**Supplementary Information:** **Realization of complex conjugate media using non-PT-symmetric photonic crystals**

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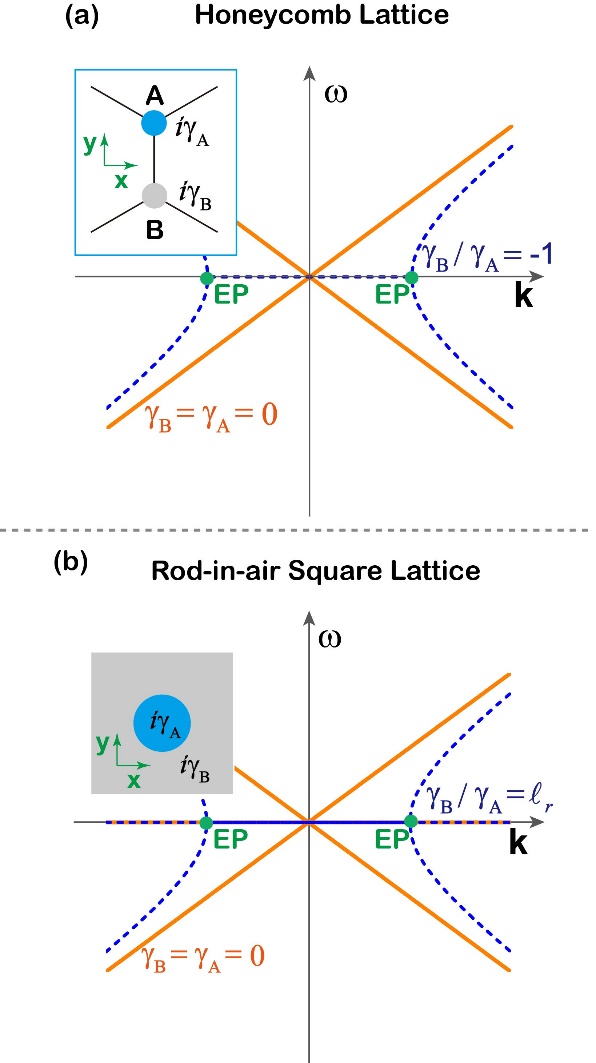


Fig. S1 Schematic band dispersions. **a** Schematic band dispersions of the Dirac cone at the Brillouin zone corner in the graphene system. **b** Schematic band dispersions of the Dirac-like cone at the Brillouin zone center in the rod-in-air square lattice photonic system. The horizontal axes represent the Bloch wave number and the vertical axes the eigenfrequencies. The inset in **a** shows the unit cell of the graphene lattice, and the non-Hermiticity of sublattice *A* (*B*) as . The inset in **b** shows the unit cell of the square lattice, and that the non-Hermiticity of the rod (the blue region) and that of the background material (the grey region) are  and  respectively. The solid orange lines show band dispersions in the Hermitian case (), while the blue dashed lines show the real parts of band dispersions in the non-Hermitian case. The green dots mark the EPs.

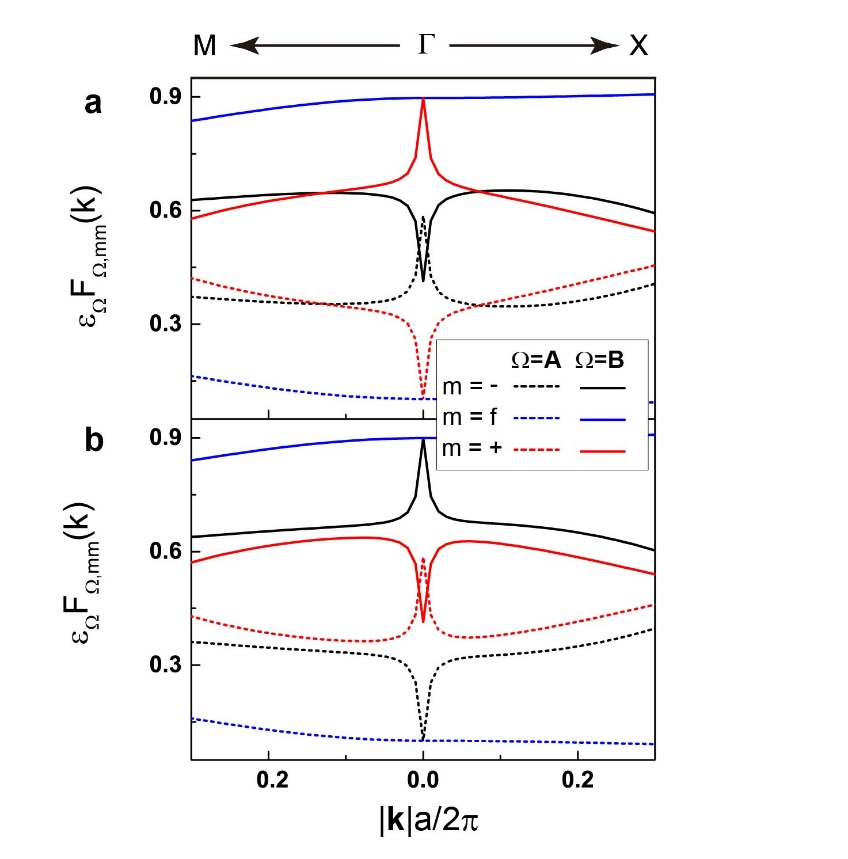


Fig. S2 The quantity  near the Dirac like cone. Amplitude distribution functions  as a function of **k** along  direction for **a**  and **b** . The relative permittivity of the cylinders is . The solid lines represent amplitude distribution functions inside the cylinder domain () and the dashed lines represent amplitude distribution functions in the background medium domain (**)**. Different colors represent different bands *m*.

