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Modelling and solvability of a class steady-state metal-forming problems

Abstract: For a class of steady-state metal-forming problems, with rigid-plastic, incompressible, strain-rate dependent material model and with unilateral contact and nonlocal Coulomb's frictional boundary conditions, a variational inequality formulation is derived and by proving the convergence of a modified secant-modulus method, existence and uniqueness results are obtained. A finite element - modified secant-modulus computational algorithm is developed and applied for solving illustrative problems.

Keywords: rigid-plastic metal-forming; nonlocal frictional unilateral contact; modified secant-modulus method; variational and finite element analysis

MSC: 74M10; 74M15; 74S05; 49J40

DOI 10.1515/nleng-2014-0021
Received August 4, 2014; accepted November 3, 2014.

1 Introduction

In this work a variational and numerical analysis of a class steady-state metal-forming problems, describing strip drawing and extrusion with unilateral contact and friction, is presented in the framework of the flow theory of plasticity [1–5], using the analogy with the frictional contact problems in linear elasticity and small deformations elasto-plasticity [6–11]. Recently, such approach has been used in [12–14], for analysis of steady and quasi-steady metal-forming problems, with rigid-plastic material and nonlocal contact frictional models. Variational inequality formulations have been derived and existence and uniqueness results have been obtained. The successive linearization secant-modulus method, developed for problems in the deformation theory of plasticity, has been successfully extended and applied for solving obtained nonlinear variational inequalities. Here, for a class of steady-state metal-forming problems with isotropic, rigid-plastic, strain-rate sensitive, incompressible material model and with unilateral contact and nonlocal Coulomb's frictional conditions, a variational inequality formulation is derived and by proving the convergence of a modified secant-modulus method, existence and uniqueness results are obtained. Finite element discretization is performed and an algorithm, combining the finite element method and the modified secant-modulus method, is proposed by which two illustrative extrusion and drawing problems are solved.

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2 Formulation of the problem

We consider a metallic workpiece occupying the domain $\Omega \subset \mathbb{R}^k$, $k = 2, 3$, with regular boundary Γ, constituting of six open, disjoint subsets (Fig. 1). A constant process velocity is prescribed on Γ_1, $\Gamma_2 \cup \Gamma_4$ is boundary free of tractions, Γ_3 is the frictional contact boundary, unilaterally constraining the material flow, on the boundary Γ_5 external forces are applied and Γ_6 is the boundary of symmetry, since due to the symmetry, only one half of the workpiece is considered. Throughout the paper $\mathbf{x} = \{x_i\}$ denotes a cartesian coordinate, δ_{ij}, $1 \le i, j \le k$, is the Kronecker symbol and the standard indicial notation and the summation convention over repeated indices are used.

Let us denote by $\mathbf{u}(\mathbf{x}) = \{u_i(\mathbf{x})\}$, $\boldsymbol{\sigma}(\mathbf{x}) = \{\sigma_{ij}(\mathbf{x})\}$, $\dot{\boldsymbol{\varepsilon}}(\mathbf{x}) = \{\dot{\varepsilon}_{ij}(\mathbf{x})\}$, the velocity vector, stress and strain-rate tensors respectively and by $\bar{\sigma} = \sqrt{\frac{3}{2} s_{ij} s_{ij}}$, $\dot{\bar{\varepsilon}} = \sqrt{\frac{2}{3} \dot{e}_{ij} \dot{e}_{ij}}$, the equivalent stress and strain-rate, where $s_{ij} = \sigma_{ij} - \sigma_H \delta_{ij}$, $\dot{e}_{ij} = \dot{\varepsilon}_{ij} - \frac{1}{3} \dot{\varepsilon}_V \delta_{ij}$, are the components of the deviatoric stress and the strain-rate tensors and $\sigma_H = \frac{1}{3}\sigma_{ii}$, $\dot{\varepsilon}_V = \dot{\varepsilon}_{ii}$ are the hydrostatic pressure and the volume dilatation strain-rate. Consider the following problem.

$(\mathcal{P}_1)$ Find the velocity $\mathbf{u}$ and stress $\boldsymbol{\sigma}$ fields, satisfying
- equation of equilibrium

$$\sigma_{ij,j} = 0 \quad \text{in} \quad \Omega, \tag{2.1}$$

- incompressibility condition

$$\dot{\varepsilon}_V = 0 \quad \text{in} \quad \Omega, \tag{2.2}$$

- strain-rate - velocity relations

$$\dot{\varepsilon}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.3}$$

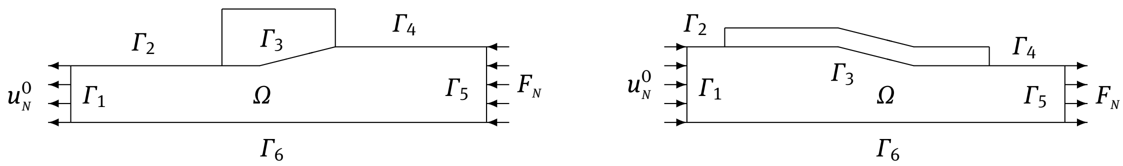

Fig. 1. Illustration of strip drawing and extrusion problems.

- yield criterion and flow rule

$$F(\sigma_{ij},\dot{\bar{\varepsilon}}) \equiv \bar{\sigma}^2 - \sigma_p^2(\dot{\bar{\varepsilon}}) = 0, \qquad \dot{e}_{ij} = \frac{3}{2}\frac{\dot{\bar{\varepsilon}}}{\bar{\sigma}} s_{ij}, \tag{2.4}$$

- boundary conditions

$$u_N = u_N^0, \quad \boldsymbol{\sigma}_T = \mathbf{0} \quad \text{on} \quad \Gamma_1, \tag{2.5}$$

$$\sigma_N = 0, \quad \boldsymbol{\sigma}_T = \mathbf{0} \quad \text{on} \quad \Gamma_2 \cup \Gamma_4, \tag{2.6}$$

$$\sigma_N = F_N, \quad \boldsymbol{\sigma}_T = \mathbf{0} \quad \text{on} \quad \Gamma_5, \tag{2.7}$$

$$u_N = 0, \quad \boldsymbol{\sigma}_T = \mathbf{0} \quad \text{on} \quad \Gamma_6, \tag{2.8}$$

$$u_N \le 0, \quad \sigma_N \le 0, \quad \text{such that} \quad u_N\sigma_N = 0 \quad \text{and} \tag{2.9}$$

$$\text{if} \quad |\boldsymbol{\sigma}_T(\mathbf{u})| < \tau_f(\mathbf{u}), \quad \text{then} \quad \mathbf{u}_T = \mathbf{0},$$
$$\text{if} \quad |\boldsymbol{\sigma}_T(\mathbf{u})| = \tau_f(\mathbf{u}), \quad \text{then} \quad \exists \quad \text{const.} \quad \lambda \ge 0,$$
$$\text{such that} \quad \mathbf{u}_T = -\lambda\boldsymbol{\sigma}_T(\mathbf{u}) \quad \text{on} \quad \Gamma_3.$$

Here and further the following notations and assumptions are also used: $\mathbf{n} = \{n_i\}$ is the unit normal vector outward to Γ; $\mathbf{u}_N = u_N\mathbf{n}$, $\mathbf{u}_T = \{u_{Ti}\}$ and $\boldsymbol{\sigma}_N = \sigma_N\mathbf{n}$, $\boldsymbol{\sigma}_T = \{\sigma_{Ti}\}$ are the normal and tangential components of the velocity and the stress vector on Γ, where $u_N = u_i n_i$, $u_{Ti} = u_i - u_N n_i$, $\sigma_N = \sigma_{ij}n_i n_j$, $\sigma_{Ti} = \sigma_{ij}n_j - \sigma_N n_i$; u_N^0 is the process velocity; F_N is the applied external force; $\tau_f(\mathbf{u}) = \mu_f(\mathbf{x})\bar{\sigma}_N(\mathbf{u})$ is the Coulomb friction bound, where $\mu_f(\mathbf{x})$ is the coefficient of friction, $\bar{\sigma}_N(\mathbf{u}) \ge 0$ is the averaged (mollified) normal stress [9], [13], [14]; $\sigma_p(\dot{\bar{\varepsilon}})$ is the strain-rate dependent, uniaxial yield limit, assumed increasing and almost everywhere differentiable function of $\dot{\bar{\varepsilon}}$, such that

$$\eta_1 \le \sigma_p'(\dot{\bar{\varepsilon}}) = \frac{d\sigma_p(\dot{\bar{\varepsilon}})}{d\dot{\bar{\varepsilon}}} \le \frac{\sigma_p(\dot{\bar{\varepsilon}})}{\dot{\bar{\varepsilon}}} \le \eta_2, \quad \forall\, \dot{\bar{\varepsilon}} \in [0,\infty), \tag{2.10}$$

where η_1, η_2 are positive constants.

