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Ali El Mfadel*, Said Melliani, Abderrazak Kassidi, and M'hamed Elomari

Existence of mild solutions for nonlocal ψ–Caputo-type fractional evolution equations with nondense domain

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Abstract: The main crux of this manuscript is to establish the existence and uniqueness of solutions for nonlocal fractional evolution equations involving ψ–Caputo fractional derivatives of an arbitrary order $\alpha \in (0, 1)$ with nondense domain. The mild solutions of our proposed model are constructed by employing generalized ψ–Laplace transform and some new density functions. The proofs are based on Krasnoselskii fixed point theorem and some basic techniques of ψ–fractional calculus. As application, a nontrivial example is given to illustrate our theoritical results.

Keywords: ψ–fractional integral, ψ–Caputo fractional derivative, ψ–mild solution

MSC: 34K37, 26A33, 34A08

1 Introduction

Fractional differential equations are a generalization of ordinary differential equations and integration of arbitrary noninteger orders. The theory of fractional calculus is an interesting and popular tool in modelling many phenomena in various fields of engineering, physics and economics. It often appears in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see[8, 13, 14, 18, 20, 22]).Due to an increase in applications to queries regarding the generalization of this form of calculation, this theory has changed in both internal structure and scope. Among these generalizations, the researchers in [2] manage to define a fractional derivative which remains the most valid, they consider a Caputo type fractional derivative with respect to another function. Some properties between fractional derivatives and fractional integrals are proved in their paper. The reader can consult articles as well [3, 9–12, 21, 25] and the references therein for more details. As is known, the nonlocal Cauchy problem is motivated by physical phenomena and abstract partial differential equations with fractional derivatives in space and time are known as fractional diffusion equations. They're valuable for modeling anomalous diffusion, which occurs when a plume of particles expands in a way that isn't predicted by the classical diffusion equation. ewski in [3] pioneered nonlocal circumstances by demonstrating the existence and uniqueness of mild and classical solutions for nonlocal Cauchy problems. According to Byszewski and Lakshmikantham [13], the nonlocal condition can be more effective in describing some physical processes than the normal initial condition. In physics, the nonlocal condition has a better effect than the classical data condition $x(0) = x_0$, for example $\varphi(x)$ might be expressed by $\varphi(x) = \sum_{i=1}^{m} c_i x(t_i)$, where c_i, $i = 1, 2, \dots, n$ are given constants and $0 < t_1 < t_2 < \dots < t_n < a$. Zhou et al in [26], studied nonlocal Cauchy problem for fractional evolution equations in an arbitrary Banach space X. Some several criteria of

***Corresponding Author: Ali El Mfadel:** Sultan Moulay Slimane University, Beni Mellal, Morocco,, E-mail: a.elmfadel@usms.ma
Said Melliani: Sultan Moulay Slimane University, Beni Mellal, Morocco, E-mail: s.melliani@usms.ma
Abderrazak Kassidi: Sultan Moulay Slimane University, Beni Mellal, Morocco, E-mail: abderrazakassidi@gmail.com
M'hamed Elomari: Sultan Moulay Slimane University, Beni Mellal, Morocco, E-mail: m.elomari@usms.ma

the existence and uniqueness for mild solutions are established.
As a result of Hille-Yosida Theorem (see [24]) it is well known that a lineare unbounded operator A is the infinitisimal generator of a C_0–semigroup $\{T(t)\}_{t\geq 0}$ if and only if

1. A is closed and $\overline{D(A)} = X$,
2. there exist two constant $\omega, M \in \mathbb{R}$ such that $(\omega,\ +\infty) \subseteq \rho(A)$ and

$$\|(\lambda I - A)^{-k}\|_{\mathcal{L}(X)} \leq \frac{M}{(\lambda - \omega)^k},\ \text{for all } \lambda > \omega, k \geq 1.$$

Now, the main question is how to investigate the fractional evolution problem with the absence of a dense domain $D(A)$ for the operator A?
Some researchers use the operator A_0 which is a restriction of A on $\overline{D(A)}$ defined as follows:

$$\begin{cases} D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\}, \\ A_0 x = Ax. \end{cases}$$

G.M. Mophou and G.M. N'Guérékata in [23] discuss the existence and uniqueness of the integral solution for a nondensely fractional semilinear differential equation with nonlocal conditions in a Banach space X. They precisely considered the following evolution problem:

$$\begin{cases} D^q x(t) = Ax(t) + f(t,\ x(t))\ ,\ t \in [0,\ T], \\ x(0) + g(x) = x_0. \end{cases}$$

Where T is a positive real constant, $0 < q < 1$, the operator A : $D(A) \subset X \to X$ is not necessarily densely defined, $f : [0,\ T] \times X \to X$ and g: $\mathcal{C}\left([0, T]\right) \to \overline{D(A)}$ are continuous functions. They used Krasnoselskii theorem to prove the existence and uniqueness results. Gu et al in [17] studied the existence of integral solutions for two classes of fractional order evolution equations with nondensely defined linear operators. First, They look at the nonhomogeneous fractional order evolution equation and use Laplace transform and probability density function to construct the integral solution. They established the existence of this integral solution by using the noncompact measure approach.
Motivated by the above works especially by [17], we study the existence and uniqueness results for the following ψ–Caputo fractional evolution equation:

$$\begin{cases} {}^C D_{0^+}^{\alpha,\psi} x(t) = Ax(t) + f\left(t, x(t)\right), \quad t \in J = [0, T], \\ x(0) + \Phi(x) = x_0. \end{cases} \tag{1}$$

Where ${}^C D_{0^+}^{\alpha,\psi}$ is the ψ–Caputo fractional derivative of order $\alpha \in (0, 1)$, $T > 0$, $x_0 \in X$, the operator $A : D(A) \subset X \to X$ is not necessarily densely defined, $f : [0, T] \times X \to X$ and for $l > 0$ there exists a positive function $\mu_l \in L^\infty(J,\ \mathbb{R}_+)$ such that $\sup_{|x|\leq l} \|f(t, x)\| \leq \mu_l(t)$. $\Phi : \mathcal{C}\left([0, T], X\right) \to \overline{D(A)}$ is a K-Lipschitz function with $K > 0$.
Our paper is organized as follows. In Section 2, we give some basic definitions and properties of ψ–fractional integral and ψ–Caputo fractional derivatives which will be used in the rest of this paper. In Section 3, we construct the mild solutions for ψ–Caputo type fractional problem (1) by using generalized Laplace transform and a new density function. We establish our main results in Section 4.As application, an illustrative example is presented in Section 5 followed by conclusion in Section 6.

2 Preliminaries

In this section, we give some notations, definitions and results on ψ-fractional derivatives and ψ-fractional integrals, for more details we refer the reader to [2, 4, 16].

Notations

• We denote by X a Banach space.

• We denote by $\mathcal{C} := C(J, X)$ the Banach space of all continuous functions endowed with the topology of uniform convergence denoted by

$$\|x\|_\infty = \sup_{t\in J} \|x(t)\|.$$

• We denote by B_r the closed ball centered at 0 with radius $r > 0$.

Definition 1. *[1] Let $q > 0$, $g \in L^1(J, \mathbb{R})$ and $\psi \in C^n(J, \mathbb{R})$ such that $\psi'(t) > 0$ for all $t \in J$. The ψ-Riemann-Liouville fractional integral at order q of the function g is given by*

$$I_{0^+}^{\alpha,\psi} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} g(s)ds.$$

Definition 2. *[1] Let $\alpha > 0$, $g \in C^{n-1}(J, \mathbb{R})$ and $\psi \in C^n(J, \mathbb{R})$ such that $\psi'(t) > 0$ for all $t \in J$. The ψ–Caputo fractional derivative at order q of the function g is given by*

$${}^C D_{0^+}^{\alpha,\psi} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1} g_\psi^{[n]}(s)ds.$$

Where

$$g_\psi^{[n]}(s) = \left(\frac{1}{\psi'(s)} \frac{d}{ds}\right)^n g(s) \quad and \quad n = [\alpha] + 1.$$

And $[q]$ denotes the integer part of the real number α.