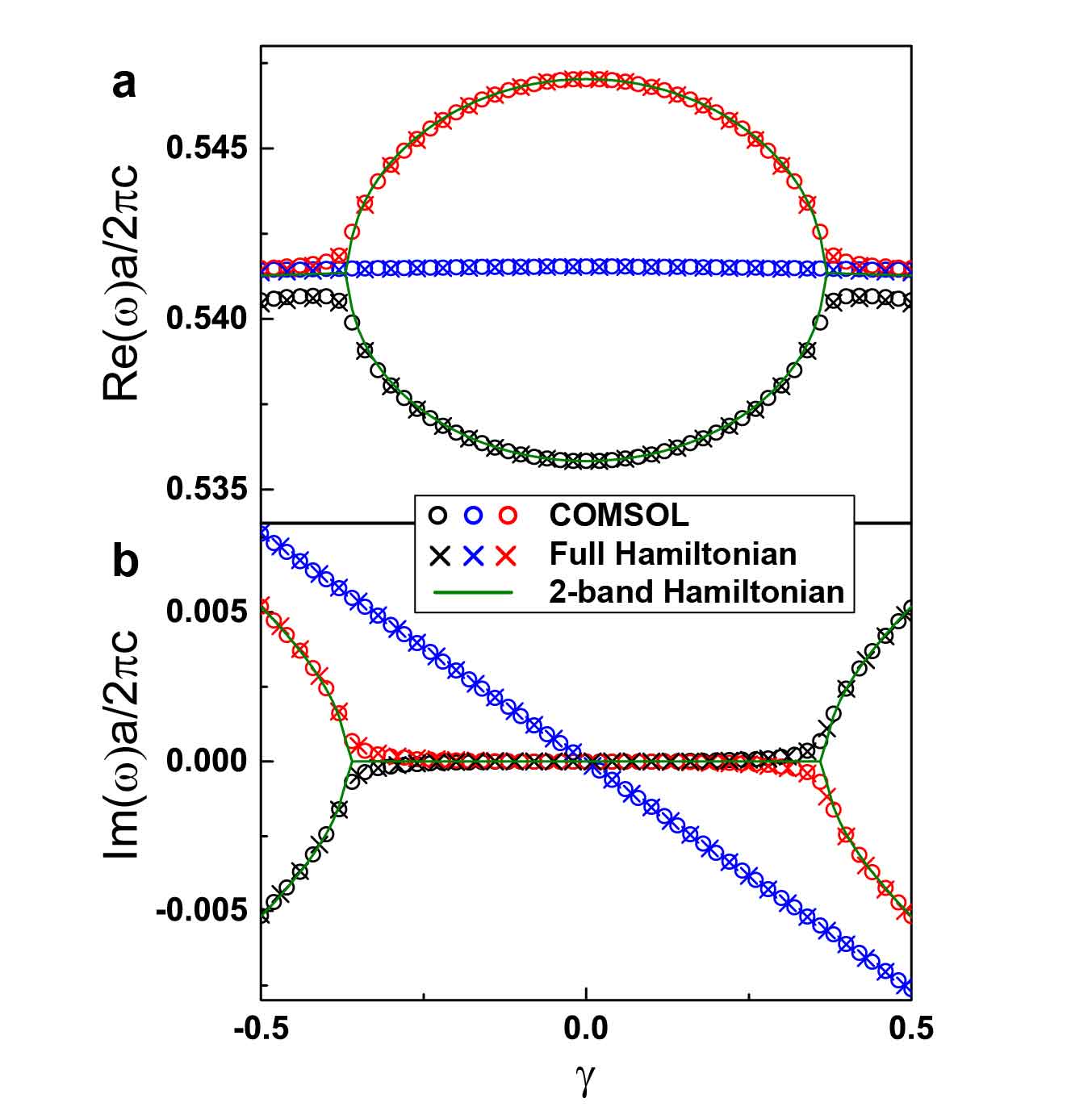


Fig. S3 Eigenfrequencies as a function of non-Hermiticity strength . **a** Real parts and **b** imaginary parts of eigenfrequencies as a function of non-Hermitian strength  at  for . The open circles represent results calculated with COMSOL, and the cross symbols represent results calculated using the full Hamiltonian expressed by Eq. . Different colors represent different bands. The solid green lines represent results from our two-band non-Hermitian Hamiltonian model. The 2D PC is constructed with cylinders () having relative permittivity  (blue areas) embedded in a background medium with relative permittivity  (yellow areas)