3 Variational formulation and solvability

Let us denote by $\mathbf{V}$ and $\mathbf{H}$ the Hilbert spaces

$$\mathbf{V} = \left\{\mathbf{v}: \ \mathbf{v} \in (H^1(\Omega))^k, \ v_N = 0 \ \text{on} \ \Gamma_6\right\},$$
$$\mathbf{H} = (H^0(\Omega))^k \equiv (L_2(\Omega))^k,$$

such that $\mathbf{V} \subset \mathbf{H} \equiv \mathbf{H}' \subset \mathbf{V}'$, where $\mathbf{V}'$ and $\mathbf{H}'$ are their dual spaces and $(H^m(\Omega))^k$, $m \ge 0$ integer, are the Hilbert spaces of vector functions defined in Ω [9], [10]. Let us equip $\mathbf{V}$ with the following inner product and norm

$$(\mathbf{u},\mathbf{v})_V = \int_\Omega \left(\dot{\varepsilon}_{ij}(\mathbf{u})\dot{\varepsilon}_{ij}(\mathbf{v}) + u_i v_i\right) d\mathbf{x},$$
$$\|\mathbf{u}\|_V = (\mathbf{u},\mathbf{u})_V^{1/2}, \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{V},$$

equivalent to the usual $(H^1(\Omega))^k$ norm $\|\mathbf{v}\|_1 = \left\{\int_\Omega (v_{i,j}v_{i,j} + v_i v_i)d\mathbf{x}\right\}^{1/2}$ and introduce the following closed, convex subsets of $\mathbf{V}$

$$\mathbf{U} = \left\{\mathbf{v}: \ \mathbf{v} \in \mathbf{V}, \ v_N = u_N^0 \ \text{on} \ \Gamma_1\right\},$$

$$\mathbf{K} = \{\mathbf{v}: \ \mathbf{v} \in \mathbf{U}, \ v_{i,i} = 0 \ \text{in} \ \Omega, \ v_N \le 0 \ \text{on} \ \Gamma_3\}$$

and the space $H^{1/2}(\Gamma) \subset L_2(\Gamma)$ of traces $v_N = \boldsymbol{y}_0(\mathbf{v}) \cdot \mathbf{n}$ of all $\mathbf{v} \in \mathbf{V}$ on Γ, with norm $\|v_N\|_{1/2,\Gamma} = \inf_{\mathbf{v}\in\mathbf{V}}\left\{\|\mathbf{v}\|_V : \ v_N = \boldsymbol{y}_0(\mathbf{v})\cdot\mathbf{n}\right\}$, where $\boldsymbol{y}_0 : (H^1(\Omega))^k \to (H^{1/2}(\Gamma))^k$ is the trace operator.

Let us now suppose that $\mu_f(\mathbf{x}) \in L_\infty(\Gamma_3)$, $F_N(\mathbf{x}) \in L_2(\Gamma_5)$. Then, for $\mathbf{u} \in \mathbf{K}$ and all $\mathbf{v} \in \mathbf{K}$, multiplying (2.1) by $(\mathbf{v} - \mathbf{u})$, in the inner product sense, applying Green's formula and taking into account the boundary conditions, we obtain

$$\int_\Omega \sigma_{ij}(\mathbf{u})(\dot{\varepsilon}_{ij}(\mathbf{v}) - \dot{\varepsilon}_{ij}(\mathbf{u}))\mathrm{d}\mathbf{x} + \int_{\Gamma_3} \tau_f(\mathbf{u})|\mathbf{v}_T|\mathrm{d}\Gamma - \int_{\Gamma_3} \tau_f(\mathbf{u})|\mathbf{u}_T|\mathrm{d}\Gamma \ge \int_{\Gamma_5} F_N(v_N - u_N)\mathrm{d}\Gamma. \tag{3.1}$$

Introducing further, for all $\mathbf{w}, \mathbf{u}, \mathbf{v} \in \mathbf{U}$, the notations

$$a(\mathbf{w};\mathbf{u},\mathbf{v}) = \int_\Omega \frac{2}{3}\frac{\sigma_p(\mathbf{w})}{\dot{\bar{\varepsilon}}(\mathbf{w})}\dot{\varepsilon}_{ij}(\mathbf{u})\dot{\varepsilon}_{ij}(\mathbf{v})\mathrm{d}\mathbf{x},$$

$$j(\mathbf{u},\mathbf{v}) = \int_{\Gamma_3} \tau_f(\mathbf{u})|\mathbf{v}_T|\mathrm{d}\Gamma, \quad f(\mathbf{v}) = \int_{\Gamma_5} F_N v_N \mathrm{d}\Gamma, \tag{3.2}$$

we obtain the following variational problem, associated with the problem ($\mathcal{P}_1$).

($\mathcal{P}_2$) Find $\mathbf{u} \in \mathbf{K}$, satisfying

$$a(\mathbf{u};\mathbf{u},\mathbf{v}-\mathbf{u}) + j(\mathbf{u},\mathbf{v}) - j(\mathbf{u},\mathbf{u}) \geq f(\mathbf{v}-\mathbf{u}), \quad \forall \mathbf{v} \in \mathbf{K}. \tag{3.3}$$

Let us assume that the following relations hold between the hydrostatic pressure and the volume dilatation strain-rate in Ω and between the normal stress and velocity on Γ_3:

$$\sigma_H(\mathbf{u}) = \frac{\dot{\varepsilon}_V(\mathbf{u})}{d}, \qquad \sigma_N(\mathbf{u}) = -\frac{u_N^+}{d_N}, \tag{3.4}$$

where $d > 0$ and $d_N > 0$ are penalty parameters (small constants), $u_N^+ = sup(u_N, 0)$, and denote

$$b(\mathbf{u};\mathbf{u},\mathbf{v}) = a(\mathbf{u};\mathbf{u},\mathbf{v}) + \int_{\Omega} \frac{1}{d}\dot{\varepsilon}_V(\mathbf{u})\dot{\varepsilon}_V(\mathbf{v})\mathrm{d}\mathbf{x} + \int_{\Gamma_3} \frac{1}{d_N} u_N^+ v_N \mathrm{d}\Gamma. \tag{3.5}$$

Then we obtain the following penalty variational formulation of problem ($\mathcal{P}_2$).

($\tilde{\mathcal{P}}_2$) Find $\mathbf{u} \in \mathbf{U}$, satisfying the inequality

$$b(\mathbf{u};\mathbf{u},\mathbf{v}-\mathbf{u}) + j(\mathbf{u},\mathbf{v}) - j(\mathbf{u},\mathbf{u}) \geq f(\mathbf{v}-\mathbf{u}), \quad \forall \mathbf{v} \in \mathbf{U}. \tag{3.6}$$

Remark 3.1. First, it is clear that the solution of problem ($\tilde{\mathcal{P}}_2$) depends on the introduced penalty parameters. It can be further shown that the functionals constituting problem ($\tilde{\mathcal{P}}_2$), possess the following properties [13], [14]. For any fixed $\mathbf{w} \in \mathbf{U}$, $b(\mathbf{w};\mathbf{u},\mathbf{v}) : \mathbf{U} \times \mathbf{U} \to \mathbb{R}$ is a symmetric, bilinear form and such that

$$b(\mathbf{w};\mathbf{u},\mathbf{u}) \geq \beta_0 \|\mathbf{u}\|_V^2, \qquad |b(\mathbf{w};\mathbf{u},\mathbf{v})| \leq \beta_1 \|\mathbf{u}\|_V \|\mathbf{v}\|_V, \tag{3.7a}$$

where β_0 and β_1 are positive constants. Also, for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{U}$ there exist positive constants m and M, such that

$$b(\mathbf{v};\mathbf{v},\mathbf{v}-\mathbf{u}) - b(\mathbf{u};\mathbf{u},\mathbf{v}-\mathbf{u}) \geq m\|\mathbf{v}-\mathbf{u}\|_V^2, \tag{3.7b}$$

$$|b(\mathbf{v};\mathbf{v},\mathbf{u}) - b(\mathbf{u};\mathbf{u},\mathbf{u})| \leq M\|\mathbf{v}-\mathbf{u}\|_V \|\mathbf{u}\|_V. \tag{3.7c}$$

