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Mathematical modeling of the dynamics of vector-borne diseases transmitted by mosquitoes : taking into account aquatic stages and gonotrophic cycle

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Abstract: In this paper, we formulate a mathematical model of vector-borne disease dynamics. The model is constructed by considering two models : a baseline model of vector population dynamics due to Lutambi et al. that takes into account the development of the aquatic stages and the female mosquitoes gonotrophic cycle and an SI-SIR model describing the interaction between mosquitoes and human hosts. We briefly study the baseline model of vectors dynamics and, for the transmission model, we explicitly compute the equilibrium points, and by using the method of Van den Driesshe and J. Watmough, we derive the basic reproduction number $\mathcal{R}_0$. Otherwise, thanks to Lyapunov's principle, Routh-Hurwitz criteria and a favorable result due to Vidyasagar, we establish the local and global stability results of the equilibrium points. Furthermore, we establish an interesting relationship between the mosquito reproduction number $\mathcal{R}_v$ and the basic reproduction number $\mathcal{R}_0$. It then follows that aquatic stages and behavior of adult mosquitoes have a significant impact on disease transmission dynamics. Finally, some numerical simulations are carried out to support the theoretical findings of the study.

Keywords: Mathematical model, mosquito population, gonotropic cycle, vector-borne disease dynamics, basic reproduction number, Lyapunov principle, numerical simulations.

MSC: 65L12, 65M20, 65N40

1 Introduction

Vector-borne diseases are central to social and economic concerns across the world. They are caused by pathogenic germs (viruses, parasites, bacteria), conveyed and inoculated by vectors (mosquitoes, ticks, bugs, sandflies, etc.), this vector having itself been infected on a viraemic host. Mosquitoes are the source of transmission of the most endemic vector-borne diseases with very high death rates. They transmit to humans malaria, dengue, Zika virus disease, yellow fever, West Nile fever also called West Nile virus, chikungunya, etc. They are among the most dangerous animals in terms of the number of deaths they cause in the world. Their ability to carry diseases and transmit them to humans results in thousands of deaths each year. It is estimated that all of the major vector-borne diseases account for about 17% of the global burden of communicable diseases and more than 700000 deaths annually. The most affected areas are the tropics and the

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subtropics. In addition, it is noted that more than 80% of the world population lives in areas exposed to the risk of at least one vector-borne disease, [1, 15].

The first mathematical contribution in epidemiology dates back to April 1760 with the model of Daniel Bernoulli, [5, 10], on smallpox. Indeed, in a thesis from the Academy of Sciences in Paris, Bernoulli presented a model and his calculations about the smallpox epidemic, rife at the time, called *"smallpox"*. Thanks to the mathematical study of his model, Bernoulli demonstrated in front of the academy of sciences that the generalization of smallpox, imported from the East, in the fight against smallpox, despite the dangers it presented, would make hope pass from 26 years and 7 months to 29 years and 9 months. Other considerable contributions to mathematical epidemiology are the work of some authors such as W. H. Hamer, A. G. Mc Kendrick and W. O. Kermack, [16]. However, the most important contribution to the body of epidemiology is that of doctor Donald Ross, [28], who can therefore be considered as the founding father of current mathematical epidemiology based on compartmental models. Compartmental models are ideally suited for modeling vector-borne diseases. This consists in dividing the host population into as many compartments as there are clinical states and connecting them to each other through flows of individuals via the different levels of transfer. Ross worked on malaria and proposed the first compartmental model of the dynamics of malaria transmission, [28]. The study of his model allowed him to demonstrate that it was not necessary to eliminate all mosquitoes to eradicate malaria, but that it was simply necessary to lower their density below a certain critical threshold. In 1957, Georges Macdonald, [24], improved Ross' model, adding additional biological hypotheses. It follows important works on the mathematical modeling of malaria in order to better understand the factors influencing its transmission. One of the greatest advancements in the modeling of infectious diseases is the inclusion of acquired immunity proposed by Dietz, Molineaux and Thomas, [4, 11]. Some works have also included environmental effects, [12, 13, 29], the spread of drug resistance, [2, 22], the evolution of immunity, [17], the treatment and impact of vaccination strategies, [3], as well as the duration of the incubation period, [7].

Even today, vector-borne diseases are a fact and several mathematical models have been further proposed. We can list among others the works of P. Zongo, B. Traoré, [32–36], F. Niyukuri, O. Koutou, [19–21] in the context of the fight against malaria, the work of A. Abdelrazec et al., [1], S. Hossain et al., [15], in the fight against dengue fever, etc. One of the greatest advances in the models proposed is the fact of taking into account the growth dynamics of the vector, and in particular that of the immature stages. Indeed, the importance of the immature stages is no longer to be demonstrated, due to the resistance of the eggs. In addition, the conditions favoring the development of these eggs are numerous and very varied. This step is essential in the transmission of the disease because it constitutes the manufacturing plant of mosquitoes. Taking these immature stages into account in mathematical models of vector-borne diseases is therefore an excellent asset for the control of these diseases. In addition, one of the innovations is the inclusion of climatic factors in the transmission dynamics of vector-borne diseases caused by mosquitoes. Indeed, the transmission cycle of these diseases is closely linked to climatic variations. Typically, the female mosquito digests blood faster and feeds more often in warmer weather, thereby, accelerating transmission. During this time, the parasite completes its life cycle faster and thereby, increases its replication. It is then in view of the importance of these factors in the evolution of these vector-borne diseases that many researchers have taken them into account in the development of their models, [2, 12].