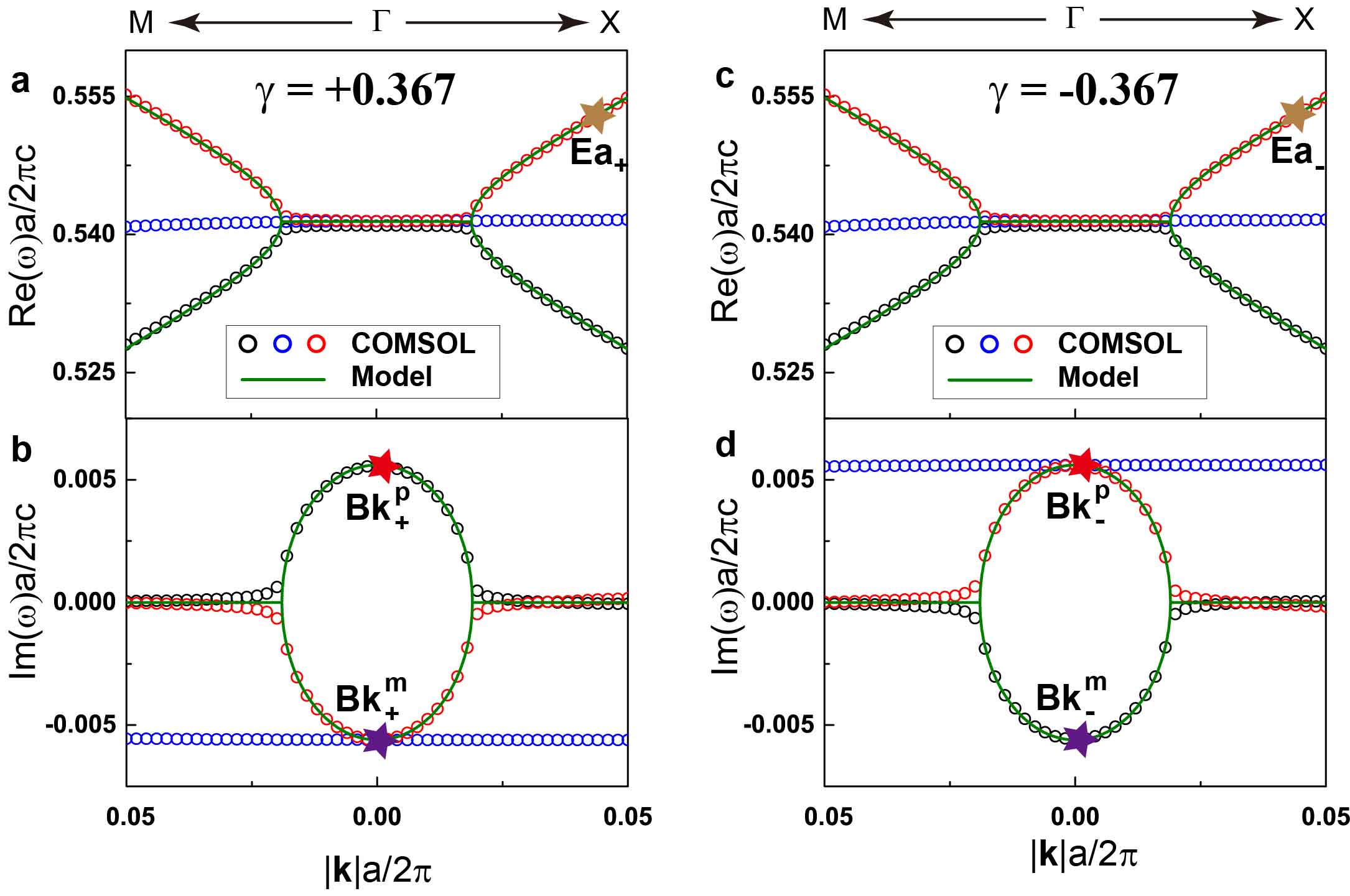


Fig. S4 Band structures of complex conjugate-paired systems. **a, c** Real parts and **b, d** imaginary parts of complex band structures for the 2D PC with **a, b**  and **c, d** . The complex eigenfrequencies are along the  direction. The open circles represent results calculated using COMSOL, while the solid green lines represent results calculated using our analytical model ( and ). The ,  and  denote the different states of two distinct regions. The other parameters are , ,  and .

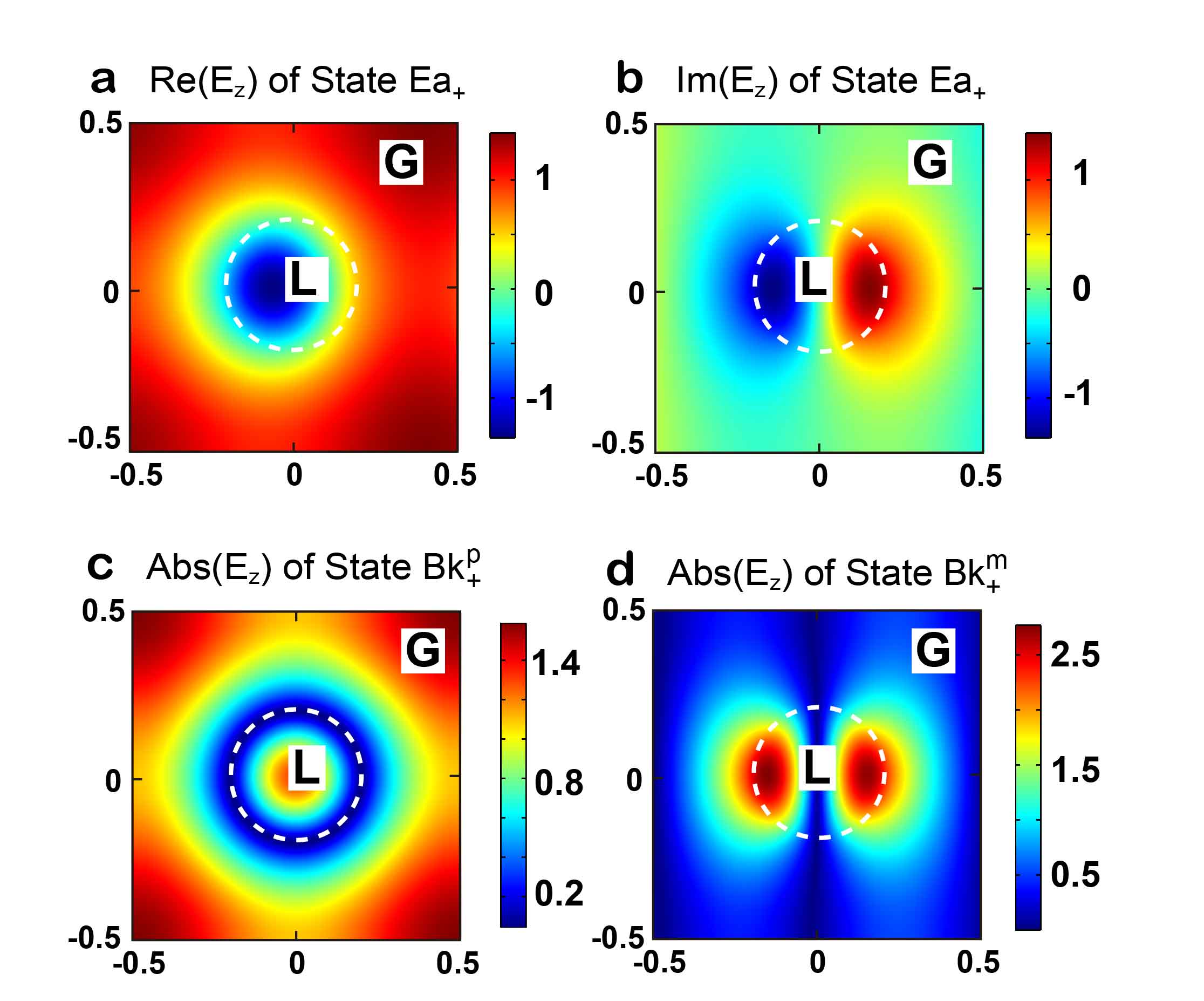


Fig. S5 Eigen field patterns of different eigenstates ( ). The eigenstates are marked by stars in **Fig. S4** for . Real parts and imaginary parts of  field for the state  are plotted in **a** and **b** respectively. The distributions of  field for the state  and  are plotted in **c** and **d**, respectively. The letter “L” and “G” denote the loss (cylinder domain) and gain (background domain).