Remark 1. In particular, if $\alpha \in]0, 1[$, then we have

$${}^C D_{0^+}^{\alpha,\psi} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} g'(s)ds.$$

And

$${}^C D_{0^+}^{\alpha,\psi} g(t) = I_{0^+}^{1-\alpha,\psi} \left(\frac{g'(t)}{\psi'(t)}\right).$$

Proposition 1. [1] Let $q > 0$, if $g \in C^{n-1}(J, \mathbb{R})$, then we have

1) ${}^C D_{0^+}^{\alpha,\psi} I_{0^+}^{\alpha,\psi} g(t) = g(t)$.
2) $I_{0^+}^{\alpha,\psi}\, {}^C D_{0^+}^{\alpha,\psi} g(t) = g(t) - \sum_{k=0}^{n-1} \frac{g_\psi^{[k]}(0)}{k!} (\psi(t) - \psi(0))^k$.
3) $I_{a^+}^{\alpha,\psi}$ is linear and bounded from $C(J, \mathbb{R})$ to $C(J, \mathbb{R})$.

Proposition 2. [1] Let $t > 0$ and $\alpha, \beta > 0$, then we have

1) $I_{0^+}^{\alpha,\psi} (\psi(t) - \psi(0))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\psi(t) - \psi(0))^{\alpha+\beta-1}$.
2) ${}^C D_{0^+}^{\alpha,\psi} (\psi(t) - \psi(0))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\psi(t) - \psi(0))^{\alpha-\beta-1}$.
3) ${}^C D_{0^+}^{\alpha,\psi} (\psi(t) - \psi(0))^n = 0$, for all $n \in \mathbb{N}$.

Definition 3. *[19] Let $x : J \to X$ be a function. The generalized Laplace transform of x is given by*

$$\mathcal{L}_\psi\{y(t)\}(s) := \widehat{x}(s) = \int_0^\infty \psi'(t) e^{-s(\psi(t)-\psi(0))} x(t)dt.$$

Definition 4. *[19] Let f and g be two functions which are piecewise continuous on J and of exponential order . The generalized ψ–convolution of f and g is defined by*

$$(f \underset{\psi}{\star} g)(t) = \int_0^t f(s)g(\psi^{-1}(\psi(t) + \psi(0) - \psi(s)))\psi'(s)ds.$$

Lemma 1. (See [19]). Let $q > 0$ and y be a piecewise continuous function on each interval $[0, t]$ and $\psi(t)$-exponential order. Then we have

1. $\mathcal{L}_\psi\{I_{0^+}^{q,\psi} y(t)\}(s) = \dfrac{\widehat{y}(s)}{s^q}$.
2. $\mathcal{L}_\psi\{{}^C D_{0^+}^{q,\psi} y(t)\}(s) = s^q \left[\mathcal{L}_\psi\{y(t)\} - \sum_{k=0}^{n-1} s^{-k-1} f^{(k)}(0)\right]$, where $n = [q] + 1$.

Definition 5. *(See [21]) Let $\rho \in [0, \infty)$. The one-sided stable probability density is defined by*

$$\omega_\alpha(\rho) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}(\psi(\rho) - \psi(0))^{-\alpha n-1}\frac{\Gamma(\alpha n + 1)}{n!}\sin(n\pi\alpha).$$

Lemma 2. [21] The Laplace transform of $\omega_\alpha(t)$ is given by

$$\int_0^\infty e^{-\lambda(\psi(t)-\psi(0))}\omega_\alpha(t)\psi'(t)dt = e^{-\lambda^\alpha}.$$

Theorem 1. (Krasnoselskii Theorem [5]) Let C be a closed convex and nonempty subset of a Banach space X. Let *F and L be two operators such that*

1. *$Fx + Ly \in C$ whenever $x, y \in C$,*
2. *F is a contraction mapping,*
3. *L is compact and continuous.*
   Then there exists $z \in C$ such that $z = Fz + Lz$.

3 Construction of mild solutions

In this section, we use the ψ-Laplace transform to construct the integral solution for the fractional evolution problem (1). For this purpose we need to prove to the following lemma.

Lemma 3. The fractional evolution problem (1) is equivalent to the following integral equation

$$x(t) = x_0 + I_{0^+}^{\alpha,\psi}\left(Ax(t) + f\left(t, x(t)\right)\right), \ \forall t \in J. \tag{2}$$

Proof. Let x be a solution of the problem (1), then we apply the ψ–fractional integral $I_{0^+}^{\alpha,\psi}$ on both sides of (1) we get

$$I_{0^+}^{\alpha,\psi}\, {}^C D_{0^+}^{\alpha,\psi} x(t) = I_{0^+}^{\alpha,\psi}\left[Ax(t) + f\left(t, x(t)\right)\right],$$

and by using Proposition 1 we obtain

$$x(t) - x(0) = I_{0^+}^{\alpha,\psi}[Ax(t) + f(t, x(t))],$$

since $x(0) + \Phi(x) = x_0$, it follows that

$$x(t) = x_0 - \Phi(x) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}[Ax(s) + f\left(s, x(s)\right)]ds.$$

Hence the integral equation (2) holds.

Conversely, by direct computation, it is clear that if x satisfies the integral equation (2), then the equation (1) holds which completes the proof. □

Definition 6. *A function $x(t)$ is said to be an integral solution of* (1) *if*
1. $x : J \to X$,
2. $I_{0^+}^{\alpha,\psi} x(t) \in D(A), \ \forall t \in J$,
3. $x(t) = x(0) + \Phi(x) + A I_{0^+}^{\alpha,\psi} x(t) + I_{0^+}^{\alpha,\psi} f\left((t, x(t))\right)$

Remark 2. We have the following remarks.
1. By using Proposition 1, we have $I^{1,\psi} x(t) = I^{1-\alpha,\psi} I_{0^+}^{\alpha,\psi} x(t)$.
2. If $x(t)$ is an integral solution of (1), then $I_{0^+}^{\alpha,\psi} x(t) \in D(A), \ \forall t \in J$, which implies that $I^{1,\psi} x(t) = I^{1-\alpha,\psi} I_{0^+}^{\alpha,\psi} x(t) \in D(A)$ for $t \in J$.
3. The limite $\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} x(s)ds \in X_0$ for $t \in J$ shows that $x(t) \in D(A)$.

Let A_0 be the part of A in $X_0 = \overline{D(A)}$ defined by

$$\begin{cases} D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\}, \\ A_0 x = Ax. \end{cases}$$

We assume the following hypotheses throughout the rest of our paper.

(H_1)The linear operator $A : D(A) \subset X \to X$ satisfies the Hille-Yosida condition, that is, there exist two constant $\omega, M \in \mathbb{R}$ such that $(\omega, \ +\infty) \subseteq \rho(A)$ and $\|(\lambda I - A)^{-k}\|_{\mathcal{L}(X)} \leq \dfrac{M}{(\lambda - \omega)^k}$, for all $\lambda > \omega, k \geq 1$.

$(H_2)T_t$ is continuous in the uniform topology for $t > 0$.

$(H_3)MK < \frac{1}{2}$.

$(H_4)M\left(K + \frac{MT^\alpha}{\Gamma(\alpha+1)} \|\mu_1\|_{L^1_{\text{loc}}(J,\mathbb{R}_+)}\right) < 1$.

Since the operator A_0 satisfies the Hille-Yosida condition, we can find the mild solution on $D(A_0)$. For this purpose, let us consider the following auxiliary problem:

$$\begin{cases} {}^C D_{0^+}^{\alpha,\psi} x(t) = A_0 x(t) + g(t), \qquad t \in J, \\ \\ x(0) + \Phi(x) = x_0. \end{cases} \tag{3}$$

Wher g is a given continuous function.