One of the recent contributions is that of A. M. Lutambi et al. [23] who took into account the constituent stages of the gonotrophic cycle in the life cycle of mosquitoes and showed the major importance of these stages in the life cycle of the vector. In this work, we are starting from this major innovation in order to deepen the mathematical study on their proposed vector model and to couple it to a model of transmission of SIR-type vector diseases. The aim is to mathematically study the impact of immature stages and the gonotrophic cycle on the transmission dynamics of vector-borne diseases transmitted by mosquitoes. Indeed, mosquito eggs are very resistant and therefore represent a reliable source of mosquito proliferation and consequently disease persistence. In addition, the immature stages as a whole constitute the generating phase of mosquitoes, so, it is important to take an interest in them in order to know their real impact in the process of transmission of these infectious diseases. As for the gonotrophic cycle, it is essential because it brings together the different classes of mosquitoes that promote the transmission and disease progression. Thus, it has a considerable impact on the dynamics of transmission and a knowledge of this impact could inspire targeted control methods adapted to each class of mosquitoes in this cycle. Finally, taking into consideration these two phases (aquatic and aerial) makes the model more realistic and could therefore make it possible to make good predictions about disease spreading.

After this introductory section in which we present the general context, the interest and the goal pursued by our work, section 2 is about the presentation of the vector model of A. M. Lutambi et al. [23], the mathematical study already done on this model and our contribution concerning this model. In this section, we highlight the impact of the gonotrophic cycle materialized by the parameter r_g, on the threshold parameter $\mathcal{R}_V$, governing the evolution of the vector density. Then, in section 3, we present our model of the transmission dynamics of vector-borne diseases of the SIR type transmitted by mosquitoes, from which we do a mathematical study. In this section, we highlight the impact of the parameter $\mathcal{R}_V$ on the parameter $\mathcal{R}_0$ that predicts the course of the disease. In section 4, we present our numerical simulations results in order to corroborate the obtained theoretical results. We conclude in section 5.

2 Baseline model of vector population dynamics: description and main results

In this section, we briefly recall some significant results of the baseline model due to Lutambi et al. [19, 21, 23, 26, 37], that describe mosquito population dynamics.

2.1 Mathematical model description

A mosquito life cycle begins with eggs (E), which hatch into larvae (L) when conditions are favorable. The larvae then mutate into pupae (P), from which adult mosquitoes emerge. After mating, female mosquitoes disperse in search of hosts to bite ; We denote these mosquitoes by A_h. By gorging themselves with blood, they recover the proteins essential for the maturation of their eggs. Once engorged with blood, the female takes refuge in a shelter until complete development of the eggs ; we denote these ones by A_r. Then it searches for a favorable place to lay eggs ; we denote the mosquitoes that lay eggs by A_0. After the laying, it leaves again to feed and the cycle resumes. This feeding cycle is called the gonotrophic cycle, and repeats until the female dies. We distinguish all of these stages because interventions may be applied to any one of them. This information on our state variables is summarized in Table 1, [32, 33, 36].

Table 1: Definition of the state variables of the female mosquitoes model.

Variables	Description
E	density of eggs
L	density of larvae
P	density of pupae
A_h	density of mosquitoes searching for hosts
A_r	density of resting mosquitoes
A_0	density of mosquitoes searching for oviposition sites

Initial conditions are : $E(0),\ \ L(0),\ \ P(0),\ \ A_h(0),\ \ A_r(0),\ \ A_0(0)$. The six subgroups have different mortality and progression rates. Each subgroup is affected by three processes: increase due to recruitment, decrease due to mortality, and development or progression of survivors into the next subgroups. Our model parameters are summarized in Table 2.

The compartmental representation of the life cycle of female mosquitoes is given in Figure 1.

Table 2: Description of mosquito model parameters.

Variables	Biological description
b	number of female eggs laid per oviposition
ρ_E	egg hatching rate into larvae
ρ_L	rate at which larvae develop into pupae
ρ_p	rate at which pupae develop into adult (emergence rate)
μ_E	natural egg mortality rate
μ_{L_1}	natural larvae mortality rate
μ_{L_2}	density-dependent larvae mortality rate
μ_p	natural pupae mortality rate
ρ_{A_h}	rate at which host seeking mosquitoes enter the resting state
ρ_{A_r}	rate at which resting mosquitoes enter oviposition site searching state
ρ_{A_0}	oviposition rate
μ_{A_h}	natural mortality rate of mosquitoes of searching for hosts
μ_{A_r}	natural mortality rate of resting mosquitoes
μ_{A_0}	natural mortality rate of mosquitoes searching for oviposition sites

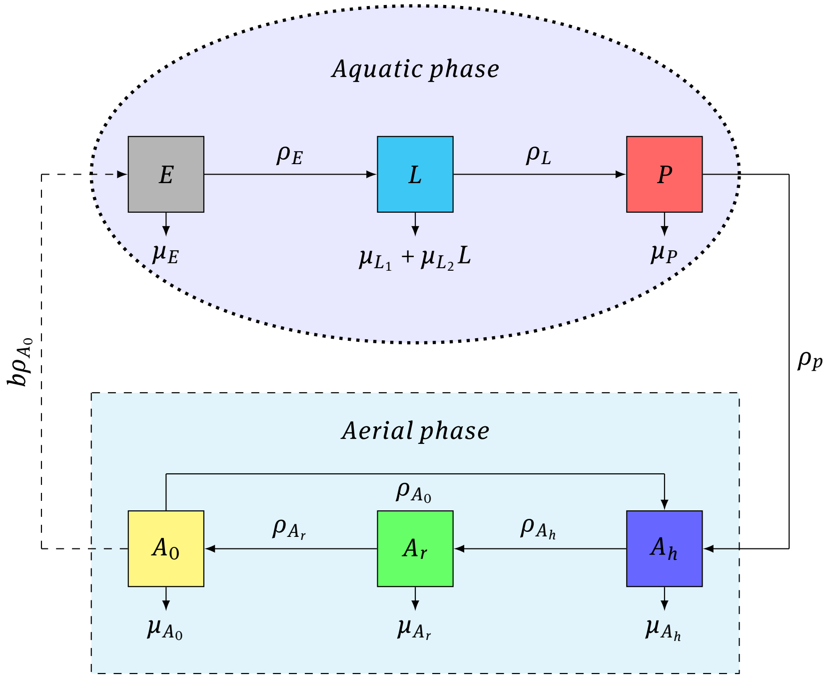

Figure 1: Compartmental representation of the different stages of growth of female mosquitoes.

