A(\hbar))$ be the approximate solution about $A(\hbar)$, then

$$\forall\epsilon > 0,\quad \bar{x}\epsilon(a_1, b_1)\,,\ \exists N_0\ and\ \exists(\hbar_1, \hbar_2)\,,\ so\ that$$

$$\forall\hbar\epsilon(\hbar_1, \hbar_2)\ and\ n \geq N_0 \Rightarrow \left|y(\bar{x}, A(\hbar)) - y^n(\bar{x}, A(\hbar))\right| < \epsilon. \tag{8}$$

Theorem 3. (see [13]) Let $A(\hbar)$ be a continuous function on $[a_1, b_1]$ and $y^n(x, A(\hbar))$ be the series of (7) about $A(\hbar)$ such that for each fixed $\hbar$is a function of x, then the curve of $y^n(x, A(\hbar))$, where $x \in (a_1, b_1)$, becomes a horizontal line when $n \to \infty$.

Theorem 4. (see [13]) Let $A(\hbar)$ be a continuous function on $[a_1, b_1]$ and $y^n(x, A(\hbar))$ be the series of (7) about $A(\hbar)$ such that for each fixed $\hbar$ is a function of x, then the curve of $\frac{d}{dx}y^n(x, A(\hbar))|_{x=\bar{x}}$, where $\bar{x} \in (a_1, b_1)$, becomes a horizontal line when $n \to \infty$.

Remark. So, it can be deduced that if for every fixed $\bar{x} \in (a_1, b_1)$ the $\hbar$-curve becomes horizontal, thus the series (7) is convergent, and according to the Theorem 1, converges to the exact solution.

3.2 Padé technique

There exist some techniques to improve the convergence rate of a given series by HAM. Among these techniques,

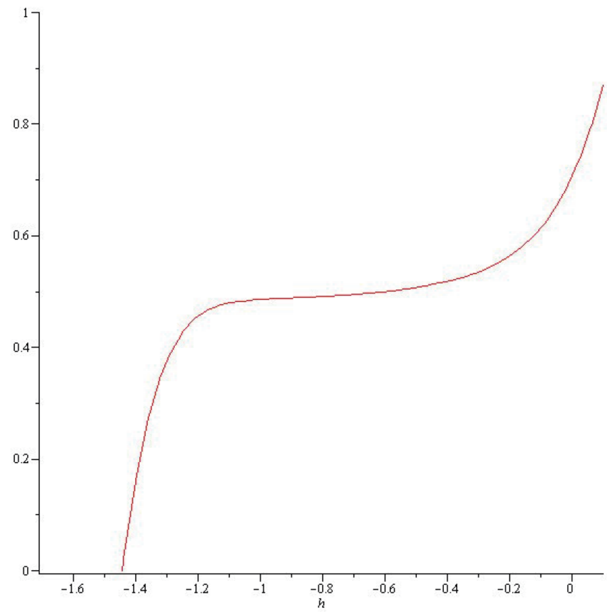

Fig. 1: The $\hbar$-curve of $y^{10}(1)$.

the so-called Padé technique is widely applied. The so-called homotopy- Padé technique was suggested by means of combining the Padé technique with HAM [1]. For a give series

$$Y=\sum_{j=0}^{\infty} y_j p^j, \tag{9}$$

the corresponding $[\frac{L}{M}]$ Padé approximant is expressed by

$$\left[\frac{L}{M}\right]=\frac{\sum_{j=1}^{L} r_i p^j}{1+\sum_{j=1}^{M} q_j p^j}, \tag{10}$$

The formal power series:

$$Y=\sum_{j=0}^{\infty} Y_j P^j \tag{11}$$

$$Y-\frac{\sum_{j=0}^{L} r_j p^j}{1+\sum_{j=1}^{M} q_j p^j}=O\left(p^{L+M+1}\right), \tag{12}$$

Determine the coefficients r_j and q_j by equating. Then, we can write out (12) in more details as:

$$\begin{cases} Y_{L+1}+Y_{Lq1}+\ldots+Y_{L-MqM}=0\,, \\ Y_{L+2}+Y_{L+1q1}+\ldots+Y_{L-M+2qM}=0 \\ \vdots \\ Y_{L+M}+Y_{L+M-1q1}+\ldots+Y_{LqM}=0 \end{cases} \tag{13}$$

$$\begin{cases} Y_0=r_0 \\ Y_1+Y_{0q1}=r_1 \\ \vdots \\ Y_L+Y_{L-1q1}+\ldots+Y_{0qL}=r_L \end{cases} \tag{14}$$

Table 1: The absolute errors of the solutions for Example 1.

x	y_{exact}	$\hbar=-0.9$	$\hbar=-0.5$
0.1	0.0481397563	4.56E−6	5.98E−6
0.2	0.0963910086	5.60E−6	5.24E−6
0.3	0.1448655114	6.05E−6	6.23E−6
0.4	0.1936755370	8.76E−6	9.87E−6
0.5	0.2429341332	9.36E−6	1.56E−5
0.6	0.2927553881	6.14E−5	7.86E−5
0.7	0.3432546921	5.21E−5	6.12E−5
0.8	0.3945490069	4.77E−4	5.87E−4
0.9	0.4467571349	5.08E−4	7.43E−4

Table 2: The $[\frac{L}{M}]$ padé approximant for Example 1.