Lemma 4. Let $\lambda > \omega$, then the resolvent R_λ of A satisfies

$$R_\lambda := (\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda(\psi(t)-\psi(0))} T\left(\psi(t) - \psi(0)\right) \psi'(t)dt.$$

Proof. Let $x \in D(A)$. From (H_2) it follows that

$$\begin{aligned} \int_0^\infty e^{-\lambda(\psi(t)-\psi(0))} T\left(\psi(t) - \psi(0)\right) x\psi'(t)dt &= \int_0^\infty e^{-\lambda t} T(t)dt, \\ &= (\lambda I - A)^{-1} x. \end{aligned}$$

That's true of all $x \in D(A)$, which implies the results. □

Proposition 3. If the fractional integral equation $x(t) = x_0 + \Phi(x) + I_{0^+}^{\alpha,\psi}\left(A_0 x(t) + g(t)\right)$ holds and g takes values in X_0, then we have

$$x(t) = S_{\alpha,\psi}(t)(x_0 - \Phi(x)) + \int_0^{\psi(t)-\psi(0)} R_{\alpha,\psi}\left(\psi(t) - \psi(s)\right) g(s)\psi^{'}(s)ds,$$

where

$$S_{\alpha,\psi}(t) = I^{1-\alpha,\psi} R_{\alpha,\psi}(t),$$

and

$$R_{\alpha,\psi}(t) = t^{\alpha-1}\int_0^{\infty} \alpha\left(\psi(\rho) - \psi(0)\right)\omega_\alpha(\rho) T_{t^\alpha\left(\psi(\rho)-\psi(0)\right)}\psi^{'}(\rho)d\rho.$$

Proof. Let $\lambda > 0$. From Lemmas 1 and 3 we have

$$\widehat{x} = \frac{1}{\lambda}\left(x_0 - \Phi(x)\right) + \frac{1}{\lambda^\alpha}\left(A_0\widehat{x} + \widehat{g}\right),$$

we obtain

$$\widehat{x} = \lambda^{\alpha-1}\left(\lambda^\alpha I - A_0\right)^{-1}\left(x_0 - \Phi(x)\right) + \left(\lambda^\alpha I - A_0\right)^{-1}\widehat{g}.$$

Let's pose $I_1 = \lambda^{\alpha-1}\left(\lambda^\alpha I - A_0\right)^{-1}\left(x_0 - \Phi(x)\right)$ and $I_2 = \left(\lambda^\alpha I - A_0\right)^{-1}\widehat{g}$ i.e $I_2 = \int_0^{+\infty} e^{-\lambda^\alpha s} T_s \widehat{g} ds.$

From Lemma 4, we get

$$\begin{aligned}
\lambda^{1-\alpha} I_1 &= \left(\lambda^\alpha I - A_0\right)^{-1}\left(x_0 - \Phi(x)\right)\\
&= \int_0^{+\infty} e^{-\lambda^\alpha\left(\psi(s)-\psi(0)\right)} T_{\psi(s)-\psi(0)}\left(x_0 - \Phi(x)\right)\psi^{'}(s)ds\\
&= \int_0^{+\infty} \alpha e^{-\left(\lambda\left(\psi(t)-\psi(0)\right)\right)^\alpha} T_{(\psi(t)-\psi(0))^\alpha}\left(x_0 - \Phi(x)\right)(\psi(t) - \psi(0))^{\alpha-1}\psi^{'}(t)dt\\
&= \alpha\int_0^{\infty}\int_0^{+\infty} e^{-\lambda\left(\psi(t)-\psi(0)\right)\left(\psi(\rho)-\psi(0)\right)}\chi_\alpha(\rho) T_{(\psi(t)-\psi(0))^\alpha}\left(x_0 - \Phi(x)\right)(\psi(t) - \psi(0))^{\alpha-1}\psi^{'}(\rho)\psi^{'}(t)dtd\rho\\
&= \alpha\int_0^{\infty}\int_0^{+\infty} e^{-\lambda\left(\psi(t)-\psi(0)\right)}\chi_\alpha(\rho) T_{\left(\frac{\psi(t)-\psi(0)}{\psi(\rho)-\psi(0)}\right)^\alpha}\left(x_0 - \Phi(x)\right)\frac{(\psi(t) - \psi(0))^{\alpha-1}}{(\psi(\rho) - \psi(0))^\alpha}\psi^{'}(\rho)\psi^{'}(t)dtd\rho\\
&= \int_0^{\infty} e^{-\lambda\left(\psi(t)-\psi(0)\right)}\Big[\int_0^{\infty}\alpha\chi_\alpha(\rho)\frac{(\psi(t) - \psi(0))^{\alpha-1}}{(\psi(\rho) - \psi(0))^\alpha} T_{\left(\frac{\psi(t)-\psi(0)}{\psi(\rho)-\psi(0)}\right)^\alpha}\left(x_0 - \Phi(x)\right)\psi^{'}(\rho)d\rho\Big]\psi^{'}(t)dt.\\
&= \mathcal{L}_\psi\left(R_{\alpha,\psi}\right)(\psi(t) - \psi(0))\left(x_0 - \Phi(x)\right).
\end{aligned}$$

Where

$$R_{\alpha,\psi}(t) = \int_0^{\infty}\alpha\chi_\alpha(\rho)\frac{(\psi(t) - \psi(0))^{\alpha-1}}{(\psi(\rho) - \psi(0))^\alpha} T_{\left(\frac{\psi(t)-\psi(0)}{\psi(\rho)-\psi(0)}\right)^\alpha}\left(x_0 - \Phi(x)\right)\psi^{'}(\rho)d\rho,$$

we can write

$$R_{\alpha,\psi}(t) = t^{\alpha-1}\int_0^{\infty}\alpha\rho\omega_\alpha(\rho) T_{t^\alpha\left(\psi(\rho)-\psi(0)\right)}\psi^{'}(\rho)d\rho,$$

where

$$\omega_\alpha(\rho)=\big(\psi(\rho)-\psi(0)\big)^{\frac{-1}{\alpha}}\chi_\alpha\left(\psi^{-1}\Big(\Big(\frac{1}{\psi(\rho)-\psi(0)}\Big)^{\frac{1}{\alpha}}+\psi(0)\Big)\right).$$

On other hand, we have

$$\mathcal{L}_\psi\left(\frac{\big(\psi(t)-\psi(0)\big)^{-\alpha}}{\Gamma(1-\alpha)}\right)(\lambda)=\lambda^{\alpha-1}.$$

Which implies that

$$\begin{aligned}I_1&=\mathcal{L}_\psi\left(\frac{\big(\psi(.)-\psi(0)\big)^{-\alpha}}{\Gamma(1-\alpha)}\star R_{\alpha,\psi}(.)\right)(t)\\&=\mathcal{L}_\psi\big(I^{1-\alpha,\psi}R_{\alpha,\psi}(t)\big).\end{aligned}$$

Let us calculate I_2.

$$\begin{aligned}I_2&=\int_0^\infty e^{-\lambda^\alpha(\psi(t)-\psi(0))}T_{\psi(t)-\psi(0)}\widehat{g}\psi'(t)dt\\
&=\int_0^\infty\int_0^\infty e^{-\lambda^\alpha(\psi(t)-\psi(0))}e^{-\lambda(\psi(s)-\psi(0))}T_{\psi(t)-\psi(0)}g(s)\psi'(s)\psi'(t)dsdt\\
&=\int_0^\infty\int_0^\infty\alpha\left(\psi(t)-\psi(0)\right)^{\alpha-1}e^{-\left(\lambda(\psi(t)-\psi(0))\right)^\alpha}e^{-\lambda(\psi(s)-\psi(0))}T_{(\psi(t)-\psi(0))^\alpha}g(s)\psi'(s)\psi'(t)dtds\\
&=\int_0^\infty\int_0^\infty\int_0^\infty\alpha\chi_\alpha e^{-\left(\lambda(\psi(t)-\psi(0))(\psi(\rho)-\psi(0))\right)}e^{-\lambda(\psi(s)-\psi(0))}(\rho)T_{(\psi(t)-\psi(0))^\alpha}\left(\psi(t)-\psi(0)\right)^{\alpha-1}\\
&\quad g(s)\psi'(\rho)\psi'(s)\psi'(t)dtdsd\rho\\
&=\int_0^\infty\int_0^\infty\int_0^\infty\alpha\chi_\alpha(\rho)e^{-\lambda(\psi(t)+\psi(s))}T_{\left(\frac{\psi(t)-\psi(0)}{\psi(\rho)-\psi(0)}\right)^\alpha}\frac{\left(\psi(t)-\psi(0)\right)^{\alpha-1}}{\left(\psi(\rho)-\psi(0)\right)^\alpha}g(s)\psi'(\rho)d\rho\psi'(s)ds\psi'(t)dt\\
&=\int_0^\infty e^{-\lambda(\psi(t)-\psi(0))}\Big[\int_0^{\psi(t)-\psi(0)}\int_0^\infty\alpha\chi_\alpha(\rho)T_{\frac{(\psi(t)-\psi(0))^{\alpha-1}}{(\psi(\rho)-\psi(0))^\alpha}(\psi(\rho)-\psi(0))^\alpha}\frac{\left(\psi(t)-\psi(s)\right)^{\alpha-1}}{\left(\psi(\rho)-\psi(0)\right)^\alpha}\\
&\quad g(s)\psi'(\rho)\psi'(s)d\rho ds\Big]\psi'(t)dt\\
&=\mathcal{L}_\psi\left(R_{\alpha,\psi}\left(\psi(t)-\psi(s)\right)\right).\end{aligned}$$