We thus obtain the following model of the complete life cycle of female mosquitoes:

$$\begin{cases} \dfrac{dE}{dt} = b\rho_{A_0}A_0 - (\mu_E + \rho_E)E, \\ \dfrac{dL}{dt} = \rho_E E - (\mu_{L_1} + \mu_{L_2}L + \rho_L)L, \\ \dfrac{dP}{dt} = \rho_L L - (\mu_P + \rho_P)P, \\ \dfrac{dA_h}{dt} = \rho_p P + \rho_{A_0}A_0 - (\mu_{A_h} + \rho_{A_h})A_h, \\ \dfrac{dA_r}{dt} = \rho_{A_h}A_h - (\mu_{A_r} + \rho_{A_r})A_r, \\ \dfrac{dA_0}{dt} = \rho_{A_r}A_r - (\mu_{A_0} + \rho_{A_0})A_0. \end{cases} \tag{1}$$

We move now to the mathematical analysis of this model. In the following, we denote the boundary of $\mathbb{D}$ by $\partial\mathbb{D}$.

2.2 Mathematical analysis

2.2.1 Existence, boundedness and uniqueness of the solutions

Let

$$X(t) = \begin{pmatrix} E \\ L \\ P \\ A_h \\ A_r \\ A_0 \end{pmatrix} \quad \text{and} \quad f(X) = \begin{pmatrix} f_1(X) \\ f_2(X) \\ f_3(X) \\ f_4(X) \\ f_5(X) \\ f_6(X) \end{pmatrix} = \begin{pmatrix} b\rho_{A_0}A_0 - (\mu_E + \rho_E)E \\ \rho_E E - (\mu_{L_1} + \mu_{L_2}L + \rho_L)L \\ \rho_L L - (\mu_P + \rho_P)P \\ \rho_p P + \rho_{A_0}A_0 - (\mu_{A_h} + \rho_{A_h})A_h \\ \rho_{A_h}A_h - (\mu_{A_r} + \rho_{A_r})A_r \\ \rho_{A_r}A_r - (\mu_{A_0} + \rho_{A_0})A_0 \end{pmatrix},$$

then when the initial conditions are $(t_0, X_0) \in \mathbb{R}_+ \times \mathbb{R}_+^6$, system (1) can be written as follows:

$$\begin{cases} \dfrac{dX(t)}{dt} = f(X(t)), \\ (t_0, X_0) \in \mathbb{R}_+ \times \mathbb{R}_+^6. \end{cases} \tag{2}$$

Theorem 2.1. (***Existence and positivity of solutions***) *The Cauchy problem (2) admits a unique maximal solution* $([0, T], X)$ *and when* $X_0 > 0$, *then this solution is positive* $(X > 0,\ \forall\, t \in [0, T])$.

Proof. Indeed:

- Function f is of class $\mathscr{C}^\infty$ so it is of class $\mathscr{C}^1$, therefore it is locally Lipschitzian on $\mathbb{R}^6$. We deduce the existence and the uniqueness of a maximal solution $([0, T], X)$ to the Cauchy problem (2). In addition, as f is $\mathscr{C}^\infty$ then this solution is also $\mathscr{C}^\infty$.
- We show by using the absurd method that $X > 0$ when $X_0 > 0$.
  Let us $\mathscr{M}(t) = \min\{E(t), L(t), P(t), A_h(t), A_r(t), A_0(t)\}$ and assume that there exists $t_1 \in]0, T[$ such as :

$$\mathscr{M}(t_1) = 0 \text{ and } \mathscr{M}(t) > 0, \quad \forall\, t \in [0, t_1[\,.$$

If $\mathscr{M}(t) = E(t)$, so $\forall\, t \in [0, t_1[\,,\ E(t) > 0$ and $A_0(t) > 0$. Consequently, according to the first equation of system (1), we have :

$$\frac{dE}{dt} > -(\mu_E + \rho_E)E,$$

which gives by integration between 0 and t_1,

$$E(t_1) > E(0)e^{-(\mu_E+\rho_E)t_1}.$$

So

$$0 = E(t_1) > E(0)e^{-(\mu_E+\rho_E)t_1} > 0,$$

which is absurd. So there is no $t_1 \in]0, T[$ such as $\mathscr{M}(t_1) = 0$.
We obtain the same contradictions when $\mathscr{M}(t) = L(t)$, $\mathscr{M}(t) = P(t)$, $\mathscr{M}(t) = A_h(t)$, $\mathscr{M}(t) = A_r(t)$, $\mathscr{M}(t) = A_0(t)$. Thus, when $X_0 > 0$, the solution X is strictly positive ($X > 0,\ \forall\, t\ \in [0, T]$).

□

For the rest, we state the following hypothesis, [32, 33, 36] :
(H1) : It is assumed that the number of female mosquitoes that lay eggs (A_0) does not exceed a number noted C, i.e

$$\forall\, t \geq 0,\ A_0(t) \leq C. \tag{3}$$

Let now :

$$M = E + L + P + A_h + A_r + A_0,\quad K = b\rho_{A_0}C \text{ and } m = \min\left\{\mu_E, \mu_{L_1}, \mu_P, \mu_{A_h}, \mu_{A_r}, \mu_{A_0}\right\}.$$

Proposition 2.1. *Under hypothesis **(H1)**, the unique solution of system (1) is bounded. Moreover the domain $\mathbb{D}_V$ containing it, defined by :*

$$\mathbb{D}_V = \left\{ \begin{pmatrix} E \\ L \\ P \\ A_h \\ A_r \\ A_0 \end{pmatrix} \in \mathbb{R}^6 \,\middle|\, 0 \leq E + L + P + A_h + A_r + A_0 \leq \frac{K}{m} \right\}$$

is positively invariant and attracts all positive orbits in $\mathbb{R}_+$.

