Supplementary Material

Analysis of the homogeneous stationary solution (HSS)

The HSS (5) of Eq. (3) is S-shaped if condition (6) holds. The values of the output intensity X at the turning points SN_1 and SN_2 are

$$X_{\mathrm{SN}_{1,2}} = \frac{2(1+\theta\Delta) \pm \sqrt{1+8\theta\Delta+\theta^2\Delta^2-3(\theta^2+\Delta^2)}}{3(1+\Delta^2)}\,. \qquad \text{(S1)}$$

If $\theta = \Delta$ the bistability condition is always satisfied, the left turning point SN_2 touches the axis in $X = 1$ and the lower branch is not accessible.

We study the stability of this solution with respect to fluctuations of the form $\delta F(\tau,\eta) = \delta F_0 \mathrm{e}^{\lambda\tau}\mathrm{e}^{iK\eta}$, $\delta F^*(\tau,\eta) = \delta F_0^* \mathrm{e}^{\lambda\tau}\mathrm{e}^{-iK\eta}$. The characteristic equation for the eigenvalue λ is

$$\lambda^2 + c_1\lambda + c_0 = 0\,, \qquad \text{(S2)}$$

with

$$c_1 = 2(2X - 1 + K^2)\,, \qquad \text{(S3)}$$

$$\begin{aligned} c_0 &= 1+\theta^2 - 4(1+\theta\Delta)X + 3(1+\Delta^2)X^2 \\ &\quad + 2\left[2(1-G\Delta)X + G\theta - 1\right]K^2 + (1+G^2)K^4\,. \end{aligned} \qquad \text{(S4)}$$

Let us consider first the single-mode limit $K = 0$. The coefficient c_1 is negative for $X < 0.5$. This means that the stationary solution is unstable due to a Hopf instability for $0 \le X \le X_{IL}$ with $X_{IL} = 0.5$. The coefficient c_0 coincides with dY/dX, and this is associated with the usual instability of the negative slope branch of the HSS.

In the general case $K \neq 0$ the threshold for the Hopf instability is lowered which means that the most unstable mode is the resonant mode $K = 0$ and the lower branch is still unstable from $X = 0$ to $X = 0.5$. Instead, the condition $c_0 = 0$ gives rise to a modulational (Turing) instability which affects the upper branch from the left turning point SN_2 up to the bifurcation point MI, with

$$X_{\mathrm{MI}} = (G+\theta)\,\frac{2\,(G+\Delta) + \sqrt{(1+G^2)(1+\Delta^2)}}{3(G^2+\Delta^2) + 8G\Delta - G^2\Delta^2 - 1}\,. \qquad \text{(S5)}$$

We assume that both the numerator and the denominator of this expression are positive. The positivity of the denominator sets some limits to the possible values of Δ and G. For instance, if $G = \Delta$ it must be $\Delta > 2 - \sqrt{3}$. An example of modulational instability domain is shown in Fig. S1 for $\Delta = 2$ and $G = 3$. The wavevector K_{MI} associated with X_{MI} is the most unstable wavevector. In a cavity

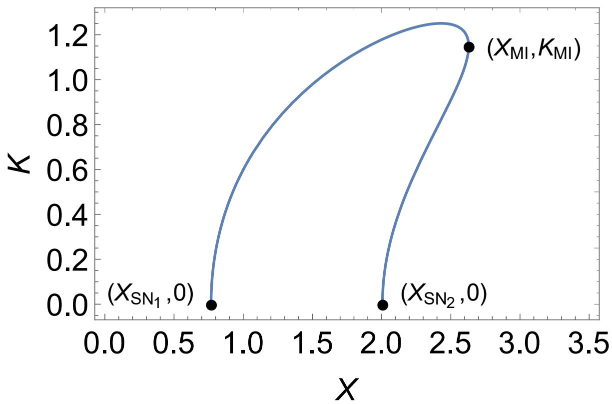

Fig. S1: Modulational instability domain of the HSS with respect to longitudinal modes of wavevector K for $\Delta = 2$ and $G = 3$. For $K = 0$ (single-mode limit) the solution is unstable only on the negative slope branch between the two turning points SN_1 and SN_2. For $K \neq 0$ the instability domain extends on the upper branch up to X_{MI} and the most unstable wavevector is K_{MI}.

of finite scaled length $\eta_{\max}$ the wavevector K can take only values which are integer multiples of the minimum wavevector $K_1 = 2\pi/\eta_{\max}$. If $K_{\bar{n}} = K_1\bar{n}$ is the wavevector closest to K_{MI}, the Turing pattern that emerges when the bifurcation point MI is crossed will consist of $\bar{n}$ rolls.