Thus $x(t)$ can be written as follows

$$x(t)=S_{\alpha,\psi}(t)(x_0-\Phi(x))+\int_0^{\psi(t)-\psi(0)}R_{\alpha,\psi}\left(\psi(t)-\psi(s)\right)g(s)\psi'(s)ds,$$

where $S_{\alpha,\psi}(t)=I^{1-\alpha,\psi}R_{\alpha,\psi}(t)$. Which completes the proof. □

Remark 3. From (H_1) we have $\|R_\lambda\|\le\dfrac{\lambda M}{\lambda-\omega}$, then we get

$$\lim_{\lambda\to+\infty}\|R_\lambda\|\le M.$$

Proposition 4. We assume that Φ is K-lipschitz and (H_2) holds, then

1. for a fixed $t > 0$, $\{R_{\alpha,\psi}(t)\}_{t>0}$ and $\{S_{\alpha,\psi}(t)\}_{t>0}$ are linear operators.
2. for $x \in X_0$, then $\left|R_{\alpha,\psi}(t)x\right| \leq \frac{t^{\alpha-1}M}{\Gamma(1+\alpha)}|x|$ and $\left|S_{\alpha,\psi}(t)x\right| \leq \frac{M}{\alpha}|x|$.
3. $\{R_{\alpha,\psi}(t)\}_{t>0}$ and $\{S_{\alpha,\psi}(t)\}_{t>0}$ are strongly continuous,
   ie. For $x \in X_0$ and $0 < h \leq T$,

$$\lim_{t_1 \to t_2} |R_{\alpha,\psi}(t_1)x - R_{\alpha,\psi}(t_2)x| = 0 \text{ and } \lim_{t_1 \to t_2} |S_{\alpha,\psi}(t_1)x - S_{\alpha,\psi}(t_2)x| = 0.$$

Proof. Since we have $\int\limits_0^\infty \alpha\rho\omega_\alpha(\rho)\psi'(\rho)d\rho = \dfrac{1}{\Gamma(1+\alpha)}$, then

$$\left|R_{\alpha,\psi}(t)x\right| \leq \frac{t^{\alpha-1}M}{\Gamma(1+\alpha)}|x|.$$

From the above inequality it follows that

$$\begin{aligned}\left|S_{\alpha,\psi}(\psi(t)-\psi(0))x\right| &= \left|I^{1-\alpha,\psi}R_{\alpha,\psi}(\psi(t)-\psi(0))x\right| \\ &\leq \frac{MI^{1-\alpha,\psi}(\psi(t)-\psi(0))^{\alpha-1}}{\Gamma(1+\alpha)}|x| \\ &\leq \frac{M\Gamma(\alpha)}{\Gamma(1+\alpha)}|x| \\ &\leq \frac{M}{\alpha}|x|,\end{aligned}$$

which implies that $\left|S_{\alpha,\psi}(t)x\right| \leq \frac{M}{\alpha}|x|$.

□

Lemma 5. The integral solution of the problem (3) is given by

$$x(t) = S_{\alpha,\psi}(t)(x_0 - \Phi(x)) + \lim_{\lambda\to\infty} \int\limits_0^{\psi(t)-\psi(0)} R_{\alpha,\psi}\left(\psi(t)-\psi(s)\right) R_\lambda g(s)\psi'(s)ds. \tag{4}$$

Proof. We have that

$$x_\lambda(t) = R_\lambda x(t)\,,\ g_\lambda(t) = R_\lambda g(t)\,,\ x_\lambda = R_\lambda x(0).$$

By applying R_λ to (3), we have

$$x_\lambda(t) = x_\lambda + A_0 I_{0^+}^{\alpha,\psi} x_\lambda(t) + I_{0^+}^{\alpha,\psi} g_\lambda(t)\,,$$

hence

$$x_\lambda(t) = S_{\alpha,\psi}(t)x_\lambda + \int\limits_0^{\psi(t)-\psi(0)} K_{\alpha,\psi}(\psi(t)-\psi(s))g_\lambda(s)ds,$$

since $x(t), x(0) \in X_0$, we have

$$x_\lambda(t) \to x(t),\ x_\lambda \to x(0),\ S_{\alpha,\psi}(t)x_\lambda \to S_{\alpha,\psi}(t)x(0),\ \text{as}\lambda \to +\infty.$$

Thus (4) holds. This completes the proof. □

Lemma 6. Let $x \in X$ and $t \geq 0$, then $\lim\limits_{\lambda\to+\infty} \int\limits_0^{\psi(t)-\psi(0)} K_{\alpha,\psi}\left(\psi(s)-\psi(s)\right) R_\lambda x\psi'(s)ds$ exists and the mapping $\eta_{\alpha,\psi}(x) = \lim\limits_{\lambda\to+\infty} \int\limits_0^{\psi(t)-\psi(0)} K_{\alpha,\psi}\left(\psi(s)-\psi(s)\right) R_\lambda x\psi'(s)ds$ define a linear operator from X_0 into X_0.

Proof. . Let $\Psi_{\alpha,\psi}(t)$ be the following operator

$$\Psi_{\alpha,\psi}(t)x_0 = \int\limits_0^{\psi(t)-\psi(0)} K_{\alpha,\psi}\left(\psi(s)-\psi(s)\right) R_\lambda x_0 \psi^{'}(s)ds,$$

for $x_0 \in X_0$ and $t \geq 0$.

Then,the following operator

$$\varsigma_{\alpha,\psi}(t) = (\lambda I - A)\Psi_{\alpha,\psi}(t)(\lambda I - A)^{-1}, \quad \lambda > \omega,$$

extends $\Psi_{\alpha,\psi}(t)$ from X_0 to X.
This definition is independent of λ due to resolvent identity. Since $\varsigma_{\alpha,\psi}(t)$ maps X into X_0, then we have

$$\begin{aligned}\varsigma_{\alpha,\psi}(t)x &= \lim_{\lambda\to+\infty} R_\lambda \varsigma_{\alpha,\psi}(t)x \\ &= \lim_{\lambda\to+\infty} \Psi_{\alpha,\psi}(t)R_\lambda x.\end{aligned}$$

This completes the proof. □

Lemma 7. Let $x \in X_0$ and $t \geq 0$, then we have ${}^C D_{0^+}^{\alpha,\psi}\Psi_{\alpha,\psi}(t)x = S_{\alpha,\psi}(t)x$ and $S_{\alpha,\psi}(t)x = A\Psi_{\alpha,\psi}(t)x + x$.

