

Research article

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Synthetic plasmonic lattice formation through invariant frequency comb excitation in graphene structures

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Abstract: Nonlinear surface-plasmon polaritons (NSPPs) in nanophotonic waveguides excite with dissimilar temporal properties due to input field modifications and material characteristics, but they possess similar nonlinear spectral evolution. In this work, we uncover the origin of this similarity and establish that the spectral dynamics is an inherent property of the system that depends on the synthetic dimension and is beyond waveguide geometrical dimensionality. To this aim, we design an ultralow loss nonlinear plasmonic waveguide, to establish the invariance of the surface plasmonic frequency combs (FCs) and phase singularities for plasmonic peregrine waves and Akhmediev breather. By finely tuning the nonlinear coefficient of the interaction interface, we uncover the conservation conditions through this plasmonic system and use the mean-value evolution of the quantum NSPP field commensurate with the Schrödinger equation to evaluate spectral dynamics of the plasmonic FCs (PFCs). Through providing suppressed interface losses and modified nonlinearity as dual requirements for conservative conditions, we propose exciting PFCs as equally spaced invariant quantities of this plasmonic scheme and prove that the spectral dynamics of the NSPPs within the interaction interface yields the formation of plasmonic analog of the synthetic photonic lattice, which we termed synthetic plasmonic lattice (SPL).

Keywords: frequency combs; plasmonic phase singularity; synthetic lattice; ultra-low loss graphene.

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1 Introduction

Synthetic lattice (SL) [1] provides a platform for photonic structures to couple the integral degree of freedom of light such as orbital angular momentum and FC with geometrical dimensions of the waveguide to form higher-order synthetic space [2, 3]. This multidimensional property observes both theoretically and experimentally in various physical systems from photonics [4, 5] and cold atoms [6] to non Hermitian systems [7] and topological circuits [8]. SL also provides artificial gauge fields for a bosonic structure, which yields control over spectral and temporal behaviors of light and hence is valuable for topological lasing [9, 10], breaking time-reversal symmetry [11], etc.

Recently, SL with the periodic-boundary condition is introduced to the reconstruction of the FCs [12], to control the light manipulation in a nonlinear waveguide [13] and to induce a synthetic Hall effect for photons [14]. In previous investigations, these lattices are constructed as photonic structures with negligible dissipation and dispersion, whose internal degree of light acts as a synthetic dimension. Besides, plasmonic structures act as nanoscopic nonlinear waveguides that transport surface-plasmon polaritons (SPPs) instead of photons. Nonlinear SPP (NSPP) wave propagation is a well-explored topic within these media both in the presence and absence of gain [15-17]. It is also well-known that the combination of nonlinear response and gain amplification can be exploited to excite and sustain nonlinear waves such as different classes of solitons. As these hybrid interfaces possess the same nonlinearity and dispersion, there must be similarities such as the internal degree of freedom between these nonlinear waves. As spatiotemporal profiles of these NSPPs are quite dissimilar, these hidden similarities should be beyond waveguide geometrical dimensionality and may be related to synthetic dimension.

Consequently, natural questions that may arise are whether we can propose an SL for plasmonic nanostructures, and what would be the consequence of this synthetic

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plasmonic lattice (SPL)? Quite generally, constructing an SPL using an internal degree of freedom of SPPs has not yet been investigated and this concept should be a subject of potential applications from quantum nanophotonics [18] to ultrafast-nanoplasmonics [19]. Note that our work is conceptually novel, as we introduce the concept of synthetic dimension to dissipative nanophotonic structures such as plasmonic waveguides, and also this work is methodologically novel, as we develop a framework based on quantum nonlinear averaging of SPP field, to uncover the similarities between various NSPPs, and to discover the invariants of a plasmonic scheme in a loss-compensated waveguide. Finally, we propose a general nonlinear plasmonic waveguide configuration that comprises a tunable nonlinear layer situated on top of a plasmonic scheme with vanishing loss; consequently, our work can be extended to other low-loss hybrid nanostructures. We justify SL formation within a nonlinear plasmonic structure in three steps, namely, (i) first, we elucidate the robust PFCs propagation, (ii) we uncover the conservative conditions, and (iii) we introduce loss suppression and careful nonlinearity modification as dual requirements to generate robust PFCs and to form SPL.

1.1 Physical picture of our scheme

In this work, we aim to uncover the mutual propagation properties between plasmonic peregrine wave and Akhmediev breather. The key result of this work is that in a plasmonic waveguide with suppressed loss and tunable nonlinear coefficient, PFCs act as an invariant across different types of nonlinear waves and we interpret them in terms of a synthetic dimension to construct an SPL. Consequently, our work introduces two novel concepts to nonlinear plasmonic nanostructures, namely, (i) unveiling PFCs as synthetic dimension, and (ii) exploiting this synthetic dimension to form an SPL. To elucidate these concepts, we design a nonlinear graphene nanostructure that fulfills conservation conditions and can be exploited to validate our results.