The functional $f(\mathbf{v}) : \mathbf{U} \to \mathbb{R}$ is linear and continuous and therefore there exists a positive constant β_2, such that for all $\mathbf{v} \in \mathbf{U}$

$$|f(\mathbf{v})| \leq \beta_2 \|\mathbf{v}\|_V. \tag{3.8}$$

For any fixed $\mathbf{w} \in \mathbf{U}$, $j(\mathbf{w},\mathbf{v}) : \mathbf{U} \to \mathbb{R}$ is proper, convex, continuous and nondifferentiable functional. Also, there exist positive constants c_f and c, depending on the friction coefficient, such that for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{U}$

$$0 \leq j(\mathbf{u},\mathbf{v}) \leq c_f \|\mathbf{u}\|_V \|\mathbf{v}\|_V, \tag{3.9a}$$

$$\left|j(\mathbf{u},\mathbf{w}) + j(\mathbf{w},\mathbf{v}) - j(\mathbf{u},\mathbf{v}) - j(\mathbf{w},\mathbf{w})\right| \leq c\|\mathbf{w}-\mathbf{u}\|_V \|\mathbf{w}-\mathbf{v}\|_V. \tag{3.9b}$$

Finally, if $\mathbf{u} \in \mathbf{U}$ is a solution of ($\mathcal{P}_2$), than it can be shown that there exists a positive constant c_0, such that

$$\|\mathbf{u}\|_V \leq c_0 |u_N^0|. \qquad \square \tag{3.10}$$

Remark 3.2. The functional $j(\mathbf{u},\mathbf{v})$ is nondifferentiable at $\mathbf{v}_T = \mathbf{0}$ and replacing it by a regularized, convex and Gâteaux differentiable functional $j_{d_T}(\mathbf{u},\mathbf{v})$ [13], [14], with Gâteaux derivative $j'_{d_T}(\mathbf{u},\mathbf{u})$ satisfying

$$\left\langle j'_{d_T}(\mathbf{u},\mathbf{u}), \mathbf{v}-\mathbf{u}\right\rangle \leq j_{d_T}(\mathbf{u},\mathbf{v}) - j_{d_T}(\mathbf{u},\mathbf{u}), \tag{3.11a}$$

$$\left\langle j'_{d_T}(\mathbf{u},\mathbf{v}) - j'_{d_T}(\mathbf{u},\mathbf{u}), \mathbf{v}-\mathbf{u}\right\rangle \geq 0, \tag{3.11b}$$

where $d_T > 0$ is the regularization parameter (small constant), we obtain the following regularized problem.

($\tilde{\mathcal{P}}'_2$) Find $\mathbf{u} \in \mathbf{U}$, satisfying the equation

$$b(\mathbf{u};\mathbf{u},\mathbf{v}-\mathbf{u}) + \left\langle j'_{d_T}(\mathbf{u},\mathbf{u}), \mathbf{v}-\mathbf{u}\right\rangle = f(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{U}, \tag{3.12}$$

the solution of which, at $d_T \to 0$, tends to the solution of problem ($\tilde{\mathcal{P}}_2$). $\square$

Let us now consider the following auxiliary problem, defining the direct secant-modulus method: find $\mathbf{u}_{n+1} \in \mathbf{U}$, $n = 0, 1, \ldots,$ satisfying for arbitrary $\mathbf{u}_0 \in \mathbf{U}$ the inequality

$$b(\mathbf{u}_n;\mathbf{u}_{n+1},\mathbf{v}-\mathbf{u}_{n+1}) + j(\mathbf{u}_n,\mathbf{v}) - j(\mathbf{u}_n,\mathbf{u}_{n+1}) \geq f(\mathbf{v}-\mathbf{u}_{n+1}), \quad \forall \mathbf{v} \in \mathbf{U}. \tag{3.13}$$

It can be proved, as in [13], [14], that the problem has a unique solution $\mathbf{u}_{n+1} \in \mathbf{U}$ and at sufficiently small coefficient of friction, the sequence of solutions $\{\mathbf{u}_n\}$, converges strongly to the unique solution $\mathbf{u} \in \mathbf{U}$ of problem ($\tilde{\mathcal{P}}_2$). Next we shall show that, when the penalty parameters d and d_N tend to zero, the solution of problem ($\tilde{\mathcal{P}}_2$) tends to the solution $\mathbf{u} \in \mathbf{K}$ of problem ($\mathcal{P}_2$). Assuming, without loss of generality, $d_N = c_N d$, where $c_N > 0$ is a constant, for all sufficiently small $d > 0$, we obtain a sequence $\{\mathbf{u}^d\}$ of solutions of problem ($\tilde{\mathcal{P}}_2$). Since $\{\mathbf{u}^d\}$ is bounded in $\mathbf{U} \subset \mathbf{V}$, then there exists a subsequence, also denoted $\{\mathbf{u}^d\}$, weakly convergent at $d \to 0$ to $\mathbf{u} \in \mathbf{K}$, such that the following result holds.

Theorem 3.1. The sequence of solutions of problem ($\tilde{\mathcal{P}}_2$) at $d \to 0$ tends to the unique solution of problem ($\mathcal{P}_2$).

Proof: For $\mathbf{u}^d \in \mathbf{U}$ and all $\mathbf{v} \in \mathbf{U}$, from (3.6) it follows that

$$\left[b(\mathbf{u}^d;\mathbf{u}^d,\mathbf{u}^d) + j(\mathbf{u}^d,\mathbf{u}^d)\right] \tag{3.14}$$

$$-\frac{1}{d}\int_{\Omega}\dot{\varepsilon}_V(\mathbf{u}^d)\dot{\varepsilon}_V(\mathbf{v})\mathrm{d}\mathbf{x} - \frac{1}{c_N d}\int_{\Gamma_3} u_N^{d+} v_N \mathrm{d}\Gamma \le a(\mathbf{u}^d;\mathbf{u}^d,\mathbf{v}) + j(\mathbf{u}^d,\mathbf{v}) - f(\mathbf{v}-\mathbf{u}^d).$$

Since the quantity in brackets in the left-hand side of (3.14) is nonnegative, we have that

$$-c_N\int_{\Omega}\dot{\varepsilon}_V(\mathbf{u}^d)\dot{\varepsilon}_V(\mathbf{v})\mathbf{dx} - \int_{\Gamma_3} u_N^{d+} v_N \mathrm{d}\Gamma \le c_N d\left[|a(\mathbf{u}^d;\mathbf{u}^d,\mathbf{v})| + j(\mathbf{u}^d,\mathbf{v}) + |f(\mathbf{v}-\mathbf{u}^d)|\right]. \tag{3.15}$$

The right-hand side of (3.15) is also bounded and from the weak convergence of $\mathbf{u}^d$ in $\mathbf{U} \subset \mathbf{V}$ it follows that

$$\dot{\varepsilon}_V(\mathbf{u}^d) \to \dot{\varepsilon}_V(\mathbf{u}) \text{ weakly in } H, \tag{3.16a}$$

$$u_N^{d+} \to u_N^{+} \text{ weakly in } H^{1/2}(\Gamma_3), \tag{3.16b}$$

Setting then $\mathbf{v} = \mathbf{v}_0 + \mathbf{u}^0 \in \mathbf{U}$ in (3.15), where $\mathbf{u}^0 \in \mathbf{U}$ is also such that $\dot{\varepsilon}_V(\mathbf{u}^0) = 0$ in Ω and $u_N^0 = \boldsymbol{y}_0(\mathbf{u}^0)\cdot\mathbf{n} = 0$ on Γ_3, for all $\pm\mathbf{v}_0 \in \mathbf{U}_0 = \{\mathbf{v}_0 : \mathbf{v}_0 \in \mathbf{V},\ v_{0N} = 0 \text{ on } \Gamma_1\}$, at $d \to 0$ we obtain

$$-c_N\int_{\Omega}\dot{\varepsilon}_V(\mathbf{u})\dot{\varepsilon}_V(\mathbf{v}_0)\mathrm{d}\mathbf{x} - \int_{\Gamma_3} u_N^{+} v_{0N} \mathrm{d}\Gamma \le 0, \tag{3.17a}$$

$$c_N\int_{\Omega}\dot{\varepsilon}_V(\mathbf{u})\dot{\varepsilon}_V(\mathbf{v}_0)\mathrm{d}\mathbf{x} + \int_{\Gamma_3} u_N^{+} v_{0N} \mathrm{d}\Gamma \le 0. \tag{3.17b}$$