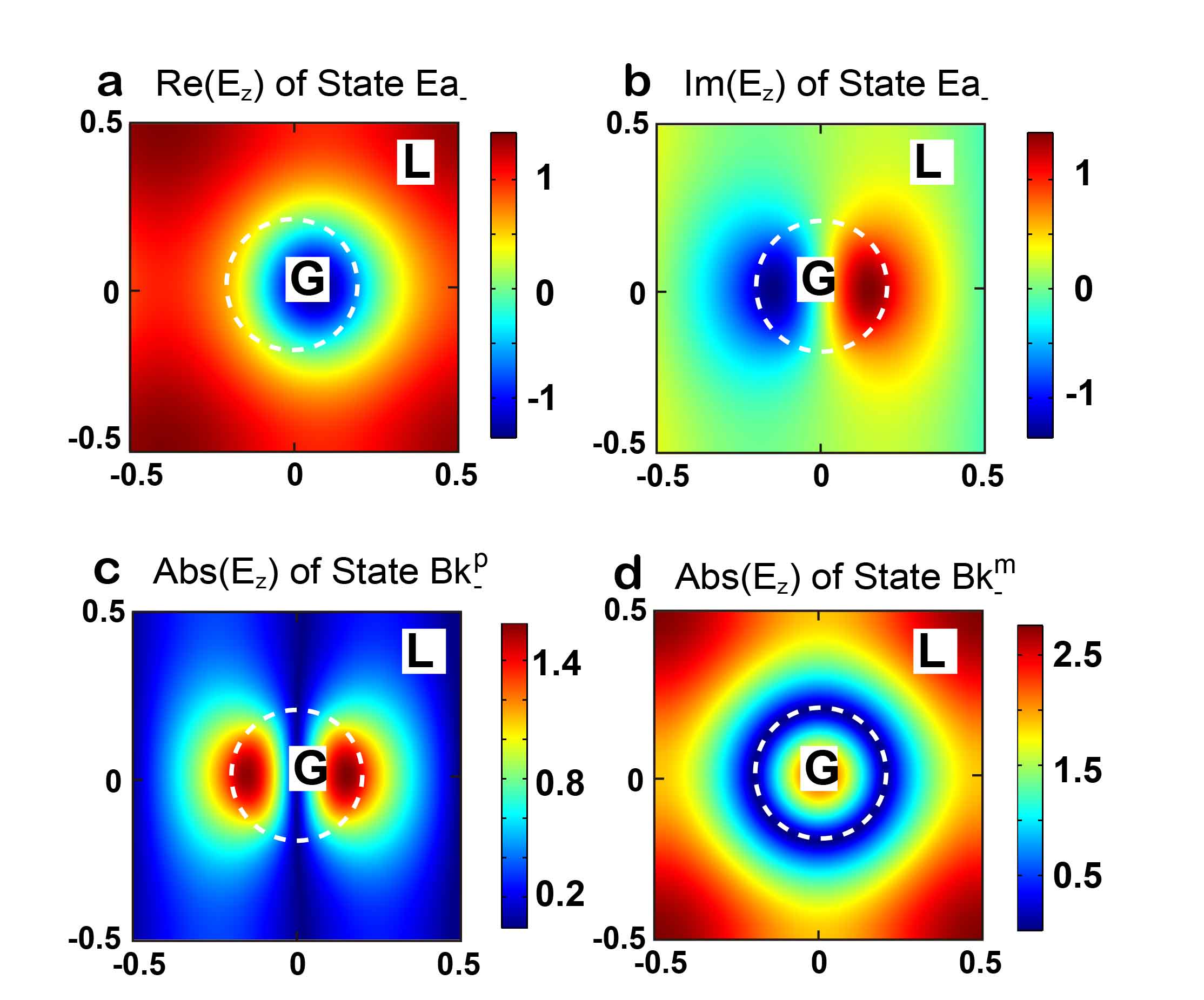


Fig. S6 Eigen field patterns of different eigenstates ( ). The eigenstates are marked by stars in **Fig. S4** for . Real parts and imaginary parts of  field for the state  are plotted in **a** and **b**, respectively. The distributions of  field for the state  and  are plotted in **c** and **d**, respectively. The letter “L” and “G” denote the loss (background domain) and gain (cylinder domain).

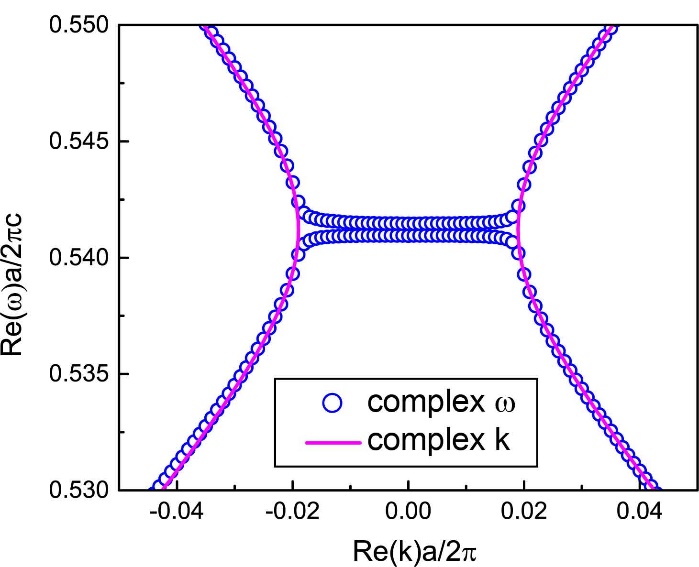


Fig. S7 Comparison between the complex-valued  and  band structure. The band structures in the real-valued **k** and real-valued  plane is plotted for the pseudo-Hermitian bands of 2D PC used in **Fig. S3** with non-Hermiticity . Band structure calculated by complex-valued  method is plotted by blue circles. Band structure calculated by complex-valued  method is plotted by magenta lines. All results are calculated using COMSOL.