2 Model

To uncover the invariant parameters of NSPPs, we suggest a nonlinear plasmonic nanostructure as it is shown in Figure 1. This scheme is a general extension to nonlinear hybrid plasmonic configurations that comprise a nonlinear medium situated on top of a plasmonic layer. In order to achieve the robust PFCs, establish the conservation conditions, and construct the SPL, we require



Figure 1: Nonlinear plasmonic waveguide configuration for exciting invariant PFC and constructing SPL. Our suggested scheme comprises a plasmonic scheme placed as the bottom layer and a nonlinear medium situated on top. This nonlinear waveguide should possess tunable nonlinearity and vanishing loss. $\chi^{(3)}(\omega)$ is the nonlinear coefficient of the medium, $\alpha_{\rm L}=0$ is its linear absorption coefficient, and we assume $\alpha_{\rm M}:={\rm Im}[\varepsilon(k,\omega)]$ as suppressed loss of the plasmonic layer.

a nonlinear waveguide with vanishing loss and tunable nonlinearity. In what follows, we elucidate the key *essential* components of each layer that validate the invariance of PFCs yielding formation of SPL.

2.1 Nonlinear material description

Various materials possess optical nonlinearity that may serve as the upper layer of our nonlinear plasmonic scheme. However, we consider a nonlinear medium that simultaneously fulfills two criteria, namely, (i) possesses suppressed linear absorption and (ii) the nonlinear parameter related to this layer is tunable. This nonlinear medium supports different classes of nonlinear waves such as Akhmedive breather and peregrine wave and provides opportunities to excite and generate robust FCs, which are required to construct an SPL.

2.2 Plasmonic layer description

Besides, our proposed plasmonic layer should possess ultra-low Ohmic loss for our wavelength of interest. Although various ultralow loss schemes can serve as the bottom layer, our scheme is double-layer graphene that can be modeled as substrate-graphene—dielectric—graphene multilayer, as it is indicated in Figure 1. We introduce gain to the bottom graphene layer by trigger laser irradiation through a photoinverted scheme and adjust a suitable laser power to suppress the Ohmic-loss of the waveguide through gain-loss competition. Our ultralow loss plasmonic scheme is an extension to previous researches [15–17], for which we exploit the gain-induced loss compensation in double-layer graphene structure (see Section S.2 of the supplementary information) instead of employing negative-index metamaterial structure.

2.3 General description of excitation

To sum up, we propose a nonlinear plasmonic waveguide that fulfills suppressed interface losses and modified nonlinearity as dual requirements for conservative conditions and supports different classes of NSPP waves such as plasmonic Akhemediev breather and peregrine waves. Robust PFCs excite as a consequence of these plasmonic fields that act as invariants of the system and can be interpreted in terms of synthetic dimension. The existence of this synthetic dimension would yield apparition of plasmonic counterparts of synthetic lattice that we termed as SPL.

3 Approach

We present our quantitative approach towards robust NSPP propagation in three steps. First, we present the key parameters required to describe our nonlinear waveguide. Next, we describe the SPP field excitation in our scheme. Finally, we elucidate the Fourier evolution of NSPP fields.

3.1 Essential parameters related to scheme

We quantitatively model our nonlinear waveguide in terms of the parameters related to the nonlinear medium and plasmonic scheme. We consider the nonlinear coefficient of the upper layer as $\chi^{(3)}(\omega)$, its linear absorption is $\alpha_L = 0$ and we further assume that the suppressed linear absorption and enhanced nonlinear coefficient are simultaneously achievable through adjusting a control parameter of the system. Besides, our proposed plasmonic structure is double-layer graphene, the bottom layer possesses gain and the upper graphene is lossy. We consider the susceptibility of the lossy graphene as $\chi^{(L)}(k,\omega)$ [20], gain-assisted graphene as $\chi^{(G)}(k,\omega)$ [21] and evaluate the effective susceptibility of the coupled system as $\chi^{(C)}(k,\omega)$. Consequently, we achieve the dielectric function of the doublelayer graphene as $\varepsilon(k,\omega) = 1 - \chi^{(C)}(k,\omega)V(k)$; V(k) the coupling matrix between two layers. This double layer excites SPP for det $\{\varepsilon(k,\omega)\}$ = 0 [22], and ultra low-loss SPP field propagates for $Im[det\{\varepsilon(k,\omega)\}] = 0$. The combination of the nonlinear material and this graphene nanostructure provides a nonlinear plasmonic waveguide that fulfills conservative conditions and hence is suitable to excite invariant PFCs.