Proof. By taking the sum of the six equations of system (1), we obtain :

$$M'(t) = b\rho_{A_0}A_0 - \left(\mu_E E + \left(\mu_{L_1} + \mu_{L_2}L\right)L + \mu_P P + \mu_{A_h}A_h + \mu_{A_r}A_r + \mu_{A_0}A_0\right).$$

So

$$M'(t) \leq b\rho_{A_0}A_0 - \left(\mu_E E + \mu_{L_1}L + \mu_P P + \mu_{A_h}A_h + \mu_{A_r}A_r + \mu_{A_0}A_0\right).$$

We then have

$$M'(t) \leq b\rho_{A_0}C - \min\left\{\mu_E, \mu_{L_1}, \mu_P, \mu_{A_h}, \mu_{A_r}, \mu_{A_0}\right\} M(t).$$

Consider

$$K = b\rho_{A_0}C \text{ and } m = \min\left\{\mu_E, \mu_{L_1}, \mu_P, \mu_{A_h}, \mu_{A_r}, \mu_{A_0}\right\},$$

so

$$M'(t) \leq K - m\, M(t).$$

First, let's solve the equation

$$N_1'(t) = K - m\, N_1(t).$$

Moreover, by applying the constant variation formula between t and t_0, a particular solution is given as follows:

$$N_1(t) = \frac{K}{m} + \left(N_1(0) - \frac{K}{m}\right) e^{-m(t-t_0)}$$

Therefore, $N_1 \longrightarrow \frac{K}{m}$ as $t \longrightarrow +\infty$ and we obtain $N_1(t) \leq \frac{K}{m}$. By applying the principle of comparison, we obtain :

$$\lim_{t \longrightarrow +\infty} \sup M(t) \leq \frac{K}{m}.$$

Therefore, $0 \leq M(t) \leq \frac{K}{m}$, then M is bounded and $T = +\infty$.

Suppose now that

$$M(t) > \frac{K}{m},$$

we then have

$$M^{'}(t) < K - m \times \frac{K}{m},$$

i.e

$$M^{'}(t) < 0.$$

In this case the mosquito population is decreasing and as the domain is compact, thus all the solutions remain there. ☐

2.2.2 Existence and stability of equilibrium points

Proposition 2.2. *Model (1) has exactly one equilibrium point on* $\partial \mathbb{D}_V$ *given by* $X_0^\star = (0, 0, 0, 0, 0, 0)$. *We label* $X_0^\star$ *the mosquito-free equilibrium point.*

Proof. Setting the right side of system (1) to zero, it is clear that $X_0^\star = (0, 0, 0, 0, 0, 0)$ is solution, therefore a point of equilibrium. ☐

In the following, we will calculate the vector reproduction number $\mathcal{R}_v$. This is the average number of female mosquitoes produced by a single female mosquito during its lifespan without any influence of density. We will use the next-generation matrix for computing this rate.
Let $x_1 = E,\ x_2 = L,\ x_3 = P,\ x_4 = A_h,\ x_5 = A_r$ et $x_6 = A_0$ be the respective densities of mosquitoes in compartments $1,\ 2,\ 3,\ 4,\ 5$ and 6, then $X = (x_1,\ x_2, \ldots,\ x_6)^T$. Consider $\mathcal{F}_i(X)$ the rate of new recruitment (egg laying), $\mathcal{V}_i^+(X)$ the transfer rate of mosquitoes into a compartment i and $\mathcal{V}_i^-(X)$ the transfer rate of mosquitoes out of the compartment i and let $V_i(X) = V_i^-(X) - V_i^+(X)$. System (1) can be written :

$$\frac{dx_i}{dt} = \mathcal{F}_i(X) - \mathcal{V}_i(X),\ i = 1,\ 2, \ldots,\ 6.$$

$\mathcal{F}$ and $\mathcal{V}$ are then given by :

$$\mathcal{F} = \begin{pmatrix} b\rho_{A_0} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \mathcal{V} = \begin{pmatrix} (\mu_E + \rho_E)E \\ (\mu_{L_1} + \mu_{L_2} L + \rho_L)L - \rho_E E \\ (\mu_P + \rho_P)P - \rho_L L \\ (\mu_{A_h} + \rho_{A_h})A_h - \rho_P P - \rho_{A_0} A_0 \\ (\mu_{A_r} + \rho_{A_r})A_r - \rho_{A_h} A_h \\ (\mu_{A_0} + \rho_{A_0})A_0 - \rho_{A_r} A_r \end{pmatrix}.$$

We calculate $F_{ij} = \left.\dfrac{\partial \mathcal{F}_i}{\partial x_j}\right|_{X_0^\star}$ and $V_{ij} = \left.\dfrac{\partial \mathcal{V}_i}{\partial x_j}\right|_{X_0^\star}$ to obtain :

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & b\rho_{A_0} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } V = \begin{pmatrix} \mu_E + \rho_E & 0 & 0 & 0 & 0 & 0 \\ -\rho_E & \mu_{L_1} + \rho_L & 0 & 0 & 0 & 0 \\ 0 & -\rho_L & \mu_P + \rho_P & 0 & 0 & 0 \\ 0 & 0 & -\rho_P & \mu_{A_h} + \rho_{A_h} & 0 & -\rho_{A_0} \\ 0 & 0 & 0 & -\rho_{A_h} & \mu_{A_r} + \rho_{A_r} & 0 \\ 0 & 0 & 0 & 0 & -\rho_{A_r} & \mu_{A_0} + \rho_{A_0} \end{pmatrix}$$