Therefore we have that

$$c_N\int_{\Omega}\dot{\varepsilon}_V(\mathbf{u})\dot{\varepsilon}_V(\mathbf{v}_0)\mathrm{d}\mathbf{x} + \int_{\Gamma_3} u_N^{+} v_{0N} \mathrm{d}\Gamma \equiv 0, \tag{3.18a}$$

or

$$\int_{\Omega}\dot{\varepsilon}_V(\mathbf{u})\dot{\varepsilon}_V(\mathbf{v}_0)\mathrm{d}\mathbf{x} \equiv 0, \quad \int_{\Gamma_3} u_N^{+} v_{0N} \mathrm{d}\Gamma \equiv 0, \quad \forall \mathbf{v}_0 \in \mathbf{U}_0, \tag{3.18b}$$

and hence $\dot{\varepsilon}_V(\mathbf{u}) \equiv 0$ and $u_N \le 0$ on Γ_3, i.e $\mathbf{u} \in \mathbf{K}$. Let us now show that this $\mathbf{u}$ is a solution of problem $(\mathcal{P}_2)$. Since for all $\mathbf{w} \in \mathbf{K}$ and $\mathbf{u}^d \in \mathbf{U}$, we have

$$\begin{aligned} & a(\mathbf{w};\mathbf{w},\mathbf{w}-\mathbf{u}^d) + j(\mathbf{w},\mathbf{w}) - j(\mathbf{w},\mathbf{u}^d) - f(\mathbf{w}-\mathbf{u}^d) \\ & = a(\mathbf{w};\mathbf{w},\mathbf{w}-\mathbf{u}^d) + j(\mathbf{w},\mathbf{w}) - j(\mathbf{w},\mathbf{u}^d) - f(\mathbf{w}-\mathbf{u}^d) \\ & - \left[b(\mathbf{u}^d;\mathbf{u}^d,\mathbf{w}-\mathbf{u}^d) + j(\mathbf{u}^d,\mathbf{w}) - j(\mathbf{u}^d,\mathbf{u}^d) - f(\mathbf{w}-\mathbf{u}^d)\right] \\ & + \left[b(\mathbf{u}^d;\mathbf{u}^d,\mathbf{w}-\mathbf{u}^d) + j(\mathbf{u}^d,\mathbf{w}) - j(\mathbf{u}^d,\mathbf{u}^d) - f(\mathbf{w}-\mathbf{u}^d)\right] \\ & \ge (m-c)\|\mathbf{w}-\mathbf{u}^d\|_V^2 \ge 0, \end{aligned} \tag{3.19}$$

taking $d \to 0$ we obtain

$$a(\mathbf{w};\mathbf{w},\mathbf{w}-\mathbf{u}) + j(\mathbf{w},\mathbf{w}) - j(\mathbf{w},\mathbf{u}) \ge f(\mathbf{w}-\mathbf{u}). \tag{3.20}$$

Setting then $\mathbf{w} = \mathbf{u}+\theta(\mathbf{v}-\mathbf{u}),\ \theta \in [0,1],\ \forall \mathbf{v} \in \mathbf{K}$ we obtain

$$\begin{aligned} 0 \le & a(\mathbf{u}+\theta(\mathbf{v}-\mathbf{u});\mathbf{u}+\theta(\mathbf{v}-\mathbf{u}),\theta(\mathbf{v}-\mathbf{u})) + j(\mathbf{u}+\theta(\mathbf{v} \\ & -\mathbf{u}),\mathbf{u}+\theta(\mathbf{v}-\mathbf{u})) - j(\mathbf{u}+\theta(\mathbf{v}-\mathbf{u}),\mathbf{u}) - f(\theta(\mathbf{v}-\mathbf{u})) \\ \le & \theta a(\mathbf{u}+\theta(\mathbf{v}-\mathbf{u});\mathbf{u}+\theta(\mathbf{v}-\mathbf{u}),\mathbf{v}-\mathbf{u}) \\ & + (1-\theta)j(\mathbf{u}+\theta(\mathbf{v}-\mathbf{u}),\mathbf{u}) + \theta j(\mathbf{u}+\theta(\mathbf{v}-\mathbf{u}),\mathbf{v}) \\ & - j(\mathbf{u}+\theta(\mathbf{v}-\mathbf{u}),\mathbf{u}) - \theta f(\mathbf{v}-\mathbf{u}) \\ = & \theta\left[a(\mathbf{u}+\theta(\mathbf{v}-\mathbf{u});\mathbf{u}+\theta(\mathbf{v}-\mathbf{u}),\mathbf{v}-\mathbf{u})\right. \\ & \left. + j(\mathbf{u}+\theta(\mathbf{v}-\mathbf{u}),\mathbf{v}) - j(\mathbf{u}+\theta(\mathbf{v}-\mathbf{u}),\mathbf{u}) - f(\mathbf{v}-\mathbf{u})\right]. \end{aligned} \tag{3.21}$$

Hence for $\theta \neq 0$ we have that

$$\begin{aligned} & a(\mathbf{u}+\theta(\mathbf{v}-\mathbf{u});\mathbf{u}+\theta(\mathbf{v}-\mathbf{u}),\mathbf{v}-\mathbf{u}) + j(\mathbf{u}+\theta(\mathbf{v}-\mathbf{u}),\mathbf{v}) \\ & - j(\mathbf{u}+\theta(\mathbf{v}-\mathbf{u}),\mathbf{u}) \ge f(\mathbf{v}-\mathbf{u}) \end{aligned} \tag{3.22}$$

and taking $\theta \to 0$ we obtain

$$a(\mathbf{u};\mathbf{u},\mathbf{v}-\mathbf{u}) + j(\mathbf{u},\mathbf{v}) - j(\mathbf{u},\mathbf{u}) \ge f(\mathbf{v}-\mathbf{u}), \qquad \forall \mathbf{v} \in \mathbf{K}, \tag{3.23}$$

which is exactly problem $(\mathcal{P}_2)$. Finally, to prove the uniqueness of the solution of problem $(\mathcal{P}_2)$, we assume that $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{K}$ are two different solutions, i.e.

$$a(\mathbf{u}_1;\mathbf{u}_1,\mathbf{v}-\mathbf{u}_1) + j(\mathbf{u}_1,\mathbf{v}) - j(\mathbf{u}_1,\mathbf{u}_1) \ge f(\mathbf{v}-\mathbf{u}_1) \tag{3.24a}$$

$$a(\mathbf{u}_2;\mathbf{u}_2,\mathbf{v}-\mathbf{u}_2) + j(\mathbf{u}_2,\mathbf{v}) - j(\mathbf{u}_2,\mathbf{u}_2) \ge f(\mathbf{v}-\mathbf{u}_2). \tag{3.24b}$$

Setting $\mathbf{v} = \mathbf{u}_2$ in (3.24a) and $\mathbf{v} = \mathbf{u}_1$ in (3.24b), after adding the inequalities and rearranging we obtain

$$\begin{aligned} & j(\mathbf{u}_1,\mathbf{u}_2) + j(\mathbf{u}_2,\mathbf{u}_1) - j(\mathbf{u}_1,\mathbf{u}_1) - j(\mathbf{u}_2,\mathbf{u}_2) \\ & \ge a(\mathbf{u}_1;\mathbf{u}_1,\mathbf{u}_1-\mathbf{u}_2) - a(\mathbf{u}_2;\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2). \end{aligned} \tag{3.25}$$

Using Remark 3.1, we obtain that for a sufficiently small coefficient of friction, i.e. for $c < m$,

$$0 \ge (m-c)\|\mathbf{u}_1-\mathbf{u}_2\|_V^2 > 0, \tag{3.26}$$

which yields $\mathbf{u}_1 \equiv \mathbf{u}_2$. □

We shall further present a modification of the secant-modulus method, introducing an additional linearization. Let us introduce the functional

$$J_0(\mathbf{v}) = \int_{\Omega}\frac{1}{2}\int_0^{\dot{\bar{\varepsilon}}^2(\mathbf{v})} G(s)ds\mathrm{d}\mathbf{x} + \int_{\Omega}\frac{1}{2d}\dot{\varepsilon}_V^2(\mathbf{v})\mathrm{d}\mathbf{x} + \int_{\Gamma_3}\frac{1}{2d_N}(v_N^+)^2\mathrm{d}\Gamma,$$

$$G(\dot{\bar{\varepsilon}}^2(\mathbf{v})) = \frac{\sigma_p(\dot{\bar{\varepsilon}}(\mathbf{v}))}{\dot{\bar{\varepsilon}}(\mathbf{v})}, \quad \forall \mathbf{v} \in \mathbf{U} \tag{3.27}$$

and consider the problem: find $\mathbf{u}_z \in \mathbf{U}$, such that for fixed $\mathbf{z} \in \mathbf{U}$, holds

$$J(\mathbf{z}; \mathbf{u}_z) = \inf_{\mathbf{v}\in\mathbf{U}} J(\mathbf{z}; \mathbf{v}), \quad J(\mathbf{z}; \mathbf{v}) = J_0(\mathbf{v}) + j(\mathbf{z}, \mathbf{v}) - f(\mathbf{v}), \quad \mathbf{v} \in \mathbf{U}, \tag{3.28}$$

or equivalently

$$b(\mathbf{u}_z; \mathbf{u}_z, \mathbf{v} - \mathbf{u}_z) + j(\mathbf{z}, \mathbf{v}) - j(\mathbf{z}, \mathbf{u}_z) \geq f(\mathbf{v} - \mathbf{u}_z), \quad \forall \mathbf{v} \in \mathbf{U}. \tag{3.29}$$