3.2 Surface plasmon-field excitation

The graphene structure-nonlinear medium interface hence excite stable SPP with reciprocal chromatic dispersion $\omega(\mathcal{K}) = \omega(-\mathcal{K})$, with constant phase $\theta = \mathcal{K}(\omega)x_1 - \omega t_1$ and with group velocity $v_{\sigma} = [\partial \mathcal{K}/\partial \omega]^{-1}$. The group-velocity dispersion of the SPP field is $K_2 = \partial^2 K / \partial \omega^2$ and the self focusing nonlinearity is W > 0.2 We tuned these nonlinear coefficient through adjusting the control parameter. Next, we consider frequency grid as $f = \sqrt{(\mathcal{K}_2 \delta \omega^2)/(W\pi^2)}$, temporal grid as $\tau_0 \sim 1/(\delta \omega)$, absorption coefficient as $\bar{\alpha} =$ $\epsilon^2 \text{Im}[\mathcal{K}(\omega)] + \text{Im}[k_{\mathcal{C}}(\omega)]; \epsilon \ll 1$ the perturbation parameter, and finally we normalize the probe pulse envelope as $u = [\Omega_{\rm p}/\xi] \exp{-\bar{\alpha}x}$. This pulse is then stable in rotated time $(\tau = t - x/v_g)$ for a few nonlinear $L_{NL} = 1/(\xi^2 W)$ and dispersion lengths $L_{\rm D} = \tau_0^2/\mathcal{K}_2$.

3.3 Spectral evolution of nonlinear SPP

The evolution of the SPP field then depends on two nonlinear parameters, i.e. dispersion $\mathcal{K}_2(\omega)$, and nonlinearity $W(\omega)$. Excited NSPPs propagate through an effective interface S_{eff} , within a characteristic time scale $t_{\rm S}$ and evolution of these nonlinear fields would yield FCs generation. In a medium with higher-order dispersion \mathcal{K}_n , we define PFCs as discrete frequency components associated with compressed plasmonic pulse envelope that excite with a central frequency ω_{SPP} , with frequency spacing δ that propagates through the interface as $\omega_n = \omega_{\text{SPP}} + \sum_n (D_n/n!) \delta^n$ for $D_n = \sum_{n=2,3,...} (\mathcal{K}_n/n!) \delta^n$. The mth PFCs has mode frequency ω_m , amplitude $\tilde{A}_m(x,\omega_m)$ that would excite and propagate through the interface. Total energy of the FCs and the total number of excited plasmon modes are $E \propto \sum_{m} |\tilde{A}_{m}(x, \omega_{m})|^{2}$ $\mathcal{N} \propto \sum_{m} |\tilde{A}_{m}(x, \omega_{m})|^{2}/(\omega_{p} + \omega_{m}),$ respectively. The total SPP field related to PFCs $\tilde{\Psi} := \tilde{\Psi}(x, \tilde{\omega}) =$ $F(\mathbf{r})\tilde{A}(x,\tilde{\omega}) \exp\{i\mathcal{K}(\tilde{\omega})x\}; F(\mathbf{r}) := F(y,z) \text{ the plasmonic}$ pulse envelope function, propagates through the interface whose dynamics described by nonlinear spectral evolution equation

$$i\frac{\partial\tilde{\Psi}}{\partial x} = \mathcal{K}(\omega)\tilde{\Psi} + \sum_{m} \int d\tilde{\omega} \, \frac{\mathcal{W}(\tilde{\omega})}{2\pi} \tilde{\Psi}_{\tilde{\omega}}^* \tilde{A}_{\tilde{\omega} - \omega_{m}} \tilde{\Psi}_{\tilde{\omega} + \omega_{m}}^*. \quad (1)$$

We note that the NSPP propagation within the nonlinear medium-graphene interface is similar to previous works [15–17] as both systems possess group-velocity dispersion and self-phase modulation. However, this work differs from our previous works due to exploiting NSPP similarities

¹ We present the detailed quantitative steps of derivation in Section S.2 of supplementary information.

² $\mathcal{W}(\omega) \propto \chi^{(3)}(\omega)$ for a nonlinear material.

to establish conservation conditions, introduce frequency comb as a synthetic dimension, and constructing SPL.

the robust spectral dynamics to a synthetic photonic lattice in Section 4.3.

3.4 Simulation parameters and scheme feasibility

Now, we introduce the simulation parameters to justify invariant PFCs excitation. In our scheme stable NSPP with $\omega_{\rm SPP}=1.57$ eV propagates with group velocity dispersion $\mathcal{K}_2=(-4.42+0.4\mathrm{i})\times 10^{-12}~\mathrm{s}^2~\mathrm{cm}^{-1}$, and with nonlinear coefficient $W=(2.98+0.6\mathrm{i})\times 10^{-11}~\mathrm{s}^2~\mathrm{cm}^{-1}$, and the group velocity of the excited SPP is $v_{\rm g}=2\times 10^4~\mathrm{m/s}$. The normalized nonlinear parameter is $g_{\mathcal{K}_2}=1.01$ and frequency grid is $f=0.045~\mathrm{s}^{-1}$. For gain-assisted graphene we have $\mathrm{Im}[k_{\rm G}]=0.07~\mathrm{cm}^{-1}$ and for coupled system we have $\mathrm{Im}[k_{\rm G}]=-0.02~\mathrm{cm}^{-1}$, hence $\bar{\alpha}=\mathrm{Im}[K_{\rm a}(\omega)+k_{\rm C}(\omega)]=0.04\approx0$. Therefore, our proposed structure is the best fit to vanishing loss and tunable nonlinear parameter and hence is a suitable candidate for invariant PFCs excitation and SPL formation.