The next-generation matrix FV^{-1} is given by :

$$FV^{-1} = \begin{pmatrix} \kappa_1 & \kappa_2 & \kappa_3 & \kappa_4 & \kappa_5 & \kappa_6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with:

$$\kappa_1 = \frac{b\prod\limits_j \left(\dfrac{\rho_j}{\mu_j + \rho_j}\right)}{1 - \prod\limits_{A_i} \left(\dfrac{\rho_{A_i}}{\mu_{A_i} + \rho_{A_j}}\right)}, \quad \text{where } j = E, L, P, A_h, A_r, A_0 \text{ and } i = h, r, 0\,;$$

$$\kappa_2 = -\frac{b\prod\limits_j \left(\dfrac{\rho_j}{\mu_j + \rho_j}\right)}{1 - \prod\limits_{A_i} \left(\dfrac{\rho_{A_i}}{\mu_{A_i} + \rho_{A_j}}\right)}, \quad \text{where } j = L, P, A_h, A_r, A_0 \text{ and } i = h, r, 0\,;$$

$$\kappa_3 = \frac{b\prod\limits_j \left(\dfrac{\rho_j}{\mu_j + \rho_j}\right)}{1 - \prod\limits_{A_i} \left(\dfrac{\rho_{A_i}}{\mu_{A_i} + \rho_{A_j}}\right)}, \quad \text{where } j = P, A_h, A_r, A_0 \text{ and } i = h, r, 0\,;$$

$$\kappa_4 = -\frac{b\prod\limits_j \left(\dfrac{\rho_j}{\mu_j + \rho_j}\right)}{1 - \prod\limits_{A_i} \left(\dfrac{\rho_{A_i}}{\mu_{A_i} + \rho_{A_j}}\right)}, \quad \text{where } j = A_h, A_r, A_0 \text{ and } i = h, r, 0\,;$$

$$\kappa_5 = \frac{b\prod\limits_j \left(\dfrac{\rho_j}{\mu_j + \rho_j}\right)}{1 - \prod\limits_{A_i} \left(\dfrac{\rho_{A_i}}{\mu_{A_i} + \rho_{A_j}}\right)}, \quad \text{where } j = A_r, A_0 \text{ and } i = h, r, 0\,;$$

$$\kappa_6 = -\frac{b\prod\limits_j \left(\dfrac{\rho_j}{\mu_j + \rho_j}\right)}{1 - \prod\limits_{A_i} \left(\dfrac{\rho_{A_i}}{\mu_{A_i} + \rho_{A_j}}\right)}, \quad \text{where } j = A_0 \text{ and } i = h, r, 0.$$

Thus :

$$\mathcal{R}_v = \rho(FV^{-1}) = \kappa_1 = \frac{b\prod_{j}\left(\frac{\rho_j}{\mu_j+\rho_j}\right)}{1-\prod_{i}\left(\frac{\rho_{A_i}}{\mu_{A_i}+\rho_{A_j}}\right)}, \quad \text{where } \; j = E, L, P, A_h, A_r, A_0 \; \text{ and } \; i = h, r, 0. \tag{4}$$

Where $\rho(FV^{-1})$ is the spectral radius of the next-generation matrix FV^{-1}.

Remark 2.1.

- *The probability that a mosquito in class j survives and move on to the next stage is* $\frac{\rho_j}{\mu_j+\rho_j}$.
- *The value* $r_g = \prod_{i}\left(\frac{\rho_{A_i}}{\mu_{A_i}+\rho_{A_i}}\right) \in [0,1]$, $i = h, r, 0$, *is the probability that a mosquito survives the feeding cycle.*
- *The value* $r_a = b\prod_{j}\left(\frac{\rho_j}{\mu_j+\rho_j}\right)$, $j = E, L, P$, *is the probability that a mosquito survives in the aquatic stage.*

  So

$$\mathcal{R}_v = \frac{r_a r_g}{1-r_g} \tag{5}$$

Although the inter-specific competition (given by $\mu_{L_2}L$*) of larvae affects the mosquito population,* $\mathcal{R}_v$ *does not depend on this quantity.*

Theorem 2.2. *When* $\mathcal{R}_v > 1$, *model (1) has a mosquito persistence equilibrium solution* $X_1^\star = (E^\star, L^\star, P^\star, A_h^\star, A_r^\star, A_0^\star)$, *whose components are given by :*

$$E^\star = \frac{b\rho_{A_0}A_0^\star}{\mu_E+\rho_E}, \quad L^\star = \frac{(\mu_{L_1}+\rho_L)(\mathcal{R}_v-1)}{\mu_{L_2}}, \quad P^\star = \frac{\rho_L L^\star}{\mu_P+\rho_P},$$

$$A_h^\star = \frac{\rho_P P^\star \mathcal{R}_v}{(\mu_{A_h}+\rho_{A_h})B_1}, \quad A_r^\star = \frac{\rho_{A_h}A_h^\star}{\mu_{A_r}+\rho_{A_r}} \quad \text{and} \quad A_0^\star = \frac{\rho_{A_r}A_r^\star}{\mu_{A_0}+\rho_{A_0}},$$

where

$$B_1 = b\prod_{j}\left(\frac{\rho_j}{\mu_j+\rho_j}\right) \quad \text{pour} \;\; j = E,\ L,\ P,\ A_h,\ A_r,\ A_0. \tag{6}$$

Proof. When we set the right side of system (1) to zero, the resolution gives us the values of $E^\star$, $L^\star$, $P^\star$, $A_h^\star$, $A_r^\star$ and $A_0^\star$ given. ☐

Theorem 2.3.

(i) The mosquito-free equilibrium $X_0^\star$ *is locally asymptotically stable when* $\mathcal{R}_v < 1$.
(ii) The mosquito persistence equilibrium $X_1^\star$ *is locally asymptotically stable when* $\mathcal{R}_v > 1$.