Proof. The proof of this Lemma derived directly from the definitions of $S_{\alpha,\psi}(t)$ and $\Psi_{\alpha,\psi}(t)$ for $t \geq 0$. □

Lemma 8. The following statements hold.
(i) Let $x \in X$ and $t \geq 0$, then

$$I_{0^+}^{\alpha,\psi}\varsigma_{\alpha,\psi}(t)x \in D(A),$$

and

$$\varsigma_{\alpha,\psi}(t)x = A(I_{0^+}^{\alpha,\psi}\varsigma_{\alpha,\psi}(\psi(t)-\psi(0))x) + \frac{(\psi(t)-\psi(0))^\alpha}{\Gamma(1+\alpha)}x.$$

(ii) If $x \in D(A)$, then

$$\varsigma_{\alpha,\psi}(t)Ax + x = S_{\alpha,\psi}(t)x.$$

Proof. To show (i), let $x \in X$ and $t \geq 0$, then we have

$$\begin{aligned}\zeta(t) &= \lambda I_{0^+}^{\alpha,\psi}\Psi_{\alpha,\psi}(t)(\lambda I - A)^{-1}x \\ &\quad + \frac{(\psi(t)-\psi(0))^\alpha}{\Gamma(1+\alpha)}(\lambda I - A)^{-1}x - \Psi_{\alpha,\psi}(t)(\lambda I - A)^{-1}x.\end{aligned}$$

Clearly $\zeta(0) = 0$. From Lemma 7 we have

$$\begin{aligned}{}^C D_{0^+}^{\alpha,\psi}\zeta(t) &= \lambda\Psi_\alpha(t)(\lambda I - A)^{-1}x + (\lambda I - A)^{-1}x - c_{{}^C D_{0^+}^{\alpha,\psi}\Psi_{\alpha,\psi}(t)(\lambda I} - A)^{-1}x \\ &= \lambda\Psi_{\alpha,\psi}(t)(\lambda I - A)^{-1}x + (\lambda I - A)^{-1}x - S_{\alpha,\psi}(t)(\lambda I - A)^{-1}x \\ &= \lambda\Psi_{\alpha,\psi}(t)(\lambda I - A)^{-1}x + (\lambda I - A)^{-1}x - A\Psi_{\alpha,\psi}(t)(\lambda I - A)^{-1}x - (\lambda I - A)^{-1}x \\ &= \lambda\Psi_{\alpha,\psi}^0(t)(\lambda I - A)^{-1}x - A\Psi_{\alpha,\psi}(t)(\lambda I - A)^{-1}x \\ &= (\lambda I - A)\Psi_{\alpha,\psi}(t)(\lambda I - A)^{-1}x \\ &= \varsigma_{\alpha,\psi}(t)x.\end{aligned}$$

It follows that

$$\begin{aligned}\zeta(t) &= I_{0^+}^{\alpha,\psi}\varsigma_{\alpha,\psi}(t)x + \zeta(0) \\ &= I_{0^+}^{\alpha,\psi}\varsigma_{\alpha,\psi}(t)x,\end{aligned}$$

and

$$
\begin{aligned}
(\lambda I - A)\zeta(t) &= (\lambda I - A) I_{0^+}^{\alpha,\psi} \varsigma_{\alpha,\psi}(t)x \\
&= \lambda I_{0^+}^{\alpha,\psi} \varsigma_{\alpha,\psi}(t)x + \frac{(\psi(t)-\psi(0))^{\alpha,\psi}}{\Gamma(1+\alpha)} x - \varsigma_{\alpha,\psi}(t)x.
\end{aligned}
$$

Thus

$$
\varsigma_{\alpha,\psi}(t)x = A(I_{0^+}^{\alpha,\psi}\varsigma_{\alpha,\psi}(t)x) + \frac{(\psi(t)-\psi(0))^{\alpha}}{\Gamma(1+q)} x.
$$

Now, we prove (ii). Let $x \in D(A)$, it follows from Lemmas 6 and 7 that

$$
\begin{aligned}
\varsigma_{\alpha,\psi}(t)Ax &= \lim_{\lambda \to +\infty} \int_0^{\psi(t)-\psi(0)} K_{\alpha,\psi}(\psi(s)-\psi(0)) R_\lambda A x \psi'(s)ds \\
&= \lim_{\lambda \to +\infty} A_0 \int_0^{\psi(t)-\psi(0)} K_{\alpha,\psi}(s) R_\lambda x \psi'(s)ds \\
&= A_0 \Psi_{\alpha,\psi}(t)x = S_{\alpha,\psi}(t)x - x.
\end{aligned}
$$

This completes the proof. □

Theorem 2. The mild solution of the evolution problem (3) is given by

$$
x(t) = S_{\alpha,\psi}\left(x_0 - \Phi(x)\right) + \lim_{\lambda \to \infty} \int_0^t K_{\alpha,\psi}\left(\psi(t)-\psi(s)\right) R_\lambda g(s)ds.
$$

Proof. The proof is given in several steps:

1) **Step 1:** Let g be always differentiable, then for $t \in J$, we have.

$$
\begin{aligned}
x_\lambda(t) &= \int_0^{\psi(t)-\psi(0)} K_{\alpha,\psi}(s) R_\lambda g(s) \psi'(s)ds \\
&= \int_0^{\psi(t)-\psi(0)} K_{\alpha,\psi}(s) R_\lambda (g(0) + \int_0^s g'(r)dr) \psi'(s)ds \\
&= \int_0^{\psi(t)-\psi(0)} K_{\alpha,\psi}(s) R_\lambda g(0) \psi'(s)ds + \int_0^{\psi(t)-\psi(0)} K_{\alpha,\psi}(s) R_\lambda \int_0^s g'(r)dr) \psi'(s)ds \\
&= \Psi_{\alpha,\psi}(t) R_\lambda g(0) + \int_0^t \varsigma^0_{\alpha,\psi}(\psi(t)-\psi(r)) R_\lambda g'(r)dr.
\end{aligned}
$$

By Lemma 7 for $t \in J$, we obtain

$$
\begin{aligned}
x(t) &= \lim_{\lambda \to +\infty} x_\lambda(t) \\
&= \varsigma_{\alpha,\psi}(t)g(0) + \int_0^t \varsigma_{\alpha,\psi}(\psi(t)-\psi(r))g(r)dr \\
&= A(I_{0^+}^{\alpha,\psi}\varsigma_{\alpha,\psi}(t)g(0)) + \frac{(\psi(t)-\psi(0))^{\alpha}}{\Gamma(1+\alpha)} g(0) \\
&+ \int_0^t [A(I_{0^+}^{\alpha,\psi}\varsigma_{\alpha,\psi}(\psi(t)-\psi(r))) + \frac{(\psi(t)-\psi(r))^{\alpha}}{\Gamma(1+\alpha)}] g'(r)dr
\end{aligned}
$$

$$= A[I_{0^+}^{\alpha,\psi}\Phi_{\alpha,\psi}(t)f(0) + \int_0^t I_{0^+}^{\alpha,\psi}\Phi_{\alpha,\psi}(\psi(t)-\psi(r))g^{'}(r)dr]$$

$$+ \frac{(\psi(t)-\psi(0))^{\alpha}}{\Gamma(1+\alpha)}g(0) + \frac{1}{\Gamma(1+\alpha)}\int_0^t(\psi(t)-\psi(r))^{\alpha}g^{'}(r)dr$$

$$= A[I_{0^+}^{\alpha,\psi}\varsigma_{\alpha,\psi}(t)g(0) + I_{0^+}^{\alpha,\psi}(\int_0^t \Phi_{\alpha,\psi}(\psi(t)-\psi(r))g^{'}(r)dr)]$$

$$+ \frac{(\psi(t)-\psi(0))^{\alpha}}{\Gamma(1+\alpha)}g(0) + \frac{1}{\Gamma(1+\alpha)}\int_0^t(\psi(t)-\psi(r))^{\alpha}g^{'}(r)dr$$

$$= A(I_{0^+}^{\alpha,\psi}x(t)) + I_{0^+}^{\alpha,\psi}g(t)\ .$$

2) **Step 2:** Now, we approach g through continuously differentiable functions g_n such that.

$$\sup_{t\in J}|g(t)-g_n(t)| \to 0, \text{ as } n\to\infty.$$

Letting

$$x_n(t) = \lim_{\lambda\to\infty}\int_0^t K_{\alpha,\psi}(\psi(s))R_\lambda g_n(s)ds,$$

we have

$$x_n(t) = A(I_{0^+}^{\alpha,\psi}x_n(t)) + I_{0^+}^{\alpha,\psi}g_n(t)\ .$$

Hence

$$\begin{aligned}\left|x_n(t)-x_m(t)\right| &= |\lim_{\lambda\to\infty}\int_0^t K_{\alpha,\psi}(s)R_\lambda[g_n(s)-g_m(s)]ds| \\ &\le \frac{MM}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}|g_n(s)-g(s)|ds \\ &\le \frac{MMT^{\alpha}}{\Gamma(\alpha)}\|g_n-g_m\|,\end{aligned}$$

which implies that $\{x_n\}$ is a Cauchy sequence and its limit is denoted by $x(t)$, thus we obtain $x(t) = A(I_{0^+}^{\alpha,\psi}x(t)) + I_{0^+}^{\alpha,\psi}f(t)$, for $t\in J$. This completes the proof.

