

# Photonic Topological Lifshitz Interfaces: supplemental document

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## A. Gaussian curvature and gap dynamics of IFS

### A.1. Gaussian curvature of IFS

In differential geometry, the Gaussian curvature of a surface is defined as  $K = \kappa_1 \kappa_2$ , where  $\kappa_1$  and  $\kappa_2$  are the principal curvatures at the given point. In the main text, we focus on the IFSs  $k_x^2/\varepsilon_y + k_y^2/\varepsilon_x + k_z^2/\varepsilon_z = k_0^2$ , especially for uniaxial anisotropic materials, namely  $\varepsilon_x \neq \varepsilon_y = \varepsilon_z$ . The Gaussian curvature of an elliptical IFS ( $\varepsilon_x > 0, \varepsilon_y = \varepsilon_z > 0$ , Fig. S1a) is

$$K = k_0^2 \frac{\varepsilon_x \varepsilon_y \varepsilon_z}{[(\varepsilon_x \varepsilon_z / \varepsilon_y)k_x^2 + (\varepsilon_y \varepsilon_z / \varepsilon_x)k_y^2 + (\varepsilon_x \varepsilon_y / \varepsilon_z)k_z^2]^2}, \quad (S1)$$

while the curvature for a hyperbolic IFS ( $\varepsilon_x < 0$  and  $\varepsilon_y = \varepsilon_z > 0$ , Fig. S1b) is

$$K = \frac{k_0^2}{\varepsilon_x \varepsilon_y \varepsilon_z} \frac{1}{(k_x^2 / \varepsilon_y^2 + k_y^2 / \varepsilon_x^2 + k_z^2 / \varepsilon_z^2)^2}. \quad (S2)$$

In the main text, we apply a geometric indicator  $G = K/|K|$  to represent the geometrical topology of IFSs.

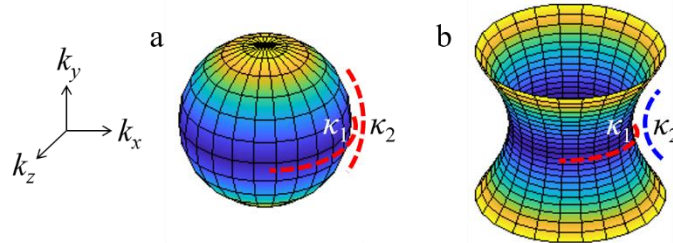


Fig. S1. Schematics of principal curvatures  $\kappa_1, \kappa_2$  for (a) elliptical and (b) hyperbolic IFSs.

### A.2. Wavevector gap of IFS

In the main text, we introduce the concept of wavevector gap ( $\Delta k_x$ ) for the  $x$ -propagating interface states in the  $k_x$ - $k_y$  two-dimensional (2D) IFC diagrams (Fig. 1b). Analogous to the energy gap in a  $k$ - $\omega$  dispersion diagram describing the forbidden range of the frequency  $\omega$ , each IFS at a constant frequency has a forbidden regime of propagating states. The position and magnitude of  $\Delta k_x$  depends on the material constants. For example, the  $\Delta k_x$  of an elliptical IFS spans the outside of the ellipse along the  $k_x$ -axis:  $\Delta k_x: \{k_x | k_x < -k_e, k_x > k_e\}$ , where  $k_e = |\varepsilon_y|k_0$  is the magnitude of the vertex of the elliptical IFC in the  $k_x$ -axis. The type-I hyperbolic IFS has a single span of  $\Delta k_x: \{(k_x | -k_h < k_x < k_h)\}$  with  $k_h = |\varepsilon_y|k_0$ , while no  $\Delta k_x$  for the type-II hyperbolic IFS. For metallic media, the  $\Delta k_x$  spans all the range of  $k_x$  in consideration except for the origin point.

For an interface,  $\Delta k_x$  is defined as the overlap of  $\Delta k_x$  in environmental layers. The propagation constant of the pure T-spin state  $\beta_{\text{pure}}$  lies inside the gap  $\Delta k_x: \{k_x | k_e < k_x < k_h\}$ , where  $k_h = |\varepsilon_{1y}|k_0$  and  $k_e = |\varepsilon_{2y}|k_0$  are the magnitude of the vertices of the hyperbolic-I and the elliptical IFCs, respectively (Fig. S2a). The material constants of this example structure for the pure T-spin states are  $\varepsilon_{1x} = -0.5$ ,  $\varepsilon_{1y} = 2.5$ ,  $\varepsilon_{2x} = 0.5$ , and  $\varepsilon_{2y} = 1.5$ . In the metal-dielectric interface, the propagation constant of the mixed T-spin state also lies inside the gap  $\Delta k_x: \{k_x | k_x < -k_e, k_x > k_e\}$ , where  $k_e = |\varepsilon_y|k_0$  (Fig. S2b). We choose an example structure for the excitation of mixed T-spin waves with  $\varepsilon_{1x} = -2.0$ ,  $\varepsilon_{1y} = -2.0$ ,  $\varepsilon_{2x} = 0.5$ ,  $\varepsilon_{2y} = 1.5$  (Fig. S2b).