If the HSS is S-shaped the bifurcation point IL typically is placed on the lower branch and it coincides with the right turning point SN_1 if $X_{\mathrm{SN}_1} = X_{\mathrm{IL}} = 0.5$. For a given Δ this happens when $\theta = \theta_{\mathrm{IL1}}(\Delta)$ with

$$\theta_{\mathrm{IL1}}(\Delta) = \Delta + \frac{\sqrt{1+\Delta^2}}{2}\,. \tag{S6}$$

As θ increases from θ_{IL1} the point IL moves to the left along the lower branch. When $\theta = \theta_{\mathrm{IL2}}(\Delta)$ with

$$\theta_{\mathrm{IL2}}(\Delta) = \frac{1}{4}\left[5\Delta + \Delta^3 + (1+\Delta^2)\sqrt{5+\Delta^2}\right]\,, \tag{S7}$$

we have $Y_{\mathrm{IL}} = Y_{\mathrm{SN}_2}$, which means that the lower branch is stable between the two turning points.

The bifurcation point MI exists only if the HSS curve is S-shaped and it is placed on the right of the left turning point SN_2. MI coincides with SN_2 when $\theta = \theta_{\mathrm{MI2}}(\Delta, G)$ with

$$\theta_{\mathrm{MI2}}(\Delta, G) = \frac{4\Delta(1+G^2) - G(1+\Delta^2) + 2(G\Delta - 1)\sqrt{(1+\Delta^2)(1+G^2)}}{4 + (3-\Delta^2)G^2}\,. \tag{S8}$$

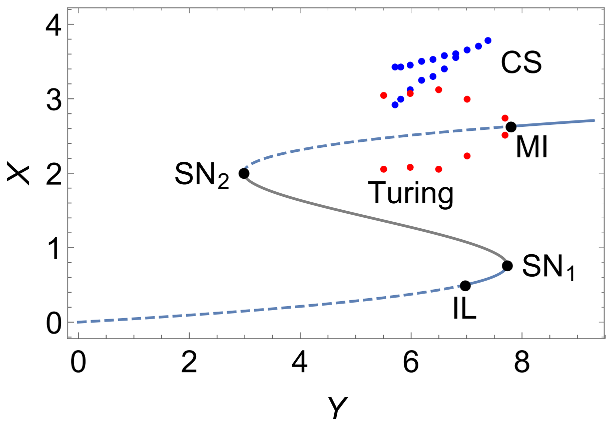

Fig. S2: Stationary homogeneous solution, where solid and dashed blue lines indicate stable and unstable configurations, cavity solitons branch (blue symbols), and Turing patterns branch (red symbols) for Eq.(3), with $\Delta = 2$, $G = 3$, $\theta = 4.7$ (point **c** of Fig. 3), and $\eta_{\max} = 200$ (figure obtained with the same parameters as in [14]).

For $G = \Delta$ we obtain the simpler expression

$$\theta_{\mathrm{MI2}}(\Delta, \Delta) = -2 + \frac{3}{2 - \Delta} \,. \tag{S9}$$

This explains the vertical asymptote at $\Delta = 2$ in Fig. 3. If we set $G = \Delta + 1$ in Eq. (S8) we find that the vertical asymptote moves to $\Delta = 1.867$.

The input intensity Y_{MI} of the point MI coincides with the input intensity of the right turning point Y_{SN_1} when $\theta = \theta_{\mathrm{MI1}}(\Delta, G)$, with

$$\begin{aligned} \theta_{\mathrm{MI1}}(\Delta, G) &= \frac{f_1(\Delta, G) + 2(G\Delta - 1)\sqrt{(1+\Delta^2)^3(1+G^2)}}{f_2(\Delta, G)} \\ f_1(\Delta, G) &= 2(3\Delta^4 + 8\Delta^3 + 2\Delta^2 - 1) - (7\Delta^4 - 18\Delta^2 + 7)G \\ &\quad +2(\Delta^4 - 2\Delta^2 + 8\Delta - 3)G^2 \,, \\ f_2(\Delta, G) &= 3\Delta^4 + 16\Delta^3 + 26\Delta^2 + 16\Delta + 7 + 4(1+\Delta)^2(3-\Delta^2)G \\ &\quad +4(3-\Delta^2)G^2 \,. \end{aligned} \tag{S10}$$

For $G = \Delta$ we obtain the simpler expression

$$\theta_{\mathrm{MI1}}(\Delta, \Delta) = 4 - \Delta - 16\frac{\Delta - 2}{\Delta^2 - 7} \,. \tag{S11}$$

This explains the vertical asymptote at $\Delta = \sqrt{7} \sim 2.646$ in Fig. 3 which shifts to $\Delta = 2.343$ if we set $G = \Delta + 1$. From Eqs. (S9) and (S11) it follows that for $G = \Delta$ the point MI is always placed on the right of SN_1 if $\Delta > \sqrt{7}$ for any θ. If instead $\Delta < \sqrt{7}$, as θ increases from $\theta_{\rm MI1}$ the point MI moves to the left of the upper branch of the stationary solution but it can touch the left turning point $\rm SN_2$ only if $\Delta < 2$ and $\theta = \theta_{\rm MI2}$.

All these results are summarized in Fig. S2 which shows the HSS and its bifurcations for $\Delta = 2$ and $G = 3$. Also shown are the branches of CSs and Turing pattern obtained assuming $\eta_{\rm max} = 100$ [14]. To the left of the bifurcation point IL the lower branch of the HSS which is the pedestal of the CS is unstable, therefore the peak intensity oscillates between a minimum and a maximum values, which are represented by the blue symbols.