It can be shown that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{U}$, the functional $J_0(\mathbf{u}) : \mathbf{U} \to \mathbb{R}$ is proper, continuous, strictly convex and Gâteaux differentiable, with strongly monotone and Lipschitz continuous Gâteaux derivative $J_0'(\mathbf{u})$, i.e.

$$\left\langle J_0'(\mathbf{u}), \mathbf{v} \right\rangle = b(\mathbf{u}; \mathbf{u}, \mathbf{v}), \tag{3.30a}$$

$$\left\langle J_0'(\mathbf{v}) - J_0'(\mathbf{u}), \mathbf{v} - \mathbf{u} \right\rangle \geq m\|\mathbf{v} - \mathbf{u}\|_V^2, \tag{3.30b}$$

$$\|J_0'(\mathbf{v}) - J_0'(\mathbf{u})\|_{V'} \leq M\|\mathbf{v} - \mathbf{u}\|_V, \tag{3.30c}$$

and satisfies the inequality

$$J_0(\mathbf{v}) - J_0(\mathbf{u}) \leq \frac{1}{2}\Big(b(\mathbf{u}; \mathbf{v}, \mathbf{v}) - b(\mathbf{u}; \mathbf{u}, \mathbf{u})\Big). \tag{3.30d}$$

The inequality (3.30d) has a geometrical interpretation and its validity can be shown by using the function

$$S(\dot{\bar{\varepsilon}}^2(\mathbf{v})) = \int_0^{\dot{\bar{\varepsilon}}^2(\mathbf{v})} G(s)\mathrm{d}s, \tag{3.31}$$

since then from (3.30d) we obtain

$$S(\dot{\bar{\varepsilon}}^2(\mathbf{v})) - S(\dot{\bar{\varepsilon}}^2(\mathbf{u})) \leq G(\dot{\bar{\varepsilon}}^2(\mathbf{u}))\Big(\dot{\bar{\varepsilon}}^2(\mathbf{v}) - \dot{\bar{\varepsilon}}^2(\mathbf{u})\Big), \tag{3.32}$$

which holds if $S(\dot{\bar{\varepsilon}}^2(\mathbf{v}))$ is a concave function, i.e. when $S''(\dot{\bar{\varepsilon}}^2(\mathbf{v})) = G'(\dot{\bar{\varepsilon}}^2(\mathbf{v})) \leq 0$, which was assumed in (2.10). Since finaly we have that, for fixed $\mathbf{z} \in \mathbf{U}$, the functional $J(\mathbf{z}; \mathbf{v}) : \mathbf{U} \to \mathbb{R}$ is proper, strictly convex, lower semicontinuous and coercive $\lim_{\|\mathbf{v}\|_V \to +\infty} J(\mathbf{z}; \mathbf{v}) = +\infty$, it follows that [8], there exists a unique element $\mathbf{u}_z \in \mathbf{U}$, satisfying (3.28), respectively (3.29). Let us now consider the following problem: find $\mathbf{u}_{n+1,m} \in \mathbf{U}$, $n = 0, 1, 2, \ldots$, satisfying for given $\mathbf{u}_{0,m} \in \mathbf{U}$, $m = 0, 1, 2, \ldots$, the inequality

$$\begin{aligned} &b(\mathbf{u}_{n,m}; \mathbf{u}_{n+1,m}, \mathbf{v} - \mathbf{u}_{n+1,m}) + j(\mathbf{u}_{0,m}, \mathbf{v}) - j(\mathbf{u}_{0,m}, \mathbf{u}_{n+1,m}) \\ &\geq f(\mathbf{v} - \mathbf{u}_{n+1,m}), \quad \forall \mathbf{v} \in \mathbf{U}, \end{aligned} \tag{3.33}$$

or the equivalent minimization problem: find $\mathbf{u}_{n+1,m} \in \mathbf{U}$, $n = 0, 1, 2, \ldots$, such that

$$J_n(\mathbf{u}_{0,m}; \mathbf{u}_{n+1,m}) = \inf_{\mathbf{v}\in\mathbf{U}} J_n(\mathbf{u}_{0,m}; \mathbf{v}),$$

$$J_n(\mathbf{u}_{0,m}; \mathbf{v}) = \left(\frac{1}{2}b(\mathbf{u}_{n,m}; \mathbf{v}, \mathbf{v}) + j(\mathbf{u}_{0,m}, \mathbf{v}) - f(\mathbf{v})\right). \tag{3.34}$$

The element $\mathbf{u}_{n+1,m} \in \mathbf{U}$, is a unique solution of problem (3.33), respectively (3.34) [8], as the following result holds.

Theorem 3.2. The sequence $\{\mathbf{u}_{n,m}\}$, defined by (3.33), or (3.34), converges strongly at $n \to 0$ to $\mathbf{u}_{0,m+1} \in \mathbf{U} \subset \mathbf{V}$, the unique solution of problems (3.28), (3.29) for fixed $\mathbf{z} = \mathbf{u}_{0,m}$. At $m \to 0$ and at sufficiently small coefficient of friction, the sequence of solutions $\{\mathbf{u}_{0,m}\}$ converges strongly to the unique solution $\mathbf{u} \in \mathbf{U} \subset \mathbf{V}$ of problem $(\tilde{\mathcal{P}}_2)$.

Proof: Setting in (3.30d) $\mathbf{v} = \mathbf{u}_{n+1,m}$ and $\mathbf{u} = \mathbf{u}_{n,m}$ and using (3.34) we get

$$J(\mathbf{u}_{0,m}; \mathbf{u}_{n+1,m}) \leq J_n(\mathbf{u}_{0,m}; \mathbf{u}_{n+1,m}) \leq J(\mathbf{u}_{0,m}; \mathbf{u}_{n,m}), \tag{3.35}$$

which implies that the numerical sequences $\{J(\mathbf{u}_{0,m}; \mathbf{u}_{n,m})\}$ and $\{J_n(\mathbf{u}_{0,m}; \mathbf{u}_{n+1,m})\}$ are decreasing and bounded from below and therefore they are convergent to one and the same limit. Then the sequence $\{\mathbf{u}_{n,m}\}$, generated by (3.33), or (3.34) is such that

$$\begin{aligned} &\frac{\beta_0}{2}\|\mathbf{u}_{n+1,m} - \mathbf{u}_{n,m}\|_V^2 \leq \frac{1}{2}b(\mathbf{u}_{n,m}; \mathbf{u}_{n+1,m} - \mathbf{u}_{n,m}, \mathbf{u}_{n+1,m} - \mathbf{u}_{n,m}) \\ &= J(\mathbf{u}_{0,m}; \mathbf{u}_{n,m}) - J_n(\mathbf{u}_{0,m}; \mathbf{u}_{n+1,m}) \\ &- \Big(b(\mathbf{u}_{n,m}; \mathbf{u}_{n+1,m}, \mathbf{u}_{n,m} - \mathbf{u}_{n+1,m}) + j(\mathbf{u}_{0,m}, \mathbf{u}_{n,m}) \\ &- j(\mathbf{u}_{0,m}, \mathbf{u}_{n+1,m}) - f(\mathbf{u}_{n,m} - \mathbf{u}_{n+1,m})\Big) \\ &\leq J(\mathbf{u}_{0,m}; \mathbf{u}_{n,m}) - J_n(\mathbf{u}_{0,m}; \mathbf{u}_{n+1,m}), \end{aligned}$$

and therefore

$$\lim_{n\to\infty} \|\mathbf{u}_{n,m} - \mathbf{u}_{n+1,m}\|_V = 0, \tag{3.36}$$

implying that $\{\mathbf{u}_{n,m}\} \in \mathbf{U}$ is a fundamental sequence in $\mathbf{V}$, i.e. there exists some element $\mathbf{u}_{0,m+1} \in \mathbf{U} \subset \mathbf{V}$, $\mathbf{u}_{0,m+1} = \lim_{n\to\infty}\{\mathbf{u}_{n,m}\}$, since $\mathbf{U}$ is closed in $\mathbf{V}$. This element is a solution of (3.33), or (3.34) at $n \to \infty$, since

$$\begin{aligned} &b(\mathbf{u}_{0,m+1}; \mathbf{u}_{0,m+1}, \mathbf{v}) + j(\mathbf{u}_{0,m}, \mathbf{v}) - f(\mathbf{v}) \\ &= \lim_{n\to\infty} \big(b(\mathbf{u}_{n,m}; \mathbf{u}_{n+1,m}, \mathbf{v}) + j(\mathbf{u}_{0,m}, \mathbf{v}) - f(\mathbf{v})\big) \\ &\geq \liminf_{n\to\infty} \big(b(\mathbf{u}_{n,m}; \mathbf{u}_{n+1,m}, \mathbf{u}_{n+1,m}) + j(\mathbf{u}_{0,m}, \mathbf{u}_{n+1,m}) - f(\mathbf{u}_{n+1,m})\big) \\ &\geq b(\mathbf{u}_{0,m+1}; \mathbf{u}_{0,m+1}, \mathbf{u}_{0,m+1}) + j(\mathbf{u}_{0,m}, \mathbf{u}_{0,m+1}) - f(\mathbf{u}_{0,m+1}). \end{aligned}$$