To simulate SPP propagation in this nonlinear plasmonic system, we assume (i) SPP waves propagate as far plasmonic fields, (ii) the plasmonic phases are constant (i.e. $\mathcal{K}x - \omega t = \text{Const.}$), and (iii) use mean field-averaging [16, 17, 23] to consider the evanescent coupling effect. Finally, we note that our constructed synthetic lattice can also be achieved in other plasmonic schemes. To this aim, the interaction interface should satisfy the dual requirement of loss suppression and nonlinearity modulation. We note that our proposal is a general extension to other nonlinear plasmonic schemes, and the formation of SPL and invariant PFCs in any other configuration can also be investigated using our theory. We exploit our method for a plasmonic configuration comprises nonlinear atomic ensemble as a specific example and elucidate all detailed elucidations such as mechanisms required to excite and probe the system within this hybrid design, the details of the nonlinear medium, and all detailed calculations of graphene nanostructure in § S.1 of the supplementary information.

4 Results

We present the results of this paper in three sections: first, we investigate the temporal and spectral dynamics of the plasmonic peregrine and Akhmediev breather phases within the interaction interface in Section 4.1. Next, in Section 4.2 we evaluate the spatial-spectral evolution of the energy flux and number of plasmon modes to achieve the conservative parameters of the system. Finally, we map

4.1 Robust FCs generation

The excited and propagated NSPPs are described by $u(x,t) = |u(x,t)| \exp\{i\phi_{NL} - \bar{\alpha}(\omega)x\}; \bar{\alpha}(\omega) = \operatorname{Im}(\mathcal{K}(\omega)),$ possess self focusing nonlinearity and weak second-order dispersion that affects the plasmonic pulse envelope. The NSPPs thereby excite as plasmonic peregrine wave and Akhmediev breather and their phases undergo nonlinear dynamical evolution that propagates as modified pattern, as it is represented in Figure 2(a) and (c). Due to SPP pulse compression, a maximum plasmonic field intensity $|u(x,t)|^2 \mapsto u_{\text{max}}$ and two dark points $|u(x,t)|^2 \mapsto 0$ is expected due to growth-return cycle for both NSPP waves. The SPP phase corresponds to these dark points are singular, as we depict in Figure 2(a) and (c), which we term as phase singularities (PSs). To achieve PSs, we assume input plasmonic field as an evanescent wave with input power $P_{\rm p}$ characterized by $U_0(x=0,t)=\sqrt{P_{\rm p}}\,\exp\{\mathrm{i}\theta_l-\bar{\alpha}x\}$ and consider the seeded noise as a perturbation with amplitude $u_{\rm N} = 0.08u_{\rm 0}$ and modulation frequency $v_{\rm mod}$ as $\Delta u_{\rm N} =$ $u_{\rm N}\cos[2\pi \ v_{\rm mod}t]$ that introduces a small perturbation to plasmonic field. We then achieve the SPP dynamics by numerically solving the nonlinear Schrödinger equation [15, 17, 23] for $u(x = 0, t) = \mathcal{U}_0(x = 0, t) + \Delta u_N$. Next, we investigate the amplitude of the plasmonic peregrine wave and Akhmediev breather in Fourier space in Figure 2(b) and (d).

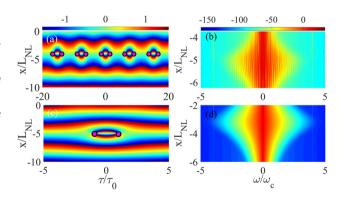


Figure 2: Panel (a) represents the phase dynamics of NSPPs through Akhmediev breather formation and panel (b) denotes corresponding spectral analysis. Panel (c) depicts the phase dynamics of NSPP through plasmonic peregrine wave and panel (d) represents its spectral evolution. For this figure $P_0=10\,\mu\text{W},~\tau_0=10\,\mu\text{s},~\delta\omega=1~\text{MHz},~u_N=0.08~\text{MHz}.$ Panels (a) and (b) are plotted for modulation parameter a=0.32 and for panels (c) and (d) we choose a=0.5. Magenta dots in panels (a) and (c) represent the coordinates of PSs. Other parameters are given in the text.

Various nonlinear plasmonic phases such as periodic (Figure 2(a)) and single PSs (Figure 2(c)) are excited by tuning the modulation parameter through plasmonic Akhmediev breather and peregrine wave formation, respectively. Figure 2(b) and (d) demonstrate that the FCs correspond to these PSs generate for both NSPP excitation and these quantities are invariant against input field modulation. Consequently, PSs are the similar features of the exciting nonlinear waves, for peregrine wave and Akhmediev breather. For a characteristic frequency $\omega_{\rm ch} = 10$ MHz, robust FCs up to $\omega_{\mathrm{comb}} pprox 3\omega_{\mathrm{ch}}$ are achieved through plasmonic PS. Consequently, similar frequency comb generation and their stable propagation through interaction interface are referred to as invariant of NSPPs. The FCs $|\omega| < 2\omega_{\rm ch}$ can propagate for a few propagation lengths $-5.5L_{\rm NL}$ < x < $-4L_{\rm NL}$ and hence would produce a robust plateau, as it is shown clearly in Figure 2(b) and (d), which we exploit this square to design a plasmonic version of SL.