Proof. For the demonstration we proceed as follows :

(i) According to Varga's theorem [38], the mosquito-free equilibrium $X_0^\star$ is locally asymptotically stable when $\mathcal{R}_v < 1$.
(ii) For the study of local stability of the equilibrium point of persistence of mosquitoes $X_1^\star$, we proceed in much the same way as in [23, 27, 29, 30]. We determined the Jacobian matrix of system (1) about this point and its characteristic polynomial. Then, by using the Routh-Hurwitz criterion [21, 23, 26, 27, 30], we show that the eigenvalues of the Jacobian matrix have their real parts strictly negative.

☐

For the rest, consider

$$a_1 = \mu_E+\rho_E, \; a_2 = \mu_{L_1}+\rho_L, \; a_3 = \mu_P+\rho_P, \; a_4 = \mu_{A_h}+\rho_{A_h}, \; a_5 = \mu_{A_r}+\rho_E \text{ and } a_6 = \mu_{A_0}+\rho_{A_0}.$$

Theorem 2.4. *The mosquito-free equilibrium $X_0^\star$ is globally asymptotically stable when $\mathcal{R}_v < 1$.*

Proof. Let the following Lyapunov candidate :

$$V(E, L, P, A_h, A_r, A_0) = \beta_1 E + \beta_2 L + \beta_3 P + \beta_4 A_h + \beta_5 A_r + \beta_6 A_0,$$

with

$$\beta_1 = \frac{\rho_E \rho_L \rho_p \rho_{A_h} \rho_{A_r}}{a_1 a_2 a_3 a_4 a_5 a_6}, \ \beta_2 = \frac{\rho_L \rho_p \rho_{A_h} \rho_{A_r}}{a_2 a_3 a_4 a_5 a_6}, \ \beta_3 = \frac{\rho_p \rho_{A_h} \rho_{A_r}}{a_3 a_4 a_5 a_6}, \ \beta_4 = \frac{\rho_{A_h} \rho_{A_r}}{a_4 a_5 a_6}, \ \beta_5 = \frac{\rho_{A_r}}{a_5 a_6} \text{ and } \beta_6 = \frac{1}{a_6}.$$

Let's pose $X = (E, L, P, A_h, A_r, A_0)^T \in \mathbb{R}^6$, we then have :

$$V(X_0^\star) = 0, \quad \text{and} \ \ \forall \ X \in \mathbb{R}^6 \setminus \left\{X_0^\star\right\}, \quad V(X) > 0.$$

On the other hand

$$\frac{dV}{dt} = \beta_1 \left(b\rho_{A_0} A_0 - a_1 E\right) + \beta_2 \left(\rho_E E - (a_2 + \mu_{L_2} L)L\right) + \beta_3 \left(\rho_L L - a_3 P\right)$$
$$+\beta_4 \left(\rho_p P + \rho_{A_0} A_0 - a_4 A_h\right) + \beta_5 \left(\rho_{A_h} A_h - a_5 A_r\right) + \beta_6 \left(\rho_{A_r} A_r - a_6 A_0\right)$$

By developing and reorganizing, we obtain :

$$\frac{dV}{dt} = (\beta_2 \rho_E - \beta_1 a_1)E + (\beta_3 \rho_{L_1} - \beta_2 a_2)L + (\beta_4 \rho_p - \beta_3 a_3)P + (\beta_5 \rho_{A_h} - \beta_4 a_4)A_h$$
$$+(\beta_6 \rho_{A_r} - \beta_5 a_5)A_r + (b\beta_1 \rho_{A_0} + \beta_4 \rho_{A_0} - \beta_6 a_6)A_0 - \beta_2 \mu_{L_2} L^2.$$

Then,

$$\beta_2 \rho_E - \beta_1 a_1 = 0, \quad \beta_3 \rho_{L_1} - \beta_2 a_2 = 0, \quad \beta_4 \rho_p - \beta_3 a_3 = 0, \quad \beta_5 \rho_{A_h} - \beta_4 a_4 = 0 \ \text{ and } \ \beta_6 \rho_{A_r} - \beta_5 a_5 = 0.$$

Also, we have :

$$\begin{aligned}
b\beta_1 \rho_{A_0} + \beta_4 \rho_{A_0} - \beta_6 a_6 &= B_1 + \frac{\rho_{A_h} \rho_{A_r} \rho_{A_0}}{(\mu_{A_h} + \rho_{A_h})(\mu_{A_r} + \rho_{A_r})(\mu_{A_0} + \rho_{A_0})} - 1 \\
&= B_1 - \frac{((\mu_{A_h} + \rho_{A_h})(\mu_{A_r} + \rho_{A_r})(\mu_{A_0} + \rho_{A_0}) - \rho_{A_h} \rho_{A_r} \rho_{A_0}}{(\mu_{A_h} + \rho_{A_h})(\mu_{A_r} + \rho_{A_r})(\mu_{A_0} + \rho_{A_0})} \\
&= B_1 - \frac{B_1}{\mathcal{R}_v} \\
&= B_1 \left(1 - \frac{1}{\mathcal{R}_v}\right) \\
b\beta_1 \rho_{A_0} + \beta_4 \rho_{A_0} - \beta_6 a_6 &= \frac{B_1(\mathcal{R}_v - 1)}{\mathcal{R}_v}.
\end{aligned}$$

So

$$\frac{dV}{dt} = \frac{B_1(\mathcal{R}_v - 1)}{\mathcal{R}_v} A_0 - \beta_2 \mu_{L_2} L^2.$$

Thereby,

$$\text{when } \mathcal{R}_v < 1, \text{ we then have } \frac{dV}{dt} < 0.$$

Therefore V is a strict Lyapunov function when $\mathcal{R}_v < 1$.