At $n \to \infty$, from

$$\begin{aligned} &m\|\mathbf{u}_{0,m+1} - \mathbf{u}_{n,m}\|_V^2 \leq b(\mathbf{u}_{0,m+1}; \mathbf{u}_{0,m+1}, \mathbf{u}_{0,m+1} - \mathbf{u}_{n,m}) \\ &- b(\mathbf{u}_{n,m}; \mathbf{u}_{n,m}, \mathbf{u}_{0,m+1} - \mathbf{u}_{n,m}) \\ &+ b(\mathbf{u}_{n,m}; \mathbf{u}_{n+1,m}, \mathbf{u}_{n+1,m} - \mathbf{u}_{n,m}) \\ &- b(\mathbf{u}_{n,m}; \mathbf{u}_{n+1,m}, \mathbf{u}_{n+1,m} - \mathbf{u}_{n,m}) \\ &+ j(\mathbf{u}_{0,m}, \mathbf{u}_{0,m+1}) - j(\mathbf{u}_{0,m}, \mathbf{u}_{0,m+1}) \end{aligned}$$

$$
\begin{aligned}
&+ j(\mathbf{u}_{0,m}, \mathbf{u}_{n,m}) - j(\mathbf{u}_{0,m}, \mathbf{u}_{n,m}) \\
&+ j(\mathbf{u}_{0,m}, \mathbf{u}_{n+1,m}) - j(\mathbf{u}_{0,m}, \mathbf{u}_{n+1,m}) \\
&+ f(\mathbf{u}_{n,m}) - f(\mathbf{u}_{n,m}) + f(\mathbf{u}_{n+1,m}) - f(\mathbf{u}_{n+1,m}) + f(\mathbf{u}_{0,m+1}) \\
&- f(\mathbf{u}_{0,m+1}) = -\Big(b(\mathbf{u}_{0,m+1}; \mathbf{u}_{0,m+1}, \mathbf{u}_{n,m} - \mathbf{u}_{0,m+1}) \\
&+ j(\mathbf{u}_{0,m}, \mathbf{u}_{n,m}) - j(\mathbf{u}_{0,m}, \mathbf{u}_{0,m+1}) - f(\mathbf{u}_{n,m} - \mathbf{u}_{0,m+1}) \Big) \\
&- \Big(b(\mathbf{u}_{n,m}; \mathbf{u}_{n+1,m}, \mathbf{u}_{0,m+1} - \mathbf{u}_{n+1,m}) + j(\mathbf{u}_{0,m}, \mathbf{u}_{0,m+1}) \\
&- j(\mathbf{u}_{0,m}, \mathbf{u}_{n+1,m}) - f(\mathbf{u}_{0,m+1} - \mathbf{u}_{n+1,m}) \Big) \\
&+ b(\mathbf{u}_{n,m}; \mathbf{u}_{n+1,m}, \mathbf{u}_{n,m} - \mathbf{u}_{n+1,m}) + j(\mathbf{u}_{0,m}, \mathbf{u}_{n,m}) \\
&- j(\mathbf{u}_{0,m}, \mathbf{u}_{n+1,m}) - f(\mathbf{u}_{n,m} - \mathbf{u}_{n+1,m}) \\
&+ b(\mathbf{u}_{n,m}; \mathbf{u}_{n+1,m} - \mathbf{u}_{n,m}, \mathbf{u}_{0,m+1} - \mathbf{u}_{n,m}) \\
&\le b(\mathbf{u}_{n,m}; \mathbf{u}_{n+1,m}, \mathbf{u}_{n,m} - \mathbf{u}_{n+1,m}) + j(\mathbf{u}_{0,m}, \mathbf{u}_{n,m}) \\
&- j(\mathbf{u}_{0,m}, \mathbf{u}_{n+1,m}) - f(\mathbf{u}_{n,m} - \mathbf{u}_{n+1,m}) \\
&+ b(\mathbf{u}_{n,m}; \mathbf{u}_{n+1,m} - \mathbf{u}_{n,m}, \mathbf{u}_{0,m+1} - \mathbf{u}_{n,m}) \\
&\le C_0 \|\mathbf{u}_{n,m} - \mathbf{u}_{n+1,m}\|_V \Big(1 + C_1 \|\mathbf{u}_{0,m+1} - \mathbf{u}_{n,m}\|_V \Big),
\end{aligned}
$$

where C_0 and C_1 are positive constants, we obtain that strong convergence also holds. Since further, the sequence $\{\mathbf{u}_{0,m}\} \in \mathbf{U}$ of solutions of (3.33), or (3.34) is bounded in $\mathbf{V}$ and since $\mathbf{U}$ is weakly closed in $\mathbf{V}$, there exists a subsequence, denoted $\{\mathbf{u}_m\} \in \mathbf{U}$, which converges weakly to some $\mathbf{u} \in \mathbf{U}$. Repeating the above analysis, it can be shown that at $m \to \infty$ strong convergence also holds and that this element is a solution of problem $(\tilde{\mathcal{P}}_2)$, unique at sufficiently small coefficient of friction [13, 14], which completes the proof.

Remark 3.3. The modified secant-modulus method is obviously slower than the direct method, but since the initial solutions are successively defined closer to the exact solution, the numerical experiments show less fluctuations around it, which makes its application useful. □

4 Finite element approximation, computational algorithm and results

Let $\mathcal{C}_h$ be a regular partition of $\bar{\Omega} = \cup_{K \in \mathcal{C}_h} K$ into finite elements K and construct the finite element spaces

$$
\mathbf{V}_h = \{\mathbf{v}^h : \ \mathbf{v}^h \in \mathbf{V} \cap (C^0(\bar{\Omega}))^k, \ \mathbf{v}^h|_K = \hat{\mathbf{v}}^h \circ F_K^{-1}, \ \hat{\mathbf{v}}^h \in (Q_l(\hat{K}))^k\},
$$

where h is the mesh parameter approaching zero, $F_K : \hat{K} \to K$, $F_K \in (Q_l(\hat{K}))^k$ is the isoparametric transformation, $\hat{K}$ is the reference element and $(Q_l(\hat{K}))^k$ is the space of polynomials on $\hat{K}$ of order not greater then $l = 1, 2$, in each variable. Let us also suppose that the standard approximation properties of $\mathbf{V}_h$ hold [9]:

$$
\forall \mathbf{v} \in (H^m(\Omega))^k \cap \mathbf{V}, \ \exists \mathbf{v}^h \in \mathbf{V}_h, \ \text{ such that}
$$

$$
\|\mathbf{v} - \mathbf{v}^h\|_s \le c_\Omega h^r \|\mathbf{v}\|_m, \quad r = \min\{l + 1 - s, m - s\}, \quad m \ge s,
$$

$$
\text{and if } \ \boldsymbol{y}_0(\mathbf{v}) \in (H^{m-1/2}(\Gamma))^k, \ \text{ then}
$$

$$
\|\boldsymbol{y}_0(\mathbf{v}) - \boldsymbol{y}_0(\mathbf{v}^h)\|_{s-1/2,\Gamma} \le c_\Gamma h^r \|\boldsymbol{y}_0(\mathbf{v})\|_{m-1/2,\Gamma},
$$

where $c_\Omega > 0$ and $c_\Gamma > 0$ are independent of h and $\mathbf{v}$ positive constants. Then from problem $(\tilde{\mathcal{P}}_2)$ we obtain the following finite-dimensional problem.

$(\tilde{\mathcal{P}}_2^h)$ Find $\mathbf{u}^h \in \mathbf{U}_h \subset \mathbf{V}_h$, satisfying for all $\mathbf{v}^h \in \mathbf{U}_h$ the inequality

$$
b\left(\mathbf{u}^h; \mathbf{u}^h, \mathbf{v}^h - \mathbf{u}^h\right) + j(\mathbf{u}^h, \mathbf{v}^h) - j(\mathbf{u}^h, \mathbf{u}^h) \ge f(\mathbf{v}^h - \mathbf{u}^h), \quad \forall \mathbf{v}^h \in \mathbf{U}_h. \tag{4.1}
$$

Remark 4.1. The considered problems are usually defined in nonconvex domains with corners, such as L-shaped, or trapeziform. Therefore the following order of regularity of the solutions could be expected $\mathbf{u} \in (H^\alpha(\Omega))^k \cap \mathbf{U}$, $\frac{5}{3} - \epsilon \le \alpha \le 3 - \epsilon$, $\epsilon > 0$ arbitrary, and the following a priori finite element error estimate holds [13].