4.2 Invariants of nonlinear system

The formation of robust PFCs and their robustness against external field modulation can be elucidated through the apparition of hidden invariants of this nonlinear system, which we termed invariant parameters. One of the key theoretical results of this work is that the PFCs act as invariant features across different types of nonlinear waves excitation. PFCs are invariant of this plasmonic system if we fulfill two conditions simultaneously, namely, (i) \mathcal{N} , which is the number of plasmonic frequency combs, remains as constant along interaction direction (x) for any kind of nonlinear plasmonic waves, and, (ii) the output spectral envelope $I_N(\omega)$ after a few nonlinear propagation lengths would be the same for both plasmonic peregrine and Akhmediev breather. In this section we establish that the PFCs stably propagate in the interaction interface and obtain a modified nonlinear evolution equation that can be used for different classes of NSPPs in the presence of conservative conditions. In Section 4.3, we prove the similarity between the spectral evolution of the plasmonic peregrine wave and Akhmediev breather and justify the formation of invariant PFCs.

We note that our robust FCs propagate for characteristic length $L_{\text{eff}} = \bar{\alpha}^{-1}[1 - \exp\{-\bar{\alpha}L\}]$ and within nonlinear timescale $t_S = \tau_0 + \partial_{\omega} [\ln\{(n_{\text{eff}}S_{\text{eff}})^{-1}\}]$. To achieve the invariant parameters, we evaluate the spatial variation of the energy flux and number of plasmon modes associated with NSPPs within spectral domain. In our analysis, we consider the FCs correspond to central SPP modes $\omega_{\rm SPP}$, $\omega_{+} = \omega_{\rm SPP} \pm \omega_{m}$ situated within the electromagnetically induced transparency window commensurate with

the suppressed Ohmic loss. The nonlinear coefficient related to transparency window is $\mathcal{W}(\omega_i)$, we introduce $\Delta := 4 \text{Im}[A_1^* A_2 A_3 A_4^* \exp\{i\Delta \mathcal{K}_t x\}]$ as SPP field detuning for four-wave mixing process, define phase-matching $\Delta \mathcal{K}_{t} = \mathcal{K}(\omega_{ch}) + \mathcal{K}(\omega_{+}) - \mathcal{K}(\omega_{0}) - \mathcal{K}_{4}(\omega_{-});$ parameter $\mathcal{K}(\omega_l) = \beta (\omega_l) + k(\omega_l) + \sum_{l} \sum_{n} (2 - \delta_{ln}) \mathcal{W}(\omega_l) |A_n(\omega_l)|^2,$ consider effective nonlinear parameter as $W_{\text{eff}} =$ $\mathcal{W}(\omega_{ch}) + \mathcal{W}(\omega_0) - \mathcal{W}(\omega_-) - \mathcal{W}(\omega_+)$ and consider k_l as corresponding wavenumber of the propagated modes. Following the technical details of derivation as in Section S.3 A2, we evaluate the spatial dynamics of energy flux as

$$\frac{\partial E}{\partial x} \propto -\mathcal{O}(\bar{\alpha}) + \sum_{m} \mathcal{W}_{\text{eff}} \Delta,$$
 (2)

and number of stable plasmon modes as

$$\frac{\partial \mathcal{N}}{\partial x} \propto -\mathcal{O}(\bar{\alpha}) + \sum_{m} \left[\frac{\mathcal{W}(\omega_{0})}{2\omega_{0}} + \frac{\mathcal{W}(\omega_{-})}{\omega_{0} + \omega_{-}} + \frac{\mathcal{W}(\omega_{+})}{\omega_{0} + \omega_{+}} \right] \Delta.$$

The dynamical evolution of the PFC, their robustness, and conservative conditions hence are limited by the nonlinear coefficient $\mathcal{W}(\omega)$ and loss of the system. We introduce loss suppression and careful nonlinear modulation are dual requirements for conservation of energy and number of plasmon modes and hence, we expect robust FCs generation only for ultra-low interface $\bar{\alpha} \mapsto 0$, and for finely tuned nonlinear coefficient. To characterize modified nonlinearity, we consider the nonlinear coefficient as $W(\tilde{\omega}) = W_0 + W_1 \delta \tilde{\omega} + W_2 \delta \tilde{\omega}^2 + \mathcal{O}(\delta \tilde{\omega}^3)$. The energy can be a conservative quantity of the system (i.e. $(\partial E/\partial x) \approx 0$) by taking $W(\tilde{\omega}) = W_0 + W_1 \delta \tilde{\omega}$ whereas we achieve the conservation of the excited FCs (i.e. $(\partial \mathcal{N}/\partial x) \approx 0$) for $W(\tilde{\omega}) = W_0 + W_1 \delta \tilde{\omega} + W_2 \delta \tilde{\omega}^2$. Consequently, the energy flux and number of plasmonic modes are simultaneous invariants of the system for

$$W(\tilde{\omega}) = W_0 \left(1 + \frac{\tilde{\omega}}{\omega_0} \right). \tag{4}$$