Finally

$$\frac{dV}{dt}(X) = 0 \Longleftrightarrow X = X_0^\star.$$

According to Lasalle's invariance principle [20, 21, 30], $X_0^\star$ is globally asymptotically stable when $\mathcal{R}_v < 1$. □

Theorem 2.5. *The mosquito persistence equilibrium point $X_1^\star$ is globally asymptotically stable when $\mathcal{R}_v > 1$.*

Proof. Let $X = (E, L, P, A_h, A_r, A_0)^T \in \mathbb{R}^6$ and consider the following function :

$$U(X) = E - E^\star - E^\star \ln \frac{E}{E^\star} + d_1 \left(L - L^\star - L^\star \ln \frac{L}{L^\star} \right) + d_2 \left(P - P^\star - P^\star \ln \frac{P}{P^\star} \right)$$

$$+ d_3 \left(A_h - A_h^\star - A_h^\star \ln \frac{A_h}{A_h^\star} \right) + d_4 \left(A_r - A_r^\star - A_r^\star \ln \frac{A_r}{A_r^\star} \right) + d_4 \left(A_0 - A_0^\star - A_0^\star \ln \frac{A_0}{A_0^\star} \right),$$

with

$$d_1 = \frac{a_1}{\rho_E}, \quad d_2 = \frac{a_1 a_2}{\rho_E \rho_L}, \quad d_3 = \frac{a_1 a_2 a_3}{\rho_E \rho_L \rho_P}, \quad d_4 = \frac{d_3 a_4}{\rho_{A_h}} \text{ and } d_5 = \frac{d_3 a_4 a_5}{\rho_{A_h} \rho_{A_0}}$$

We then have :

$$U(X_1^\star) = 0, \quad \text{and} \quad \forall \; X \in \mathbb{R}^6 \setminus \left\{ X_1^\star \right\}, \quad U(X) > 0.$$

Moreover, we obtain :

$$\frac{dU}{dt} = \left(1 - \frac{E^\star}{E} \right) (b\rho_{A_0} A_0 - a_1 E) + d_1 \left(1 - \frac{L^\star}{L} \right) (\rho_E E - a_2 L) + d_2 \left(1 - \frac{P^\star}{P} \right) (\rho_L L - a_3 P)$$

$$+ d_3 \left(1 - \frac{A_h^\star}{A_h} \right) (\rho_P P + \rho_{A_0} A_0 - a_4 A_h) + d_4 \left(1 - \frac{A_r^\star}{A_r} \right) (\rho_{A_h} A_h - a_5 A_r) + d_5 \left(1 - \frac{A_0^\star}{A_0} \right) (\rho_{A_r} A_r - a_6 A_0).$$

Taking into account that

$$b\rho_{A_0} A_0^\star = a_1 E^\star, \; \rho_E E^\star = a_2 L^\star, \; \rho_L L^\star = a_3 P^\star, \; \rho_P P^\star + \rho_{A_0} A_0^\star = a_4 A_h^\star, \; \rho_{A_h} A_h^\star = a_5 A_r^\star \text{ and } \rho_{A_r} A_r^\star = a_6 A_0^\star,$$

we obtain

$$\frac{dU}{dt} = -b\rho_{A_0} A_0 \left(1 - \frac{1}{\mathcal{R}_V} \right) + a_1 E^\star \left(6 - \frac{EL^\star}{LE^\star} - \frac{LP^\star}{PL^\star} - \frac{PA_h^\star}{A_h P^\star} - \frac{A_h A_r^\star}{A_r A_h^\star} - \frac{A_r A_0^\star}{A_0 A_r^\star} - \frac{A_0 E^\star}{EA_0^\star} \right)$$

$$+ d_3 \rho_{A_0} A_0^\star \left(3 - \frac{A_h A_r^\star}{A_r A_h^\star} - \frac{A_r A_0^\star}{A_0 A_r^\star} - \frac{A_0 A_h^\star}{A_h A_0^\star} \right).$$

The first term is automatically negative when $\mathcal{R}_v > 1$. Furthermore, since the arithmetic mean exceeds the geometric mean, it follows that the second and third terms are also negative. Hence $\frac{dU}{dt} < 0$, so U is a strict Lyapunov function when $\mathcal{R}_v > 1$.

On the other hand

$$\frac{dU}{dt}(X) = 0 \Longleftrightarrow X = X_1^\star.$$

According to Lasalle's invariance principle [20, 21, 27, 29], $X_1^\star$ is globally asymptotically stable when $\mathcal{R}_v > 1$. ☐

Remark 2.2. *It emerges from this study that the asymptotic behavior of the model is linked to the threshold parameter $\mathcal{R}_v$. Thus, if this threshold is less than or equal to one, then the mosquito population converges towards a state of extinction, while if it is strictly greater than one, there is a proliferation of the vector population, which demonstrates the importance of taking into account the complete life cycle of female mosquitoes in the formulation of mathematical models of vector-borne diseases. It is then clear that $\mathcal{R}_v$ is a good vector density control parameter.*

2.2.3 Impact of the gonotrophic cycle on the vector reproduction rate $\mathcal{R}_v$

We have obtained in (5) the relation $\mathcal{R}_v = \dfrac{r_a r_g}{1 - r_g}$.
Let study the function $\mathcal{R}_v$ with respect to the unknown $r_g \in [0\,;1[$. We then have :

$$\frac{d\mathcal{R}_v}{dr_g} = \frac{r_a}{1 - r_g} \geq 0 \quad \text{so} \quad \mathcal{R}_v \quad \text{is increasing from} \quad [0\,;1[\quad \text{to} \quad [0\,;+\infty[.$$

Moreover

$$0 \leq \mathcal{R}_v \leq 1 \Longrightarrow 0 \leq r_g \leq \frac{1}{r_a + 1} \text{ and } \mathcal{R}_v > 1 \Longrightarrow \frac{1}{r_a + 1} < r_g < 1.$$

Theorem 2.6. *We have the following results :*

(i) If $0 \leq r_g \leq \dfrac{1}{r_a + 1}$ *then* $0 \leq \mathcal{R}_v < 1$ *and the mosquito-free equilibrium point* $X_0^\star$ *is globally asymptotically stable.*