□

Corollary 1. By using Definition 6, Remark 2 and Theorem 2 we can give the mild solution of the fractional evolution problem (1) as follows:

$$x(t) = S_{\alpha,\psi}\left(x_0-\Phi(x)\right) + \lim_{\lambda\to\infty}\int_0^t K_{\alpha,\psi}\left(\psi(t)-\psi(s)\right)R_\lambda f(s,x(s)))\psi^{'}(s)ds.$$

4 Main results

In this section, we use Krasnoselskii [5] fixed point theorem to prove existence and uniqueness results of (1).

Theorem 3. Assume that the hypotheses $(H_3)-(H_4)$ are satisfied, then the fractional evolution problem (1) has a unique mild solution defined on J.

Proof. We give this proof in two several steps.

Step 1: Existence of solution.

Consider the following operators F and L.

$$\begin{cases} Fx(t) = S_{\alpha,\psi}(t)(x_0 - g(x)), \\ Lx(t) = \lim\limits_{\lambda\to+\infty} \int\limits_0^{\psi(t)-\psi(s)} (\psi(t)-\psi(s))^{\alpha-1} P_{\alpha,\psi}(\psi(t)-\psi(s)) R_\lambda f(s,\ x(s)) ds. \end{cases}$$

where $P_{\alpha,\psi}(t) = \int\limits_0^{\infty} \alpha \left(\psi(\rho)-\psi(0)\right) \omega_\alpha(\rho) T_{t^\alpha\left(\psi(\rho)-\psi(0)\right)} \psi^{'}(\rho) d\rho.$

• Choose a ball of radius r. If $x, y \in B_r$, then

$$\begin{aligned} \|(Fx)+(Ly)\| &\le \|S_{\alpha,\psi}(x_0 - \Phi(x))\| \\ &+ \lim_{\lambda\to+\infty} \| \int\limits_0^{\psi(t)-\psi(s)} (\psi(t)-\psi(s))^{\alpha-1} P_{\alpha,\psi}(\psi(t)-\psi(s)) R_\lambda f(s,\ y(s)) \psi^{'}(s) ds \| \\ &\le \|S_{\alpha,\psi}(\mathrm{f})\| \|x_0 - \Phi(x)\| \\ &+ \lim_{\lambda\to+\infty} \int\limits_0^{\psi(t)-\psi(s)} (\psi(t)-\psi(s))^{\alpha-1} \|P_{\alpha,\psi}(\psi(t)-\psi(s))\ \|R_\lambda\| \|f(s,\ y(s))\| \psi^{'}(s) ds \\ &\le \frac{M}{\alpha} \|x_0 - \Phi(x)\| \\ &+ \frac{\left(\psi(T)-\psi(0)\right)^\alpha}{\Gamma(\alpha+1)} M \|\mu\|_{L^\infty}. \end{aligned}$$

Since Φ is Lipschitzian with a positive constant K, then

$$\|Fx + Ly\| \le M\left(\|x_0\| + \|\Phi(0)\| + Kr\right) + \frac{\left(\psi(T)-\psi(0)\right)^\alpha}{\Gamma(\alpha+1)} M \|\mu\|_{L^\infty}.$$

So just one can takes $r \ge 2M\left(\|x_0\| + \|\Phi(0)\| + \frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)} M \|\mu\|_{L^\infty(J,\mathbb{R}_+)}\right)$ to find the result.

• F is a contraction mapping.

for any $t \in J$, $x, y \in B_r$ we have

$$\begin{aligned} \|Fx(t) - Fy(t)\| &\le \|S_{\alpha,\psi}(t)\left(\Phi(x) - \Phi(y)\right)\| \\ &\le \|S_{\alpha,\psi}(t)\| \|\Phi(x) - \Phi(y)\| \\ &\le MK \|x - y\|_\infty. \end{aligned}$$

$\|Fx - Fy\|_\infty \le MK\|x-y\|_\infty$ Which implies that By the hypothesis (H_3), $MK < 1$, F is a contraction mapping.

• L is continuous and compact.

Continuity of L.

Let (x_n) be a sequence in $B_r = \{x \in X,\ \|x\| \le r\}$ such that $x_n \to x$ in B_r. Since f is a continuous mapping, we have

$$f(s,\ x_{\mathrm{n}}(s)) \to f(s,\ x(s))\ ,\ n \to \infty.$$

For all $t \in J$, we have

$$\|Lx_n - Lx\| \le \lim_{\lambda\to+\infty} \int\limits_0^{\psi(t)-\psi(0)} (\psi(t)-\psi(s))^{\alpha-1} \|P_{\alpha,\psi}(t-s)\| R_\lambda\| \|f(s,\ x_n(s)) - f(s,\ x(s))\| \psi^{'}(s) ds,$$

we get

$$\|Lx_n - Lx\| \leq \frac{1}{\Gamma(\alpha+1)} M \int_0^{\psi(t)-\psi(0)} (\psi(t)-\psi(s))^{\alpha-1} \|f(s,\ x_n(s)) - f(s,\ x(s))\| ds,$$

for all $t \in J$. Therefore, using on the one hand the fact that

$(\psi(t)-\psi(s))^{\alpha-1}\|f(s,\ x_n(s))-f(s,\ x(s))\| \leq 2\mu(t)(\psi(t)-\psi(0))^{\alpha-1} \in L^1(J,\ \mathbb{R}_+)$, for $s \in J$, by using Lebesgue dominated convergence Theorem we obtain that

$$\lim_{n\to\infty} \int_0^{\psi(t)-\psi(0)} (\psi(t)-\psi(s))^{\alpha-1} \|f(s,\ x_n(s)) - f(s,\ x(s))\| \psi^{'}(s) ds = 0.$$

Consequently,

$$\lim_{n\to\infty} \|Lx_n - Lx\|_\infty = 0.$$

In other words L is continuous. It is easy to check that (Lx_n) is uniformly bounded.

• Compactness of L.

By using Ascoli-Arzela Theorem [15], we first prove that $\{Lx\ :\ x \in B_r\}$ is relatively compact in X. Obviously, $\{Lx(0)\ :\ x \in B_r\}$ is compact. Fix $t \in [0,\ T]$ each $h \in [0, T]$ and $x \in B_r$, define the operator L^h by

$$\begin{aligned} L^h x &= \lim_{\lambda\to+\infty} \int_0^{\psi(t)-\psi(h)} (\psi(t)-\psi(s))^{\alpha-1} P_{\alpha,\psi}(\psi(t)-\psi(s)) R_\lambda f(s,\ x(s)) \psi^{'}(s) ds \\ &= \lim_{\lambda\to+\infty} \|P_{\alpha,\psi}(\psi(h)-\psi(0))\| \int_0^{\psi(t)-\psi(h)} (\psi(t)-\psi(s))^{\alpha-1} P_{\alpha,\psi}(\psi(t)-\psi(h)-\psi(s)) \\ &\quad \times R_\lambda f(s,\ x(s)) \psi^{'}(s) ds. \end{aligned}$$

Since the operators $P_{\alpha,\psi}$ are compact in X for $t > 0$, then the sets $\{L^h x\ :\ x \in B_r\}$ are relatively compact in X for each $t \in J$. Moreover, now let's use the fact that A_0 is an infinitesimal generator of a C_0semigroup, we have

$$\begin{aligned} \|Lx - L^h x\| &\leq \lim_{\lambda\to+\infty} \int_{\psi(t)-\psi(h)}^{\psi(t)} (\psi(t)-\psi(s))^{\alpha-1} \|P_{\alpha,\psi}(\psi(t)-\psi(s))\ \|R_\lambda\| \|f(s,\ x(s))\| \psi^{'}(s) ds \\ &\leq \frac{T^\alpha}{\Gamma(\alpha+1)} M \|\mu_r\|_{L^\infty(J,\mathbb{R}_+)} \left(\psi(h)-\psi(0)\right). \end{aligned}$$

Therefore, we deduce that $\{Lx\ :\ x \in B_r\}$ is relatively compact in X for all $t \in [0,\ T]$ and since it is compact at $t = 0$ we have the relative compactness in X for all $t \in J$.