This equation establishes that the apparition of conservation within a plasmonic system is independent of the SPP field dispersion/dissipation. The conservation of PFCs across different classes of NSPPs is an essential step to justify the invariance of FCs within our system (see Section S.3 A2 of supplementary information for more details). In what follows, we describe the nonlinear spectral evolution of the NSPP field in the presence of conserved \mathcal{N} , E, and use our modified nonlinear equation to establish the apparition of invariant PFCs.

To describe the NSPP evolution, we should solve nonlinear Schrödinger equation considering conservative conditions and by assuming nonlinearity as Eq. (4), which is challenging due to the plasmonic field being propagated within the dispersive interface. To remedy this limitation, we consider SPP field as electromagnetic waves that can be quantized through interaction interface and use the quantum SPP approach that is independent of interface dispersion. Here we consider the NSPP field as time-harmonic profile $J(\mathbf{r},t) := J_{\omega} \exp\{i\omega t\} + \text{c.c.}; \ J \in \{\mathbf{E},\mathbf{B}\}, \text{ next,}$ we consider Weyl gauge field and evaluate vector potential A(r,t). Following Ref. [24], we expand this potential in terms of amplitude and frequency of each mode and evaluate the time average classic energy for this SPP field. The quantum counterpart of this energy yields the quantization of PFCs with bosonic annihilation creation operators $\hat{\boldsymbol{b}}$ ($\hat{\boldsymbol{b}}^{\dagger}$).³ We consider the nonlinear coefficient of the interface as W^0 , and rewrite the interaction Hamiltonian for the stable nonlinear quantum plasmon mode as $\mathcal{H}_{\mathrm{I}} = \sum_{m} \iint d\tilde{\omega} d\omega (\mathcal{W}^{0}/2) \hat{\boldsymbol{b}}_{\omega}^{\dagger} \hat{\boldsymbol{b}}_{\tilde{\omega}}^{\dagger} \hat{\boldsymbol{b}}_{\tilde{\omega} - \omega_{m}} \hat{\boldsymbol{b}}_{\tilde{\omega} + \omega_{m}}$. Next, we use Eq. (4) and use the Heisenberg equation of motion [25] to evaluate the dynamics of the mean-field value associated with the stable plasmon mode $(\partial \langle \hat{A}_m \rangle / \partial x)$ [26], and include the dispersion term due to plasmonic field. We then achieve the evolution of NSPPs similar to Eq. (1) but with including invariant parameters as

$$\frac{\partial \tilde{A}}{\partial x} = i\mathcal{K}(\omega)\tilde{A} + \sum_{m} \mathcal{F}\left[\Lambda(\tilde{\omega})|A_{m}|^{2}A_{m}\right] + \text{c.c.}, \quad (5)$$

for \mathcal{F} the Fourier transform operator and Λ is the non-linearity that is linearized in the presence of conserved energy and invariant number of excited SPP modes (see supplementary material § S.3 B).

4.2.1 Discussion on the universality of FCs

In this work, we uncover the invariants only for plasmonic peregrine wave and Akhemediev breather, which means that our approach falls short of providing a universal description of nonlinear SPPs and frequency comb generation. Here, PFCs propagation and SPL formation are highly limited by loss suppression and nonlinear modification. Consequently, our predicted effects would depend on robust PFC generation and conservation conditions, and hence are device-dependent. The extension of stable FC formation to other classes of solitons needs careful investigations and goes beyond the scope of this work.

4.3 Nonlinear spectral dynamics and synthetic lattice formation

In this section, first, we achieve the output spectral envelope function of the NSPP waves for peregrine and Akhmediev breather cases with and without the existence of conservatives, to establish the invariance of PFCs in our hybrid system that can be interpreted as a synthetic dimension. Next, we exploit this synthetic space to construct an SPL.

To investigate the spectral dynamics and evaluate the output envelope profiles of the NSPPs, we assume SPP field as $A(x,\omega) := \sqrt{P_p} \exp\{i\phi_S - i\mathcal{K}(\omega)x\}$ and numerically solve Eq. (5) considering $x_0 = -10L_{\rm NL}$. For a lossy plasmonic interface with $\chi^{(3)}(\omega) := \mathcal{W}^0$, the plasmonic peregrine waves and Akhmediev breather have different output spectral envelope, as it is clearly shown in Figure 3(b) and (d) (dashed blue curves), indicating that in the interaction interface with constant nonlinearity, PFCs are not robust and invariant of the system. We also investigate the spectral-spatial dynamics of phase $\phi(x,\omega)$ (Figure 3(a) for Akhmediev breather and Figure 3(c) for plasmonic peregrine wave) and output spectral envelope of the corresponding NSPP waves in the presence of conservation conditions, representing that the output spectral profile of the NSPPs for both plasmonic Akhmediev breather (Figure 3(b) red curve) and peregrine wave (Figure 3(b) red curve) becomes the same.