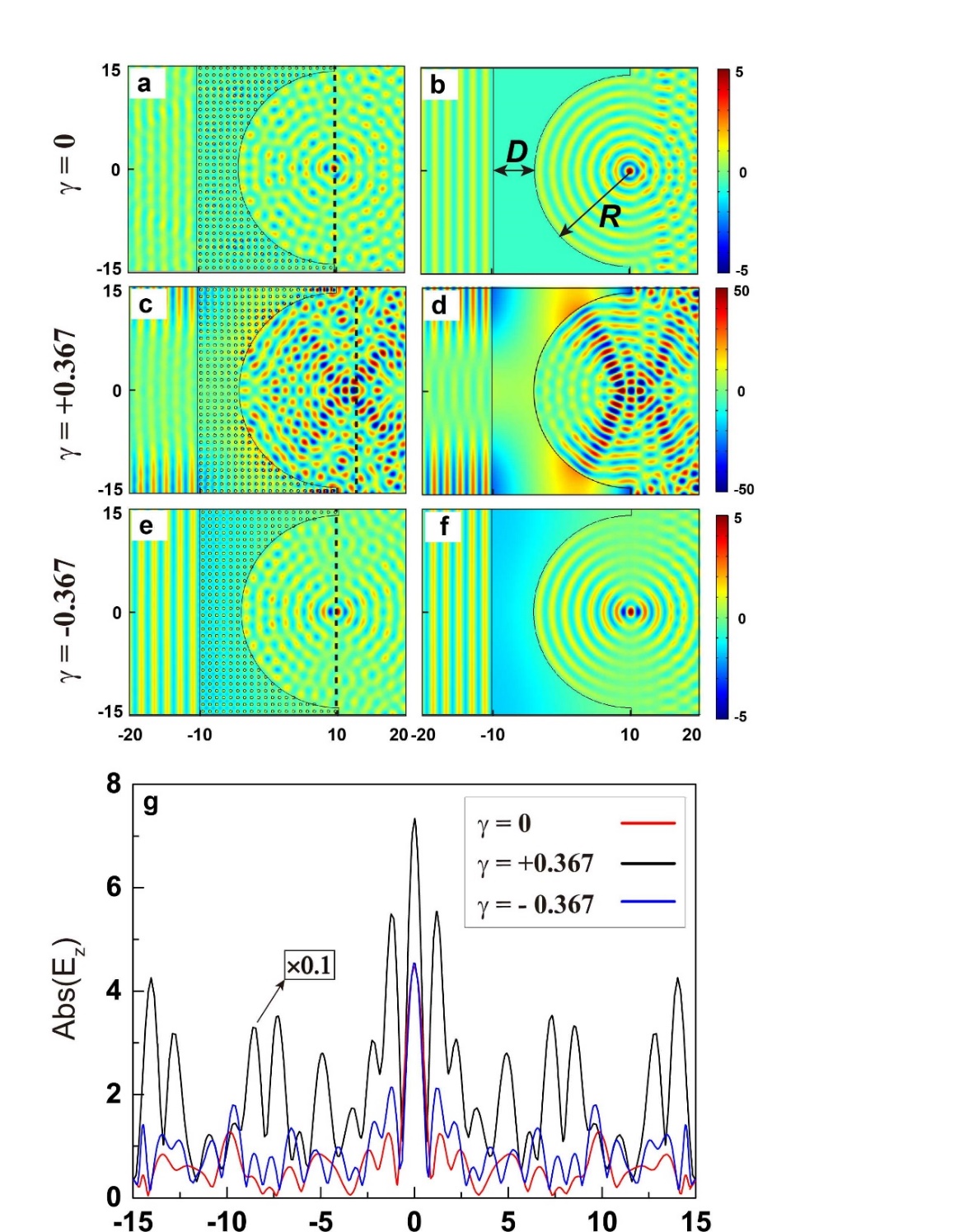


Fig. S8 The focusing effect of PC as a lens. Calculated electric field distributions for PC (**a, c, e**) and EM (**b, d, f**) as a lens. The frequency of the incident wave is . The effective parameters used in **b** are , corresponding to the PC with in **a**. The effective parameters used in **d** are  corresponding to the PC with  in **c**. The effective parameters used in **f** are  corresponding to the PC with  in **e**. The radius of the lens is , and the thickness is .  (dashed lines in **a** and **e**) and  (dashed lines in **c**) at the focusing plane for PCs are plotted in **g**.

Supplementary Notes

Supplementary Note 1: A brief review of pseudo-Hermitian Hamiltonian

Traditional quantum mechanics theory requires the operators of the physical observables to be Hermitian, which guarantees that the system is total energy conserved and the time evolution is unitary. However, non-conservative elements are ubiquitous because dissipation and energy leakage to the surrounding environment are inevitable. In the past decades, there is a surge of interest in studying the physics of non-Hermitian systems. Among various non-Hermitian systems, the parity-time (PT)-symmetric system [1,2] where *P* represents space reflection and *T* represents time reversal, is the most studied one because it can support real spectra. Many novel phenomena have been realized in electronic and classical wave systems with PT-symmetry. As we change the parameters of a PT-symmetric Hamiltonian, a phase transition occurs in the eigenvalues from purely real (exact PT-symmetry) to complex numbers (broken PT-symmetry). The phase transition point is called as an exceptional point (EP), a non-Hermitian degenerate point at which the eigenvalues and eigenstates simultaneously coalesce. However, PT-symmetry is not a necessary condition for achieving a real spectrum [3-6]. A non-Hermitian system possessing any antilinear symmetries is referred to as pseudo-Hermitian system whose eigenfrequencies are purely real in the exact symmetry regime and form complex-conjugate pairs in the broken symmetry regime.

Let  be a linear operator acting in a Hilbert space ** and  be a linear Hermitian invertible operator. Then the operator *H* satisfies



is called -pseudo-Hermitian operator [3]. Pseudo-Hermitian operators have many properties [3-6]. We introduce some basic properties we used in this paper. Every diagonalizable linear operator  with a discrete spectrum is anti-pseudo-Hermitian: , where  is an antilinear anti-Hermitian operator [5]. Let  be a diagonalizable linear operator acting in a Hilbert space  with a discrete spectrum. Then the following are equivalent [5]:

1. The eigenvalues of *H* are real or come in complex-conjugate pairs.
2. *H* is pseudo-Hermitian.
3. *H* has an antilinear symmetry .

If we choose the linear operator as space reflection operator  and the antilinear operator as time reversal operator , the pseudo-Hermitian Hamiltonian is the PT-symmetric Hamiltonian. This indicates that PT symmetry can be considered a subset of the antilinear symmetry class, and real spectra can also be realized in non-PT-symmetric system

Supplementary Note 2: Non-Hermitian Hamiltonian of the graphene system

In this supplementary note, through an analytical model of the Dirac cone of a graphene system, we show that the non-Hermitian graphene system can support real spectra as long as the average non-Hermiticity is zero. In such condition, the loss-gain distribution of the graphene system happens to be PT-symmetric. In fact, the condition that the average non-Hermiticity is zero is automatically satisfied in a PT-symmetric system, because the loss and gain are always equal. The graphene system is a good example to illustrate the relationship between the pseudo-Hermiticity and the PT-symmetry.