Now, let us prove that $\mathrm{L}(B_r)$ is equicontinuous. The functions $Lx,\ x \in B_r$ are equicontinuous at $t = 0$.

For $0 < t_2 < t_1 \le T$, we have

$$\begin{aligned}\|Lx(t_1)-Lx(t_2)\| \le\ & \lim_{\lambda\to+\infty}\Big\|\int_0^{\psi(t_2)-\psi(0)}(\psi(t_1)-\psi(s))^{\alpha-1}(P_{\alpha,\psi}(\psi(t_1)-\psi(s))-P_{\alpha,\psi}(\psi(t_2)-\psi(s)))\\ & R_\lambda f(s,\ x(s))\psi'(s)ds\Big\|\\ &+\lim_{\lambda\to+\infty}\Big\|\int_0^{\psi(t_2)-\psi(0)}\Big((\psi(t_1)-\psi(s))^{\alpha-1}-(\psi(t_2)-\psi(s))^{\alpha-1}\Big))\\ & S_{\alpha,\psi}(\psi(t_2)-\psi(s))R_\lambda f(s,\ x(s))\psi'(s)ds\Big\|\\ &+\lim_{\lambda\to+\infty}\Big\|\int_{\psi(t_2)-\psi(0)}^{\psi(t_1)-\psi(0)}(\psi(t_1)-\psi(s))^{\alpha-1}S_{\alpha,\psi}(\psi(t_1)-\psi(s))R_\lambda f(s,\ x(s))\psi'(s)ds\Big\|\\ &\le I_1+I_2+I_3.\end{aligned}$$

Where

$$I_1=\lim_{\lambda\to+\infty}\Big\|\int_0^{\psi(t_2)-\psi(0)}(\psi(t_1)-\psi(0))^{q-1}(P_{\alpha,\psi}(\psi(t_1)-\psi(s))-P_{\alpha,\psi}(\psi(t_2)-\psi(s)))R_\lambda f(s,\ x(s))\psi'(s)ds\Big\|,$$

$$I_2=\lim_{\lambda\to+\infty}\Big\|\int_0^{\psi(t_2)-\psi(0)}((\psi(t_1)-\psi(s))^{\alpha-1}-(\psi(t_2)-\psi(s))^{\alpha-1})P_{\alpha,\psi}(\psi(t_2)-\psi(0))R_\lambda f(s,\ x(s))\psi'(s)ds\Big\|,$$

and

$$I_3=\lim_{\lambda\to+\infty}\Big\|\int_{\psi(t_2)-\psi(0)}^{\psi(t_1)-\psi(0)}(\psi(t_1)-\psi(s))^{\alpha-1}P_{\alpha,\psi}(\psi(t_1)-\psi(s))R_\lambda f(s,\ x(s))\psi'(s)ds\Big\|.$$

I_1, I_2 and I_3 tend to 0 independently of $x \in B_{\mathrm{r}}$ when $t_2 \to t_1$. Indeed, let $x \in B_r$. We have

$$\begin{aligned}I_1 \le\ & \lim_{\lambda\to+\infty}\int_0^{\psi(t_2)-\psi(0)}\|(\psi(t_1)-\psi(0))^{\alpha-1}(P_{\alpha,\psi}(\psi(t_1)-\psi(s))-P_{\alpha,\psi}(\psi(t_2)-\psi(0)))\\ & R_\lambda f(s,\ x(s))\|\psi'(s)ds\\ \le\ & \lim_{\lambda\to+\infty}\int_0^{\psi(t_2)-\psi(0)}\|(\psi(t_1)-\psi(0))^{\alpha-1}\\ & (P_{\alpha,\psi}((\psi(t_1)-\psi(t_2))+(\psi(t_2)-\psi(s)))-P_{\alpha,\psi}(\psi(t_2)-\psi(0)))\\ & R_\lambda f(s,\ x(s))\|\psi'(s)ds\\ \le\ & \Big(P_{\alpha,\psi}(\psi(t_1)-\psi(t_2)-I)\Big)\lim_{\lambda\to+\infty}\int_0^{\psi(t_2)-\psi(0)}\|(\psi(t_2)-\psi(0))^{\alpha-1}\|S_{\alpha,\psi}(\psi(t_2)-\psi(s))\\ & \|R_\lambda\|f(s,\ x(s))\|\psi'(s)ds\\ \le\ & \Big(S_{\alpha,\psi}(\psi(t_1)-\psi(t_2)-I)\Big)\lim_{\lambda\to+\infty}\frac{M}{\Gamma(\alpha+1)}T^\alpha\|\mu_t\|_{L^\infty(J,\mathbb{R}_+)}.\end{aligned}$$

Therefore the continuity of the function $t \mapsto \|S_{\alpha,\psi}\|$ for $t \in (0,\ T)$ allows us to conclude that

$$\lim_{t_2\to t_1} I_1 = 0.$$

On the other hand,

$$\begin{aligned}
I_2 &\leq \lim_{\lambda\to+\infty} \int_0^{\psi(t_2)-\psi(0)} \|((\psi(t_2)-\psi(s))^{\alpha-1} - (\psi(t_1)-\psi(s))^{\alpha-1})S_{\alpha,\psi}(\psi(t_2)-\psi(s))R_\lambda f(s,\ x(s))\|\psi'(s)ds \\
&\leq \lim_{\lambda\to+\infty} \int_0^{\psi(t_2)-\psi(0)} |(\psi(t_2)-\psi(s))^{\alpha-1} - (\psi(t_2)-\psi(s))^{\alpha-1}|\|S_{\alpha,\psi}(\psi(t_2)-\psi(s))\ \|R_\lambda\|\|f(s,\ x(s))\|\psi'(s)ds \\
&\leq M \int_0^{\psi(t_2)-\psi(0)} |(\psi(t_2)-\psi(s))^{\alpha-1} - (\psi(t_2)-\psi(s))^{\alpha-1}|\mu_r|(s)\psi'(s)ds \\
&\leq M\|\mu_r\|_{L^\infty(J,\mathbb{R}_+)} \int_0^{\psi(t_2)-\psi(0)} |(\psi(t_2)-\psi(s))^{\alpha-1} - (\psi(t_1)-\psi(s))^{\alpha-1}|\psi'(s)ds \\
&\leq \frac{1}{\Gamma(\alpha+1)}\|\mu_r\|_{L^\infty(J,\mathbb{R}_+)}|\psi(t_1)-\psi(t_2)|^\alpha.
\end{aligned}$$

Hence $\lim_{t_2\to t_1} I_2 = 0$.

$$\begin{aligned}
I_3 &\leq \lim_{\lambda\to+\infty} \int_{\psi(t_2)-\psi(0)}^{\psi(t_1)-\psi(0)} \|(\psi(t_1)-\psi(s))^{\alpha-1}S_{\alpha,\psi}(\psi(t_1)-\psi(0))R_\lambda f(s,\ x(s))\|\psi'(s)ds \\
&\leq \lim_{\lambda\to+\infty} \int_{\psi(t_2)-\psi(0)}^{\psi(t_1)-\psi(0)} (\psi(t_1)-\psi(s))^{\alpha-1}\|S_{\alpha,\psi}(\psi(t_1)-\psi(s))\ \|R_\lambda\|\|f(s,\ x(s))\|\psi'(s)ds \\
&\leq \frac{M}{\Gamma(\mathrm{q})} \int_{\psi(t_2)-\psi(0)}^{\psi(t_1)-\psi(0)} (\psi(t_1)-\psi(s))^{\alpha-1}\mu(s)\psi'(s)ds \\
&\leq \frac{1}{\Gamma(\alpha+1)}\|\mu_s\|_{L^\infty(J,\mathbb{R}_+)}|\psi(t_1)-\psi(t_2)|^\alpha.
\end{aligned}$$

Consequently, $\lim_{t_2\to t_1} I_3 = 0$.