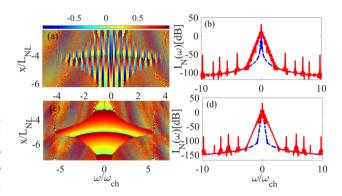


Figure 3: Spectral evolution of NSPPs through the interaction interface: panel (a) is the spectral phase variation $\phi(x,\omega)$ panel (b) represents the logarithmic spectral harmonic intensity of the plasmonic breather as a function of perturbation frequency. Panel (c) denotes the phase variation and (d) is spectral logarithmic power density for plasmonic peregrine wave excitation. In both panels (b) and (d) the blue dotted-dashed line represents the input field and red solid-line denotes the NSPP excitation in the presence of invariants [Eq. (4)]. Despite the dissimilar phase variation, excited FCs are the invariant of the nonlinear system. See the text for more details.

³ As SPP wave conservatives are independent of interface dispersion, without the loss of generality, we consider $\mathcal{K}(\omega) = 0$.

⁴ We termed this dual requirement as conservative condition.

Our results hence indicate that in a nonlinear waveguide fulfilling the vanishing loss and tunable nonlinearity, (\mathcal{N}, E) are conserved parameters of the systems. In this case, the interaction interface supports the stable propagation of plasmonic peregrine wave and Akhmediev breather for a few propagation lengths with similar output spectral probe envelope. Consequently, FC is the hidden conserved parameter among both NSPPs, and propagated PFCs are interpreted as a synthetic dimension in our scheme. We exploit this synthetic dimension to construct an SPL. We present our qualitative and quantitative approach towards SPL formation in two steps, namely, (i) well-defined synthetic dimension and (ii) constructing synthetic lattice. In what follows we describe these two steps in detail.

4.3.1 Qualitative description

Qualitatively, we prove that PFCs are invariant of excited NSPPs and they propagate as discrete frequency components with equal frequency spacing $\delta:=\omega_{\rm EIT}/\mathcal{N}$, they act as an internal degree of freedom for SPPs and can be considered as a synthetic dimension [1]. Consequently, we assume $x/L_{\rm NL}$ and $\omega/\omega_{\rm ch}$ to construct SPL as it is qualitatively represented in Figure 4(b). To SPL formation, we need to characterize lattice sites and hopping between these sites. In our scheme, discrete FCs effectively act as lattice sites, and we consider SPP field overlapping between FCs and their nearest neighbor as hopping between PFCs, which qualitatively describes SPL formation. As these combs are stable for a limited propagation

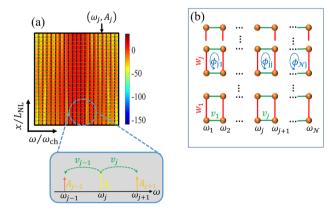


Figure 4: Mapping between the stable FCs to a synthetic dimension: panel (a) represents the power spectrum of the spectral harmonic side-bands for $|\omega| < 2\omega_{\rm ch}$ and $-5.5L_{\rm NL} < x < -4L_{\rm NL}$ correspond to robust propagation of FCs. The inset of this figure represents the correlation between the FCs. Panel (b) is the qualitative description of the synthetic lattice corresponding to frequency comb excitation.

length, our synthetic structure can be considered as a two-dimensional lattice in $x-\omega$ plane.

4.3.2 Quantitative description

We assume PFCs excite with frequency components $\omega_n =$ $\pm n\delta$, and with propagation constant $\xi_n = \xi_{NL} + \Delta \xi_L$; $\xi_{\rm NL} = \sum_n C_n \cos(k_n)$ is the nonlinear part of propagation constant for $C_n := \mathcal{W} \sum_m A_m(0) A_{m-n}^*(0)$ the correlation between different harmonic side-bands. We assume \mathcal{W} as Eq. (4) to include the conservative conditions and consider only the nearest neighbor effect in our analysis. The total electric field for each PFCs are then $|\mathbf{E}| = \sum a_n(x) \exp\{i[\omega_n t - \xi_n x]\}$. We then achieve the phase matching condition for PFCs in terms of system parameters as $\Delta \xi_{\rm I} = \delta n_{\rm o}/{\rm c.}^5$ These characterized PFCs then act as a synthetic dimension for two NSPP waves. The dynamical evolution of a_n is then achieved through coupled-mode equation $\partial_x a_n(x) =$ $-(\alpha_{\text{eff}}/2)a_n(x) + i[Ca_{n+1}(x) - C^*a_{n-1}(x)].$ Following Ref. [1], the Hamiltonian describing this coupled-mode equation is $\mathcal{H}_0 = \kappa \sum_n [|n+1\rangle\langle n| + |n-1\rangle\langle n|]$ (see further mathematical details in Section S.4). To extend this Hamiltonian to a two-dimensional case, we need to consider hopping along x-axis as well. Within a characteristic $x-\omega$ plane presented in Figure 4(a), we assume that the amplitude of PFCs are robust and consequently, we consider the phase variation along this direction.