The schematic of the graphene system is shown in the inset of Fig. S1a. There are two atoms, labeled as A and B, inside the unit cell, and Dirac cone dispersions appear in the corners of the Brillouin zone. The Hamiltonian describing the Dirac cone dispersion can be written as , where  and  are Pauli matrices and the solid orange lines in Fig. S1a show the dispersion. The eigenfrequencies of  are real numbers, as  is Hermitian. When we introduce the non-Hermiticity into the graphene model, the Hamiltonian will be non-Hermitian, and the eigenfrequencies are complex numbers in general. However, the eigenfrequencies can still be purely real if we tune the ratio of loss and gain properly. To see this, we add non-Hermiticity  to sites A and B, and the Hamiltonian can then be written as



The eigenfrequencies and eigenvectors of the Hermitian part  are

,

and

,

where .

To facilitate the discussion of the underlying pseudo-Hermiticity, we introduce the ratio  and define the non-Hermitian strength . Using these notations, we can write the non-Hermitian part *V* in a simple form as , where



Now let's rotate the non-Hermitian Hamiltonian into the diagonal representation of . We first diagonalize the Hermitian Hamiltonian  using a unitary transformation,



where  is expressed as

,

and

.

Operating the same unitary transformation on the non-Hermitian part *V*, we obtain that

,

where

.

Then, the non-Hermitian Hamiltonian after unitary transformation becomes



where  is the identity matrix. In Eq. ,



is the average non-Hermiticity of these two states () within the primitive unit cell, and



is the coupling between these two states induced by non-Hermiticity.

The central quantity



characterizes the eigenmode profiles.  () expresses the overlapping between the eigenmodes  and in the domain  (either site *A* or site *B*). And  expresses the amplitude distribution of the eigenmode  in the domain  (). The eigenfrequencies of in Eq. are

.

Equation shows that the necessary and sufficient condition for real spectra (or complex conjugate-paired eigenfrequencies) is

,

which is also the criterion of pseudo-Hermiticity because eigenvalues of Hamiltonian being real is equivalent to pseudo-Hermiticity. In the physical sense,  mean that the average non-Hermiticity within the unit cell must be zero for these two states . According to Eq. and the orthonormal relation of the eigenvectors of : , where  is the Kronecker delta function , we solve a specific value of  for  as

.

The value of  should be equal for both states,  and . Therefore, we need

,

which means that the amplitude distribution of the eigenmodes for these two bands () in the domain  () must be equal.

In the graphene system, by substituting the eigenvectors Eq. into Eq. , we obtain

.

Therefore, we solve , and Eq. is reduced to

.

The eigenvalues of the above Hamiltonian are . When , the EPs marked by the green dots in Fig. S1a will emerge. Two bands will coalesce, and the Hamiltonian is defective at those EPs. Figure S1a shows that a ring ( ) formed by EPs will divide the wave vector space  into two regions. Eigenfrequencies are real in the PT-symmetry regime (), and form complex conjugate pairs in the broken PT-symmetry regime (). Therefore, the Hamiltonian is pseudo-Hermitian; and thus the related antilinear symmetry should exist (see Supplementary Note 1 for details). If we choose the linear operator , we find:

.

Hence, we can say  is -pseudo-Hermitian.

The values of  deserve additional comments. According to Eq. , the values of  depend not only on the distributions of non-Hermiticity (described by ) in the unit cell but also on the eigenmode profiles (as described by). The criteria of the non-Hermitian system being pseudo-Hermitian (possessing real spectra) are , which requires . In the graphene system, the eigenmode profiles of the Dirac cone satisfy , resulting in the loss-gain ratio . Thus, the non-Hermiticity distribution in the graphene system is , and the graphene is PT-symmetric in real space, where *P* represents space reflection and *T* represents time reversal. However, for a general non-Hermitian lattice without eigenmode profile like , the relative ratio of non-Hermiticity  should be modified, and the non-Hermitian lattice may not be *PT*-symmetric in real space. The non-PT-symmetric PC in the main text is such an example.

Supplementary Note 3: The formulation of the non-Hermitian Hamiltonian model of a non-Hermitian 2D PC

In this paper, we focus on TM polarization and the eigenvalue problem of the 2D PC shown in the inset of Fig. S1b, which can be described by a Helmholtz equation as

,

where  is the angular frequency and  the speed of light. In principle, the eigenfrequency and eigenfunction of Bloch states for the 2D PC with complex permittivity can be obtained by solving Eq. numerically. However, we are aiming to formulate a non-Hermitian Hamiltonian model to study analytically the band structure of the non-Hermitian PC.

To calculate the complex band structure of a non-Hermitian () PC, we construct a model Hamiltonian using Hermitian () system’s Bloch states obtained at a fixed value of **k** as the bases. The Bloch states for Hermitian system can be expressed as , where *n* denotes the band index and is a positive integer. The periodic function  and the corresponding eigenfrequency  can be obtained numerically using COMSOL. To construct the Hamiltonian for any given **k**, we express Bloch wave functions of the non-Hermitian system as  with . Substituting this expansion into Eq. , we arrive at



where  and  are respectively the real parts and imaginary parts of permittivity, and  and  are eigenfrequencies for  and  case. Multiplying Eq. by  and integrating it within a unit cell, we obtain 

in which

,

is the integration within a unit cell for a specific Bloch **k**.

We use the orthonormal relation



to normalize the eigenvector . By analogy with Eq. of the graphene system, in order to characterize the eigenmode profiles, we introduce a quantity



where

,

is an indicator function (or a high dimensional step function) and  denotes the rod ( ) or air () domain. Then the orthonormal relation in Eq. can be rewritten as

,

where  denotes the relative permittivity of the rod domain and  represents the relative permittivity of the air domain.

The eigenvalue problem described by Eq. can be rewritten as a generalized eigenvalue problem for a specific **k** as



where  are the eigenvectors. The matrices in Eq. are

. For convenience, we omit the subscript **k** for simplicity and rewrite Eq. as



where  is generally a complex number, and

.

This Hamiltonian model formulation only needs the eigenfrequencies and the eigenfunctions of the system in Hermitian limit.

Since  involves the spatial integrations of eigenfunctions of Hermitian systems, symmetry plays an important role here. In our system, we choose the center of the rod as origin. If the states  and  have the same spatial symmetry (for example, both of them are even about the *x* axis),  is non-zero. If the states  and  have opposite spatial symmetry (one is odd and the other is even),  will be zero. This symmetry-based argument can further simplify our Hamiltonian and offers us a good tool to analyze the non-Hermiticity.