In short, we have proven that $L(B_\mathrm{r})$ is relatively compact, for $r \in (0,1)$, $\{Lx \ : \ x \in B_\mathrm{r}\}$ is a family of equicontinuous functions. Hence by using the Arzela-Ascoli Theorem [15], L is compact.

Finally, since F is a contraction , L is compact and $Fx + Ly \in B_r$ for $x, y \in B_r$, then by using Krasnoselskii Theorem [7] we conclude that (1) has at least one mild solution on J.

Step 2: Uniqueness of solution:

To prove the uniqueness of solution we use Banach fixed point Theorem. For this purpose, let us define the operator $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ by

$$\mathcal{T}x = S_{\alpha,\psi}(x_0 - \Phi(x)) + \lim_{\lambda\to+\infty} \int_0^{\psi(t)-\psi(0)} (\psi(t)-\psi(s))^{\alpha-1}P_{\alpha,\psi}(\psi(t)-\psi(0))R_\lambda f(s,\ x(s))\psi'(s)ds.$$

Note that $\mathcal{T}$ is well defined on $\mathcal{C}$. Let $t \in J$ and $x, y \in \mathcal{C}$, We have

$$\begin{aligned} \|\mathcal{T}x - \mathcal{T}y\| &\leq \|S_{\alpha,\psi}(\mathrm{r})(\Phi(x) - \Phi(y))\| + \lim_{\lambda \to +\infty} \int_0^{\psi(t)-\psi(0)} (\psi(t) - \psi(0))^{\alpha-1} \|S_{\alpha,\psi}(\psi(t) - \psi(s)) \\ &\quad R_\lambda(f(s,\ x(s)) - f(s,\ y(s)))\|\psi'(s)ds \\ &\leq \|S_{\alpha,\psi}(t)\|\|\Phi(x) - \Phi(y)\| \\ &\quad + \lim_{\lambda \to +\infty} \int_0^{\psi(t)-\psi(0)} (\psi(t) - \psi(s))^{\alpha-1} \|S_{\alpha,\psi}(\psi(t) - \psi(s)) \\ &\quad \|R_\lambda\| \|f(s,\ x(s)) - f(s,\ y(s))\|\psi'(s)ds. \end{aligned}$$

It follows that

$$\begin{aligned} \|\mathcal{T}x(t) - \mathcal{T}y(t)\| &\leq ML\|x - y\|_\infty \\ &\quad + M \int_0^{\psi(t)-\psi(0)} (\psi(t) - \psi(s))^{\alpha-1} \mu_1(s)\|x - y\|\mathrm{d}s \\ &\leq Mb\|x - y\|_G + \frac{M}{\Gamma(q)}\|\mu_1\|_{L^1(J,\mathbb{R}_+)}\|x - y\|_\infty + \int_0^{\psi(t)-\psi(0)} (\psi(t) - \psi(s))^{\alpha-1}\psi'(s)s \\ &\leq \left(MK + \frac{MT^\alpha}{\Gamma(\alpha+1)}\|\mu_1\|_{L^1_{\text{loc}}(J,\mathbb{R}_+)}\right)\|x - y\|_\infty. \end{aligned}$$

So we get

$$\|\mathcal{T}x - \mathcal{T}y\|_\infty \leq \left(MK + \frac{MMT^\alpha}{\Gamma(\alpha+1)}\|\mu_1\|_{L^1_{\text{loc}}(J,\mathbb{R}_+)}\right)\|x - y\|_\infty.$$

Therefore, from (H_4) we get $MK + \dfrac{MMT^\alpha}{\Gamma(\alpha+1)}\|\mu_1\|_{L^1_{\text{loc}}(J,\mathbb{R}_+)} < 1$ and in view of Banach contraction mapping principle, we can conclude that $\mathcal{T}$ has a unique fixed point in $\mathcal{C}$. □

5 An illustrative example

In this section, we give a nontrivial example to illustrate our main result.
Consider the following fractional evolution problem:

$$\begin{cases} {}^C D_{0^+}^{\frac{1}{2},t} u(t,x) = \dfrac{\partial^2}{\partial x^2}u(t,x) + \dfrac{e^{-t}}{9+e^t}cos(u(t,x)), \ \ (t,x) \in [0,1]\times[0,1], \\ u(t,0) = u(t,1) = 0, \quad t \in [0,1], \\ u(0,x) = \displaystyle\sum_{i=1}^{10} \beta_i |u(t_i,x)|, \ \ \beta_i > 0, \ \ 0 < t_i < T, \ \ i = 1,2,..,10 \text{ and } x \in [0,1]. \end{cases} \tag{5}$$

We choose $X = \mathcal{C}([0,1]\times[0,1],\ \mathbb{R})$ and we consider the operator A : $D(A) \subset X \to X$ defined by

$$D(A) = \left\{u \in X : \frac{\partial^2}{\partial x^2}u \in X \text{ and } u(0,x) = u(0,1) = 0\right\},$$

$$Au = \frac{\partial^2}{\partial x^2}u.$$

Then, we have

$$\overline{D(A)} = \left\{ u \in X : u(t,0) = u(t,1) = 0 \right\}.$$

$\overline{D(A)} \neq X, \rho(A) = (0,\ +\infty)$ and for $\lambda > 0$, $\|R(\lambda, A)\| \leq \dfrac{1}{\lambda}$.

This implies that A satisfies (H_1) with $M = 1$. Since it is well known that A generates a compact C_0–semigroup $\left(S_{\alpha,\psi}\right)_{t\geq 0}$ on $\overline{D(A)}$ such that $\|S_{\alpha,\psi}\| \leq 1$. Hence (H_2) is satisfied with $M = 1$.

In this example we set $\alpha = \frac{1}{2}$, $T = 1$, $\psi(t) = t$, $f(t,u) = \dfrac{e^{-t}}{9+e^t} cos(u(t,x))$

and $\Phi(x) = \sum\limits_{i=1}^{10} \beta_i |x(t_i)|$ with $\sum\limits_{i=1}^{10} \beta_i < 1$.

It is clear that $\left|f(t,x(t))\right| \leq \mu(t) = \frac{e^{-t}}{9+e^t}$ and f is continuos, we have

$$|\Phi(x(t))| = \left| \sum_{i=1}^{10} \beta_i |x(t_i)| \right|,$$

it follows that

$$|\Phi(x)| \leq \sum_{i=1}^{10} \beta_i \, \|x\|,$$

On the other hand, we have

$$|\Phi(x(t)) - \Phi(y(t))| = \left| \sum_{i=1}^{10} \beta_i |x(t_i) - \sum_{i=1}^{10} \beta_i |y(t_i)| \right|,$$

from which, we have

$$|\Phi(x) - \Phi(y)| \leq \sum_{i=1}^{10} \beta_i \, |x-y|,$$

thus $K = \sum\limits_{i=1}^{10} \beta_i$.

We take $\mu(t) = \frac{e^{-t}}{9+e^t}$ and $MK + \frac{MT^\alpha}{\Gamma(\alpha+1)} \|\mu_1\|_{L^1_{\mathrm{loc}}(J,\mathbb{R}_+)} < 1$, whenever $0 < K < 1 - \frac{3}{4\sqrt{\pi}}$.

Finally, all the conditions of Theorem 3 are satisfied, thus it is easy to see that the fractional nonlinear problem (5) has one solution defined on $[0,1]$.

6 Conclusion

In the present paper, we studied the existence and uniqueness of solutions for nonlocal fractional evolution problem with nondense domain involving Caputo type fractional derivative with respect to another function ψ. The forme of mild solutions is given by using semi-group and a density function. Our proofs of the existence results are based on Krasnoselskii fixed point theorem. As application, an example is given to illustrate the obtained results.

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