We write the Hamiltonian of SPL in $x-\omega$ plane in terms of hoppings $w_{i,j}$ and $v_{i,j}$ in the most general case⁶

$$\mathcal{H}_{SPL} = \sum_{i,j,n} w_{i,j} \hat{a}_{i,j} \hat{a}_{i+n,j}^{\dagger} + \sum_{i,j} v_{i,j} \hat{a}_{i,j} \hat{a}_{i,j+1}^{\dagger} + \text{H.C.}, \quad (6)$$

and consider the evolution of this lattice along the interaction interface as $|\psi(x)\rangle=\exp\{i\mathcal{H}_{\mathrm{SPL}}x\}|\psi(x=x_0)\rangle$. This system is a synthetic lattice corresponding to the nonlinear interaction within a plasmonic interface, hence its dynamics are described through the annihilation-creation operators \hat{a} (\hat{a}^{\dagger}), and complex hoppings $w_{i,j}, v_{i,j}$. To achieve the SPL Hamiltonian, we also include the dissipation to the SPP wave dispersion $\mathcal{K}(\omega) \mapsto \mathcal{K}(\omega) + \mathrm{i}\bar{\alpha}$. Equation (6) indicate that SPL has square-type structure but with complex coefficients that yield *anomalous* hopping phase between lattice sites i,j, we termed as $\phi_{i,j}$, which is nonzero for specific plateau within our SPL.

⁵ *c* is the speed of light in vacuum.

⁶ $i \in \{-N, ..., N\}$ and j are dummy variables along ω and x directions respectively, and $\{n\}$ represents any set of neighbors along the frequency direction, however we only consider n = 1.

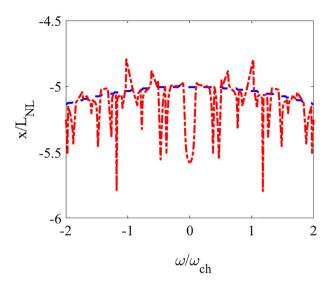


Figure 5: Observation of anomalous phase hopping, and nonzero phase trajectories through different NSPP field excitation: Dashed blue curve is the trajectory of nonzero phase hoppings $\phi_{i,j} \neq 0$ for the plasmonic rogue wave formation, and red dashed-dotted curve is the corresponding phase trajectory for the breather formation.

For our nonlinear plasmonic system, this hopping phase depends on the modulation parameter, is different for various nonlinear field excitation and we achieve this non-zero flux $(\delta\phi_{i,j}\neq 0)$ only for specific spatial-spectral trajectories, as it is clearly shown in Figure 5. These trajectories are reciprocal, frequency dependent and can be exploited to introduce anomalous artificial gauge field, and is suitable to produce nonuniform synthetic magnetic fields.

5 Conclusions

To sum up, we exploit the spectral SPP field evolution to discover the hidden invariant parameter of the nonlinear plasmonic wave systems. The key theoretical result of this work is that in a plasmonic waveguide with suppressed loss and tunable nonlinear coefficient, PFCs act as an invariant across different types of nonlinear waves and are interpreted in terms of a synthetic dimension to construct an SPL. Invariant PFCs apparition and SPL formation are based on assumptions related to nonlinear waveguides and assumptions related to the SPP field. For the SPP field, we assume far-field approximation, neglect its phase variation, and consider the effect of evanescent coupling through mean-field averaging. For plasmonic configuration, we assume the interface with low dispersion and controllable nonlinearity. Our loss compensation scheme and field propagation methodology in our proposed scheme is then an extension to the ultra low-loss plasmonic scheme represented in [15–17] that the gain compensated graphene structure play the role of the metamaterial layer. In particular, we uncover that the FCs and PSs are similar features for plasmonic peregrine waves and Akhmediev breather.

Next, we prove that the linearized nonlinear coefficient provides similar PFCs that can be interpreted as the internal degrees of freedom and thereby act as synthetic dimensions. Qualitatively, we achieve this finely tuned nonlinearity as control parameters. Our analysis thereby indicates that dual requirement of the vanishing losses, linearizing the interface nonlinearity, and the existences of robust FCs with preserved number of excitation are requirements to constructing an SPL. We quantitatively describe the robust FCs generation through quantum NSPP field formalism commensurate with the Schrödinger approach and we also perform the mean-value averaging to achieve the Fourier dynamics of NSPPs in the presence of conservative conditions. Our approach justifies the existence of the anomalous hopping phase through characteristic reciprocal trajectories, which depend on input field modulation and can be exploited to various effects, from nonuniform magnetic flux to anomalous gauge field.

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