Theorem 4.1. Let $\mathbf{u} \in (H^\alpha(\Omega))^k \cap \mathbf{U}$ and $\mathbf{u}^h \in \mathbf{U}_h$ be the solutions of problem $(\tilde{\mathcal{P}}_2)$ and problem $(\tilde{\mathcal{P}}_2^h)$ respectively and let also $\sigma_N(\mathbf{u}) \in H^{\alpha-3/2}(\Gamma_3)$ and $\boldsymbol{\sigma}_T(\mathbf{u}) \in (H^{\alpha-3/2}(\Gamma_3))^k$. Then there exists a positive constant C, independent of h, such that

$$
\|\mathbf{u} - \mathbf{u}^h\|_V \le C h^r \|\mathbf{u}\|_\alpha, \quad r = \min\{l, \alpha - 1\}. \qquad \square \tag{4.2}
$$

Regularizing the nondifferentiable terms in $(\tilde{\mathcal{P}}_2^h)$ and applying the modified secant-modulus method, we obtain the following problem.

$((\tilde{\mathcal{P}}_2^h)'_{nm})$ Find $\mathbf{u}^h_{n+1,m} \in \mathbf{U}_h$, $m, n = 0, 1, 2, \ldots$, satisfying for arbitrary initial $\mathbf{u}^h_{0,0} \in \mathbf{U}_h$ and every $\mathbf{v}^h \in \mathbf{U}_h$ the equation

$$
b(\mathbf{u}^h_{n,m}; \mathbf{u}^h_{n+1,m}, \mathbf{v}^h - \mathbf{u}^h_{n+1,m}) + \langle j'_{d_T}(\mathbf{u}^h_{0,m}, \mathbf{u}^h_{n+1,m}), \mathbf{v}^h - \mathbf{u}^h_{n+1,m} \rangle = f(\mathbf{v}^h - \mathbf{u}^h_{n+1,m}), \tag{4.3}
$$

until $\|\mathbf{u}^h_{n+1,m} - \mathbf{u}^h_{n,m}\| / \|\mathbf{u}^h_{n+1,m}\| < \delta$, where $\|\cdot\|$ is a vector norm and δ is the accuracy tolerance. This problem defines the following algorithm.

Algorithm: Find $\{u^h_{n+1,m}\}$, $m, n = 0, 1, 2, \ldots$, satisfying for arbitrary initial $\{u^h_{0,0}\}$ the system of equations

$$
\mathbf{K}(u^h_{n,m}, u^h_{0,m})\{u^h_{n+1,m}\} = \mathbf{F}(u^h_{n,m}, u^h_{0,m}), \tag{4.4}
$$

until $\|u^h_{n+1,m} - u^h_{n,m}\| / \|u^h_{n+1,m}\| < \delta$.

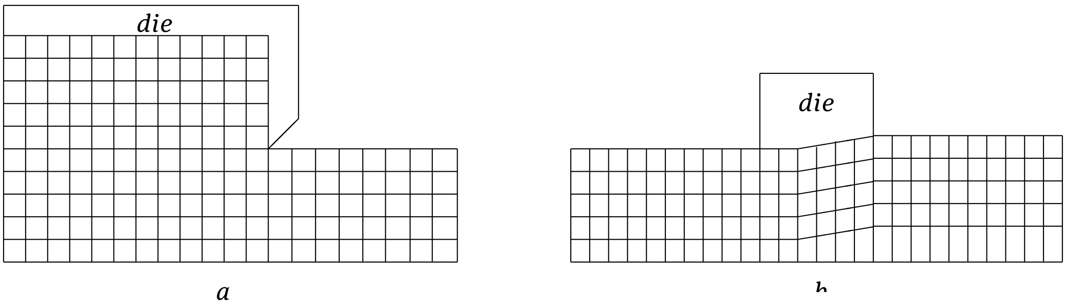

Fig. 2. Finite element meshes for the example extrusion and drawing pr

Here **K** and **F** are velocity dependent stiffness matrix and load vector, obtained by using complete Gauss integration, except for the hydrostatic pressure - volumetric strain rate terms, for which complete or reduced Gauss integration is used. The vector of nodal velocities is denoted by $\{u^h_{n+1,m}\}$. The algorithm is applied for solving two dimensionless, example extrusion [3, 13] and drawing [4, 14] problems. The following yield limit expression, satisfying (2.10) is used:

$$\sigma_p(\dot{\bar{\varepsilon}}) = \begin{cases} \frac{\sigma_p(\dot{\bar{\varepsilon}}_1)}{\dot{\bar{\varepsilon}}_1}\dot{\bar{\varepsilon}} & \text{if } \dot{\bar{\varepsilon}} \in [0, \dot{\bar{\varepsilon}}_1], \\ A\dot{\bar{\varepsilon}}^{\alpha} & \text{if } \dot{\bar{\varepsilon}} \in [\dot{\bar{\varepsilon}}_1, \dot{\bar{\varepsilon}}_2], \\ \frac{\sigma_p(\dot{\bar{\varepsilon}}_2)}{\dot{\bar{\varepsilon}}_2}\dot{\bar{\varepsilon}} & \text{if } \dot{\bar{\varepsilon}} \in [\dot{\bar{\varepsilon}}_2, \infty), \end{cases} \tag{4.5}$$

where $A > 0$, $\alpha \in (0, 1]$, $\dot{\bar{\varepsilon}}_1$ and $\dot{\bar{\varepsilon}}_2$ are material constants, depending on the process conditions. In both examples $\alpha = 10^{-3}$, $\dot{\bar{\varepsilon}}_1 = 10^{-3}$, $\dot{\bar{\varepsilon}}_2 = 10^3$ are used. The choice of the regular finite element meshes Fig. 2, the values of the regularization and penalty constants, mesh dependent for the discrete penalty method, has been made on the base of computational experiments. In both examples, the following value of the regularization constant $d_T = 10^{-6}$ and averaging of the effective strain-rates, contact and hydrostatic pressures and friction stresses at finite element nodes (centers) is used. The computational experiments show that the algorithm converges for up to 35 iterations, with accuracy $\delta = 10^{-4}$ and depending on the used friction coefficient, which is about twice the number of iterations performed by the direct secant-modulus method.

Example 1 ([3, 13]): A two-dimensional workpiece with length 20, initial and final thicknesses 10 and 5 respectively, is extruded through a square die with ram velocity $u^0_N = 1$, $F_N = 0$. The following values of the material and penalty constants and friction coefficients are used: $A = \sqrt{3}$, $d = 0.001$ and $d_N = 10^{-6}$; $\mu_f = 0.0$ and $\mu_f = 0.1$.

Nine-noded biquadratic, isoparametric finite elements, with reduced 2×2-Gauss integration for hydrostatic pressure - volumetric strain-rate terms, are used. The distributions of the velocity vectors and the effective strain-

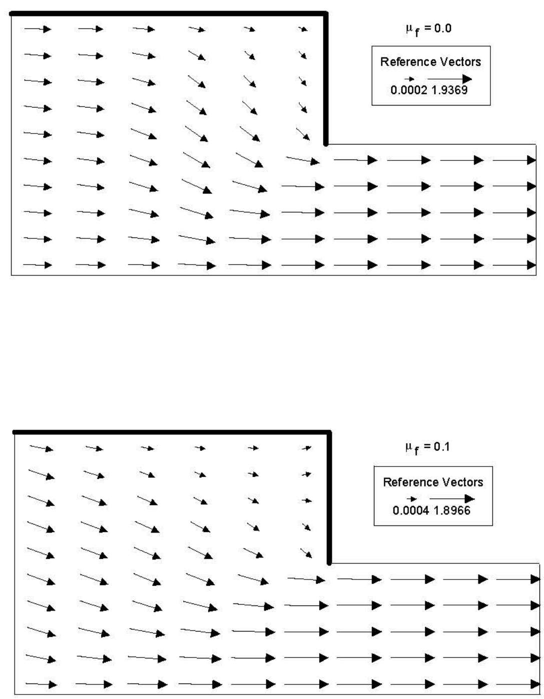

Fig. 3. Velocity vectors at friction coefficients $\mu_f = 0.0$ and $\mu_f = 0.1$.

rates, for the two friction coefficients, are illustrated on Fig. 3 and Fig. 4 and very well demonstrate the influence of the friction. Since the frictionless, analytical slip-line solution for the extrusion pressure is $p = 2.6\tau_p$, where $p = |\sigma_H|$ and $\tau_p = \sigma_p/\sqrt{3}$ is the shear yield limit [1], we have that $p = 2.6$ in our case, which is closely approached by the computed average hydrostatic pressure values $p_{av} = 2.65$ and $p_{av} = 2.62$ for the two friction coefficients.