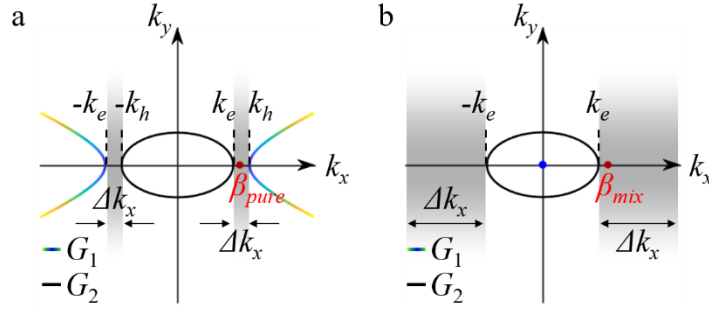


Fig. S2. Wavevector gaps of IFCs for (a) pure T-spin states and (b) mixed T-spin states.

### B. Wave quantities for bound interface states

In the main text, we focus on a transverse magnetic (TM) mode that propagates along the interface ( $x$ -axis) between two nonmagnetic ( $\mu = \mu_0$ ) anisotropic media with permittivity tensors  $\epsilon_1$  and  $\epsilon_2$ . For each medium, we assume uniaxial anisotropic medium for the formulation of the permittivity tensor with  $\epsilon_y = \epsilon_z$  (Fig. S3).

#### B.1. Dispersion relation of the $x$ -propagating interface state

The dispersion relation of this TM interface mode is

$$\beta^2 = k_0^2 \frac{\epsilon_{1y}\epsilon_{2y}(\epsilon_{1x} - \epsilon_{2x})}{\epsilon_{1x}\epsilon_{1y} - \epsilon_{2x}\epsilon_{2y}}, \quad (\text{S3})$$

where  $\epsilon_{x,y,z}$  is the component of the permittivity tensor and  $k_0$  is the free-space wavenumber.

#### B.2. Transverse field-decay factors for evanescent interface states

The transverse decay factors of an evanescent wave are  $\alpha_1 = [\epsilon_{1x}(\beta^2/\epsilon_{1y} - k_0^2)]^{1/2}/k_0$  for  $y > 0$ , and  $\alpha_2 = -[\epsilon_{2x}(\beta^2/\epsilon_{2y} - k_0^2)]^{1/2}/k_0$  for  $y < 0$ . The signs of these decay factors determine the confinement of the excited wave at the interface. When the decay factors have the same sign of values  $\text{sgn}(\alpha_1) = \text{sgn}(\alpha_2)$ , the excited interface wave decays away from the interface, corresponding to a bound evanescent wave. The other regions with different signs  $\text{sgn}(\alpha_1) \neq \text{sgn}(\alpha_2)$  denote unbound modes.

#### B.3. Spinor representation for T-spin states:

The electrical fields have the form  $\mathbf{E}(x, y) = \mathbf{\Psi}(y)e^{-ikx}$  with the amplitude vector  $\mathbf{\Psi} = E_+\mathbf{e}_+ + E_-\mathbf{e}_-$ , where  $\mathbf{e}_\pm = (\mathbf{e}_x \pm i\mathbf{e}_y)$  represent two opposite transverse spinors along the  $\pm z$ -axis. We note that our representation of the spin component in terms of the electric field follows the Abraham representation of the momentum of light in an anisotropic medium [1, 2]. Because of the duality between electric and magnetic fields, all of the results in our work can be applied to the spinning of a magnetic field by considering uniaxial anisotropic materials with the permeability tensor [3]. The local T-spin density is defined as  $\sigma_z = (|E_+|^2 - |E_-|^2) / (|E_+|^2 + |E_-|^2)$  for a  $+x$ -propagating ( $\beta > 0$ ) interface state, where  $E_\pm = E_x \pm iE_y$ . The spin density map of  $\sigma_{z1}$  and  $\sigma_{z2}$  in the main text represents the spatial average of the local spin density along  $y$ -axis in the layer 1 and 2, respectively.

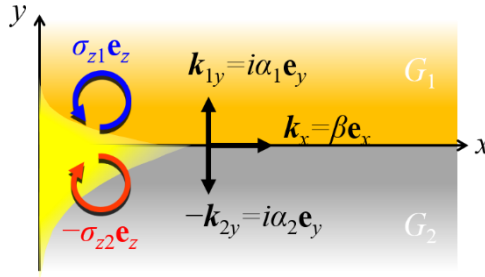


Fig. S3. Wave quantities ( $\beta$ ,  $\alpha_1$ , and  $\alpha_2$ ) and spinor representations ( $\sigma_{z1}$ ,  $\sigma_{z2}$ ) for T-spin interface states.

### References

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