x	$[\frac{6}{6}]$	$[\frac{7}{7}]$	$[\frac{8}{8}]$
0.1	0.0481399145	0.0481398235	0.0481398143
0.2	0.0963913376	0.0963912365	0.0963911255
0.3	0.1448658458	0.1448657355	0.1448656331
0.4	0.1936759781	0.1936758684	0.1936757566
0.5	0.2429344883	0.2429343784	0.2429332645
0.6	0.2927585751	0.2927574832	0.2927563297
0.7	0.3432578831	0.3432567985	0.3432557102
0.8	0.3945754189	0.3945653129	0.3945562789
0.9	0.4467995789	0.4467873909	0.4467772909

Setting $p=1$ provides the $[\frac{L}{M}]$ Padé approximant

$$\left[\frac{L}{M}\right]=\frac{\begin{vmatrix} Y_{L-M+1} & \ldots & Y_{L+1} \\ \ldots & \ddots & \ldots \\ Y_L & \ldots & Y_{L+M} \\ \sum_{j=M}^{L} Y_{j-M} & \ldots & \sum_{j=0}^{L} Y_j \end{vmatrix}}{\begin{vmatrix} Y_{L-M+1} & \ldots & Y_{L+1} \\ \ldots & \ddots & \ldots \\ Y_L & \ldots & Y_{l+M} \\ 1 & \ldots & \sum_{j=0}^{L} Y_j \end{vmatrix}} \tag{15}$$

For more details refer to [14].

4 Test examples

In this section, we solve a test problem to demonstrate the accurate nature of the proposed method. The validity of the method is based on assumption that the series (5) converges at $q=1$. There is the convergence control parameter $A(\hbar)$ which guarantees that this assumption can be satisfied. We need to concentrate on the convergence of the obtained results by properly choosing $\hbar$.

Example 1 Consider the problem of finding the minimum of the functional

$$J[y] = \int_0^1 \frac{1 + y^2(x)}{y'^2(x)} dx\,, \tag{16}$$

with boundary conditions

$$y(0) = 0\,,\; y(1) = 0.5\,. \tag{17}$$

The exact solution of this problem is $y(x) = sinh(0.4812118250596x)$.

We choose $y_0(x) = \frac{\pi}{2}$ as initial approximation guess. We study the influence of $\hbar$ on the convergence of $y^{10}(1)$. We can investigate the influence of $\hbar$ on the convergence region of y^{10} (1) by means of $\hbar$ -curve as shown in Fig. 1. From Fig. 1, the convergence region of y^{10} (1) is $[-1.2, -0.3]$. The absolute values of the errors are given in Table 1 which proves the accuracy of the solution.

We employ the Padé technique to gain more accurate approximations of the solution, as shown in Table 2.

5 Concluding remarks

In this paper, we studied the application of the HAM for solving the variational problems. One advantage of this method is the use of a computational algorithm with fast convergence. We also used Padé approximant to improve accuracy of the HAM. An example is presented to illustrate the accuracy of the present method. The given example reveal that the HAM yields a very effective and convenient technique to the approximate solutions of the variational problems.

References

[1] S. J. Liao: Beyond Perturbation: Introduction to Homotopy Analysis Method, Chapman Hall/ CRC Press, Boca Raton (2003).

[2] S. Abbasbandy, Y. Tan and S. J. Liao: Newton-homotopy analysis method for nonlinear equations, Appl. Math. Comput. 188(2007) 1794-1800.

[3] S. Abbasbandy: Solitary wave solutions to the Kuramoto Sivashinsky equation by means of the homotopy analysis method, Nonlinear Dynam. 52 (2008) 35—40.

[4] S. Abbasbandy and E.J. Parkes: Solitary smooth hump solutions of the Camassa-Holm equation by means of the homotopy analysis method, Chaos Solitons Fract. 36 (2008) 581—591.

[5] F. M. Allan: Construction of analytic solution to chaotic dynamical systems using the Homotopy analysis method, Chaos Solitons Fractals 39(2009) 1744-1752.

[6] A. S. Bataineh, M. S. M. Nooraniand and I. Hashim: Solving systems of ODEs by homotopy analysis method, Commun. Nonlinear Sci. Numer. Simul. 13(2008) 2060-2070.

[7] A. S. Bataineh, M. S. M. Noorani and I. Hashim: On a new reliable modification of homotopy analysis method, Commun. Nonlinear Sci. Numer. Simul. 14(2009) 409-423.

[8] A. K. Alomari, M. S. M. Noorani and R. Nazar: Solution of delay differential equation by means of homotopy analysis method, Acta. Appl. Math. 108(2009) 395-412.

[9] Y. M. Chen and J. K. Liu: Improving convergence of incremental harmonic balance method using homotopy analysis method, Acta. Mech. Sin. 25(2009) 707-712.

[10] S.J. Liao: The proposed homotopy analysis technique for the solution of nonlinear problems, PHD thesis, Shanghai Jiao Tong University (1992).

[11] S. J. Liao: An approximate solution technique not depending on small parameters: a special example, Int. J. Non-linear Mech. 30(1995) 371-380.

[12] T. Odzijewicz, A. B. Malinowska and D. F. M. Torres: Fractional variational calculus with classical and combined Caputo derivatives, Nonlinear Anal . Theory Methods Appl. 75(3) (2012) 1507-1515.

[13] M. Ghasemi, M. Fardi and R. K. Ghaziani: Solution of system of the mixed Volterra-Fredholm integral equations by an analytical method, Math. Comput. Model. 58(2013) 1522-1530.

[14] G. A. Baker: Essentials of Padé approximants, Academic Press, London (1975).