Example 2 ([4], [14]): A two-dimensional workpiece with length 7.8, initial and final thicknesses 1.625 and 1.5 respectively, is drawn with velocity $u^0_N = 1000$, $F_N = 0$, through a die with semi-angle 6°. The following values of the material and penalty constants and friction coefficients are used: $A = 100\sqrt{3}$, $d = 0.005$ and $d_N = 0.01$; $\mu_f = 0.0$, $\mu_f = 0.2$ and $\mu_f = 0.4$.

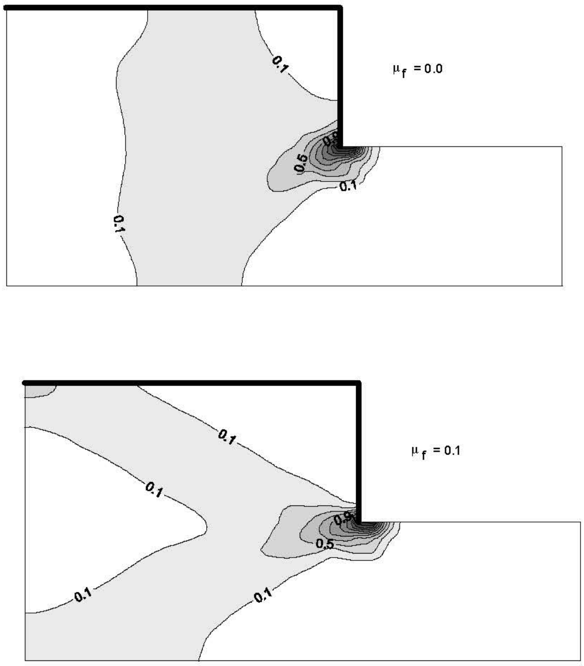

Fig. 4. Equivalent strain-rates at friction coefficients $\mu_f = 0.0$ and $\mu_f = 0.1$.

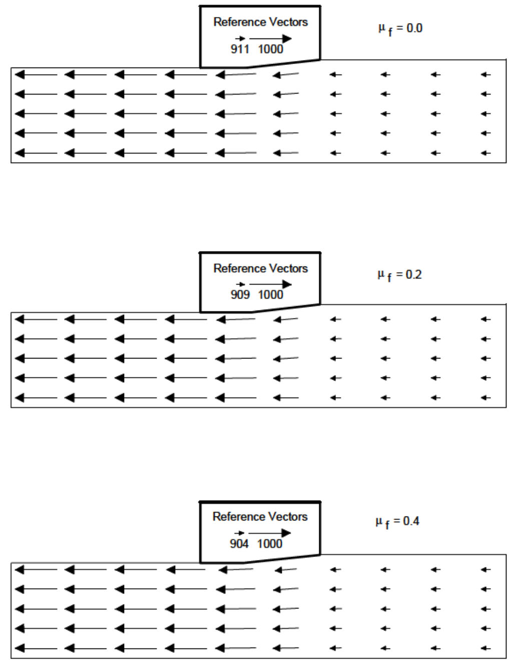

Fig. 5. Velocity vectors at friction coefficients $\mu_f = 0.0$, $\mu_f = 0.2$ and $\mu_f = 0.4$.

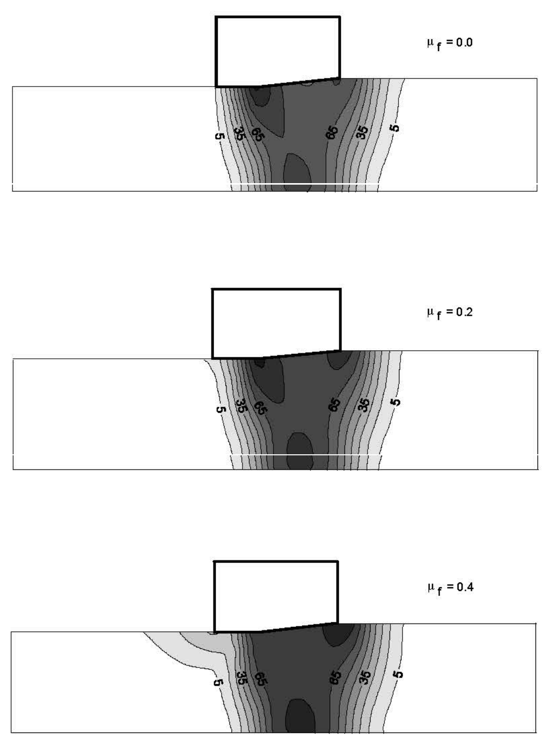

Fig. 6. Equivalent strain-rates at friction coefficients $\mu_f = 0.0$, $\mu_f = 0.2$ and $\mu_f = 0.4$.

Here four-noded bilinear, isoparametric finite elements, with complete integration, are used. The distributions of the velocity vectors and the effective strain-rates, for the three friction coefficients, are illustrated on Fig. 5 and Fig. 6 and show the influence of the friction. The analytical, slip-line die-pressure solution, for the frictionless case of this example, is given by the expression $q = 2.5\tau_p$, where $q = |\sigma_N|$ is the normal pressure and $\tau_p = \sigma_p/\sqrt{3}$ is the shear yield limit [1]. This gives $q = 250$, which is closely approached by the average normal pressure $q_{av} = 243$, computed for $\mu_f = 0.0$. The average normal pressures, computed for the other friction coefficients are correspondingly $q_{av} = 255$ and $q_{av} = 273$, as the computational experiments show that close to the exact solution are also those obtained for small friction coefficients, which corresponds to the experimental observations outlined in [1].

5 Conclusion

In this work a variational inequality approach, for analysis of a class of unilateral contact problems with friction in the flow theory of plasticity, describing steady-state metal-forming processes, is proposed. The convergence of a modified secant-modulus method is proved and existence and uniqueness results are obtained. An algorithm, based on the finite element and modified secant modulus method, is proposed by which two example problems are solved. The computational experiments support the theoretical results and show the applicability of the proposed method of approach.

References

[1] R. Hill, The Mathematical Theory of Plasticity, Oxford University Press, Oxford, 1950.

[2] L.M. Kachanov, Fundamentals of the Theory of Plasticity, North-Holland, Amsterdam, 1971.

[3] O.C. Zienkiewicz, P.C. Jain, E. Onate, Flow of solids during forming and extrusion: Some aspects of numerical solutions, *International Journal of Solids and Structures*, 14(1), 15-38, 1978.

[4] J. van der Lugt, J. Huetink, Thermal mechanically coupled finite element analysis in metal-forming processes, *Computer Methods in Applied Mechanics and Engineering*, 54(2), 145-160, 1986.

[5] S. Kobayashi, S.-I. Oh, T. Altan, Metal Forming and the Finite Element Method, Oxford University Press, Oxford, 1989.

[6] G. Duvaut, J.-L. Lions, Inequalities in Mechanics and Physics, Springer-Verlag, Berlin, 1976.

[7] J. Nečas, I. Hlavaček, Mathematical Theory of Elastic and Elasto-Plastic Bodies: An Introduction, Elsevier, Amsterdam, 1981.

[8] G. Glowinski, Numerical Methods for Nonlinear Variational Problems, Springer-Verlag, Berlin, 1984.

[9] N. Kikuchi, J.T. Oden, Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods, SIAM, Philadephia PA, 1988.

[10] W. Han, M. Sofonea, Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity, AMS-Intl. Press, Providence RI, 2002.

[11] M. Shillor, M. Sofonea, J.J. Telega, Models and Variational Analysis of Quasistatic Contact, Lecture Notes in Physics, Vol. 655, Springer, Berlin, 2004.

[12] T.A. Angelov, The Kachanov method for some metal forming problems, *In: Nonsmooth/Nonconvex Mechanics with Applications in Engineering, Proc. Int. Conf. in Memoriam of Prof. P.D. Panagiotopoulos*, ZITI, Thessaloniki, 425-430, 2002.

[13] T.A. Angelov, Modelling and analysis of a class of metal-forming problems, *Advances in Applied Mathematics and Mechanics*, 2(6), 722-745, 2010.

[14] T.A. Angelov, Modelling and numerical approach to a class of metal-forming problems - quasi-steady case, *Mathematical Methods in the Applied Sciences*, 34(11), 1330-1338, 2011.