We first derive an analytical form of the Hamiltonian using only three bands, namely the three bands of the Dirac-like cone. According to group theory, we know that along the  direction (the subgroup is ), the eigenfunctions for the flat band are odd functions () under the mirror reflection , while the eigenfunctions for the two linear bands are even functions (). Since the two linearly dispersive bands () are even while the flat band () is an odd mode, the matrix elements  and  in  are all zero, implying that there is no non-Hermiticity-induced coupling between the flat band and the linear bands. Therefore, the Hamiltonian matrices  and  in Eq. can be formally written as





where . In Eq.



is a real number, representing the average non-Hermiticity of state *m* within the primitive unit cell, and



is a complex number, representing an overlapping between two eigenmodes . As some elements in Eq. are zero, we can decouple the flat band from the Hamiltonian, and write the Hamiltonian as

,

where .

Supplementary Note 4: Proof of pseudo-Hermitian criteria for the Dirac-like cone

In the absence of non-Hermiticity (), the Hamiltonian is a diagonal matrix, and the eigenvalues are . It is the existence of  that makes the Hamiltonian non-Hermitian and the eigenvalues are

.

If we want to make Hamiltonian pseudo-Hermitian, the eigenvalues must be purely real or come in complex conjugate pairs. Therefore, we must have

.

Because  and  are not zero, we have the pseudo-Hermitian criteria

.

Under such conditions,  is also a real number, and the eigenvalues in Eq. are either real or complex conjugate pairs.

Supplementary Note 5: Group symmetry of Eigenstates at the  point

In this note, we try to show that the symmetry of the eigenstates at the  point guarantees that all three bands are decoupled. It is known that at the  point, according to the group theory, there are two degenerate modes of *E* representation, and one single mode *A*1 representation [7]. Based on these mode symmetries, we can analyze the amplitude distribution function of these modes and the related couplings between them. We express Bloch eigenfunctions of two degenerate *E* modes as  and , and the *A*1 mode as . For a square lattice PC with the  symmetry, we know operating  (rotation by ) on these two *E* mode eigenfunctions transforms them into the following forms [7],

.

For the *A*1 mode, we have

.

Similarly, operating  (rotation by ) on these two *E* modes functions transforms them into the following forms

,

.

For the *A*1 mode, we have

.

Based on the above symmetry relations, we can prove that the following amplitude distribution functions must be zero. The proof is in the following:

,

.

In the above equations, the subscript denotes the integration range. Equations and show that all couplings between these three modes at the  point are zero. Furthermore, these symmetry relations also give the following identity as

,

.

Equation states that amplitude distribution functions for two degenerate *E* modes are always equal in different domains (rod or background) separately.

As illustrated in the main text, the symmetry at the  point is higher than that at other **k** points, and we can analyze its eigenfrequency behavior separately. The symmetry of the eigenstates at the  point guarantees that all three bands are decoupled, so that the coupling terms . Given that three bands are degenerate at the Dirac-like point, i.e., , the eigenvalues of the Hamiltonian at the  point become . If we wish to achieve pseudo-Hermiticity in these bands, then the upper and lower eigenvalues must form a complex conjugate pair, namely . This requires , which is the pseudo-Hermitian condition at the  point. This criterion is in fact less demanding than  for the other **k** points due to the higher symmetry at the  point.

Supplementary Note 6: Pseudo-Hermitian criteria can always be satisfied near the Dirac-like cone

From the main text, we know that in order to obtain , we must tune the system parameters so that the system satisfies . Therefore, we plot the amplitude distribution function  for filling ratio  and  in Figs. S2a and S2b, respectively. The Dirac-like cone appears between these two filling ratios. The solid lines represent amplitude distributions in the cylinder () and the dashed lines represent amplitude distributions in the background medium (). We note that when  (Fig. S2a), the flat band and the upper band are degenerate at the  point in the band structure, which means that they are two degenerate *E* modes. And as shown by Eq. , the amplitude distribution functions of these two bands (red and blue lines) are also equal. If we increase the filling ratio  to , the flat band is degenerate with the lower band, and these two bands are the two *E* modes. And then the amplitude distribution function of the flat band (blue line) is equal to that of the lower band (black line), as shown in Fig. S2b.

Therefore, there must exist intersection between the amplitude distribution functions of the two linearly dispersive bands near Dirac-like cone during the band inversion process, indicating that  can always be satisfied for some particular . To verify this, we plot the amplitude distribution functions of the three bands for the Dirac-like cone, as shown in Fig. 1B of the main text. We can see that there indeed exists an intersection at  (marked by the dashed line).

Supplementary Note 7: Complex conjugate-paired systems

In this note, we try to find the relations of eigenfrequency and eigenstate between the system with , *i.e.* ,  and its conjugate system , i.e., . We will analytically derive the relations of eigenfrequency and eigenstate protected by all these properties.

Figure. S3 plots the eigenfrequencies as a function of non-Hermiticity strength . The solid green lines show results calculated using the analytical model, which agrees well with the results calculated by COMSOL and full Hamiltonian represented by Eq.. Figure. S3 suggests that there are EPs at  and . We first show the complex band structure in Figs. S4a and S4b for  (loss in the cylinder). Operating time reversal on the system with , we can obtain a system with non-Hermiticity . The eigenfrequencies of the system with  and  are the same according to Eq. (9) of the main text, but the Hamiltonian denoted by Eq. (7) of the main text will be different. In other words, these two systems ( and ) possess the same band dispersion, but different eigenfunctions. This non-Hermitian system does not have time reversal symmetry. To show the correspondence of complex conjugate-paired systems, in Figs. S4c and S4d, we plot the eigenfrequencies as a function of **k** for  (the cylinder is gain medium). Comparing Figs. S4a with S4c, we see that when we change the sign of , the eigenfrequencies of the two bands denoted by solid green lines stay unchanged. This is consistent with our analytical model shown in Eq. (9) of the main text. For the flat band (blue circles), the imaginary parts in Fig. S4b are negative, while the imaginary parts in Fig. S4d are positive. This is because the flat band is not pseudo-Hermitian, and its eigenfrequencies are . The sign change of  just transforms eigenfrequencies into their complex conjugate paired counterpart. The imaginary parts of the bands denoted by red and black circles in Figs. S4b and S4d illustrate .

To understand this relationship better, we plot field patterns of some eigenstates in Figs. S5 and S6. For interpretation purpose, we denote the states with real eigenfrequencies as exact (Ea) states. On the contrary, the states with complex eigenfrequency are labeled as broken (Bk) states. As shown in Figs. S4, some Ea-states and Bk-states are labeled out, the subscript  denotes the sign of , and the superscript  denotes the sign of imaginary eigenfrequency, where “p” represents “plus” and “m” represents “minus”.

In Figs. S5a and S5b, the real part and imaginary part of  field for the state  (brown star () in Fig. S4a) behave like a monopole and a dipole respectively. The monopole’s field mainly concentrates in the background medium (gain medium), and the dipole’s field mainly concentrates in the cylinder medium (loss medium). Since the field of the state  is a combination of monopole and dipole, the eigenfrequency of state  can be a purely real number, with almost zero imaginary parts. This is because the electric field distributes almost equally in the loss and gain domain. A similar statement holds true for the state  (brown star in Fig. S4c), as shown in Figs. S6a and S6b. The field of the state  is also a combination of the dipole and monopole, and then the electric field distributes almost equally in the loss and gain domain. Therefore, the imaginary parts of the eigenfrequencies are almost zero.

In contrast to the state  and , which lie outside the ring of EPs, the eigenfunctions for the states inside the ring of EPs have different properties. Figure S5c shows the pattern of  field of the state  (red star () in Fig. S4b). This state is a monopolar mode with field localizing mostly inside the background medium (gain domain), resulting in an eigenfrequency with the positive imaginary part. For comparison, we also plot the field of the state  (red star in Fig. S4d) in Fig. S6c. This is a dipolar mode, whose field concentrates in the cylinder (gain domain), resulting in a positive imaginary part of eigenfrequencies also. The state  and  have the same eigenfrequencies, but the eigenfunctions of them are quite different. Figure S5d shows the field  of state  (purple star () in Fig. S4b). This state is a dipolar mode whose field concentrates in the cylinder (lossy domain), resulting in eigenfrequencies with negative imaginary parts. We see that the eigenfunctions of state  and  have the same character, which confirms . The similar conclusion holds true for the state  and  too, as shown in Figs. S5c and S6d.

Supplementary Note 8: Effective medium theory of complex conjugate-paired systems

In this note, we try to calculate the effective parameters of the pseudo-Hermitian bands of the PC. When studying the non-Hermitian Hamiltonian model, we usually calculate the “complex-valued (**k**) vs. real-valued **k**” band structure for the given real **k**. Fig. S4 is such an example. However, the effective parameters should be calculated as a function of real-valued . Therefore, in order to calculate effective parameters of the PCs, we need to solve the Helmholtz equation for TM polarization in the “complex-valued () vs. real-valued ” plane. For a non-Hermitian PC, we cannot build a relationship between “complex-valued ”band and “complex-valued ” band in general. But for bands with real eigenfrequencies, as shown in Fig. S7, the bands calculated by “complex-valued ”method should coincide with the bands calculated by “complex-valued ” method.

Next, let’s consider the effective parameters calculated by “complex-valued ” method. Substituting the Bloch state into the Helmholtz equation and taking complex conjugation, we obtain

,

for the system with . Taking into account , we obtain

.

Equations and give the following relations between the conjugate-paired systems as

,

and the corresponding electric fields are

.

Considering the Bloch vector , and using , we obtain the magnetic field



By using Eq., we can give the relation of the field averaging ratio (defined in the main text) between conjugate-paired systems as





Therefore, the effective relative permittivity and permeability obey the following relation

.

Supplementary Note 9: The focusing effect of PC as a lens

The zero-refractive-index medium can be used to realize a focusing lens [8]. In this note, we will show that the focusing effect of the CCM as a lens, since the refractive index is very close to zero. As we add non-Hermiticity into this PC, the effective refractive index is very small as shown in Fig. 3C of the main text. Therefore, the phase change in the PC is also very small, and we can expect a good focusing effect of our PC as a lens. However, the numerically simulated field distributions for the PC as a lens shows that the results depend on the details.

For comparison, we first plot the focusing effect for Hermitian case  in Figs. S8a and S8b. Figures S8c and S8d show the focusing effect of PC with  and the corresponding EM with  respectively. Figures S8e and S8f show the focusing effect of PC with  and the corresponding EM with  respectively. Comparing with the Hermitian case, we see that the focusing is still fairly good when , but complicated field patterns are observed for . This is because the lens’ thickness is very close to the *S* matrix pole for , but far away from the *S* matrix pole for . Therefore, the focusing effect for  is marked by strong scattering fields due to the *S* matrix pole. To show this further, in Fig. S8g, we plot the distribution of  on the focusing plane. The fields for non-Hermitian PC with  (black lines) are about 10 times stronger than the non-Hermitian PC  (blue lines). Although the refractive index satisfies , the effective permeability , leading to a quite different focusing effect.

Supplementary References

1. Bender, C. M., Brody, D. C. & Jones, H. F. Complex extension of quantum mechanics. *Phys. Rev. Lett.* **89,** 270401 (2002).
2. Bender, C. M. Making sense of non-Hermitian Hamiltonians. *Reports Prog. Phys.* **70,** 947–1018 (2007).
3. Mostafazadeh, A. Pseudo-Hermiticity versus PT symmetry: The necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian. *J. Math. Phys.* **43,** 205–214 (2002).
4. Mostafazadeh, A. Pseudo-Hermiticity versus PT symmetry II: A complete characterization of non-Hermitian Hamiltonians with a real spectrum. *J. Math. Phys.* **43,** 2814 (2002).
5. Mostafazadeh, A. Pseudo-Hermiticity versus PT-symmetry III: Equivalence of pseudo-Hermiticity and the presence of antilinear symmetries. *J. Math. Phys.* **43,** 3944–3951 (2002)
6. Mostafazadeh, A. Exact PT-symmetry is equivalent to Hermiticity. *J. Phys. A. Math. Gen.* **36,** 7081–7091 (2003)
7. Sakoda K., *Optical Properties of Photonic Crystals,* (Springer Berlin Heidelberg, 2001).
8. Huang, X., Lai, Y., Hang, Z. H., Zheng, H. & Chan, C. T. Dirac cones induced by accidental degeneracy in photonic crystals and zero-refractive-index materials. *Nat. Mater.* **10,** 582–586 (2011).