(ii) If $\dfrac{1}{r_a + 1} < r_g < 1$ *then* $\mathcal{R}_v > 1$ *and the mosquito persistence equilibrium point* $X_1^\star$ *is globally asymptotically stable.*

Remark 2.3. *This study allowed us to highlight the impact of the gonotrophic cycle materialized by the parameter* r_g *on the vector reproduction rate* $\mathcal{R}_v$*. The gonotrophic cycle is then of major importance in the life of vector; it is therefore essential to take it into account in the process of controlling the vector density and the fight against diseases transmitted by mosquitoes.*

3 About the transmission dynamics model

3.1 Mathematical formulation of the model

In this section we build a model of the dynamics of vector-borne diseases transmitted by mosquitoes. We took into account the compartment eggs (E), larvae (L) and pupae (P) of aquatic stage. Then, the compartments of healthy mosquitoes searching for hosts (A_{hs}), of resting healthy mosquitoes (A_{rs}), and of healthy mosquitoes seeking to lay eggs (A_{0s}), of healthy mosquitoes gonotrophic cycle. Otherwise, the compartments of infectious mosquitoes searching for hosts (I_V), of resting infectious mosquitoes (A_{ri}), and of infectious mosquitoes seeking to lay eggs (A_{0i}), of infectious mosquitoes gonotrophic cycle. Finally, the compartments (S_H) of susceptible humans, (I_H) of infectious humans and (R_H) of recovered humans, representing humans. Due to their short lifespan, we do not consider the class of recovered mosquitoes in the construction of our model. This information about our state variables is summarized in Table 3. The initial conditions of the model are :

$$E(0),\ L(0),\ P(0),\ A_{hs}(0),\ A_{rs}(0),\ A_{0s}(0),\ A_{ri}(0),\ A_{0i}(0),\ I_V(0),\ S_H(0),\ I_H(0),\ R_H(0).$$

We also have the following definitions :

- the population of mosquitoes searching for hosts to take a blood meal for egg maturation noted A_h is defined by : $A_h = A_{hs} + I_V$,
- the population of resting mosquitoes noted A_r is defined by : $A_r = A_{rs} + A_{ri}$
- the population of mosquitoes seeking to lay eggs noted A_0 is defined by : $A_0 = A_{0s} + A_{0i}$.

The total population of adult mosquitoes denoted N_V is defined by :

$$N_V = A_{hs} + A_{rs} + A_{0s} + A_{ri} + A_{0i} + I_V = A_h + A_r + A_0.$$

The total population of humans denoted N_H is defined by :

$$N_H = S_H + I_H + R_H.$$

We now give the following definitions :

Table 3: Definition of the state variables of the transmission model.

Variables	Biological description
E	density of eggs
L	density of larvae
P	density of pupae
A_{hs}	density of healthy mosquitoes searching for hosts
A_{rs}	density of resting healthy mosquitoes
A_{0s}	density of healthy mosquitoes seeking to lay eggs
A_{ri}	density of resting infectious mosquitoes
A_{0i}	density of infectious mosquitoes seeking to lay eggs
I_V	density of infectious mosquitoes searching for hosts
S_H	density of susceptible humans
I_H	density of infectious humans
R_H	density of recovered humans

- β_V is the rate of mosquito bites on human hosts per unit time,
- β_H is the rate at which a human is bitten by mosquitoes per unit time,
- α_{VH} is the probability that contacts with a human is infectious for a mosquito,
- α_{HV} is the probability that contacts with a mosquito is infectious for a human.

The rate of transmission of the disease from mosquitoes to humans that we note β_{VH} is defined by $\beta_{VH} = \beta_V\alpha_{VH}$ and that of transmission of the disease from humans to mosquitoes that we note β_{HV} is defined by $\beta_{HV} = \beta_H\alpha_{HV}$.

Thus, the force of infection for mosquitoes A_{hs} that we note λ_V is defined by :

$$\lambda_V(I_H, I_V) = \beta_{VH}\frac{I_H}{N_H} = \frac{\beta_V\alpha_{VH}I_H}{N_H}.$$

We recall that ρ_{A_h} is the rate of female mosquitoes that rest after taking a blood meal.

The rate of uninfected mosquitoes after a blood meal that we note $\lambda_V^{'}$ is then defined by :

$$\lambda_V^{'}(I_H, I_V) = \rho_{A_h} - \lambda_V(I_H, I_V).$$

The force of infection for humans induced by infectious mosquitoes that we note λ_H is defined by :

$$\lambda_H(I_H, I_V) = \beta_{HV}\frac{I_V}{N_H} = \frac{\beta_H\alpha_{HV}I_V}{N_H}.$$

In addition to the parameters listed in Table 2, we use the parameters from Table 4. When a mosquito in the A_{hs} compartment bites a human, it has a probability α_{VH} to be infected. Two scenarios can then arise :

Table 4: Description of transmission model parameters.

Variables	Description
Λ_H	constant recruitment of susceptible humans
μ_H	natural death rate of humans
γ_H	rate of infected humans

- if not infected, it follows the cycle $A_{hs},\ A_{rs},\ A_{0s},\ A_{hs}$ of healthy adult mosquitoes. Indeed, it passes to class A_{rs} of healthy mosquitoes which rest, then class A_{0s} of healthy mosquitoes which lay and finally it returns to class A_{hs} of healthy mosquitoes searching hosts and so on,
- if it is infected, it integrates the cycle $A_{ri},\ A_{0i},\ I_V,\ A_{ri}$ of infected adult mosquitoes. In fact, it passes into compartment A_{ri} of infected mosquitoes which rest for the maturation of eggs, then compartment A_{0i}

of infected mosquitoes which lay eggs and finally it integrates compartment I_V of infectious mosquitoes searching for hosts and the cycle resumes.

When a susceptible human is bitten by an infectious mosquito, it may be infected with a probability α_{HV}. Once a human is infected, it passes after a certain time to class I_H of infectious humans and is in turn capable of infecting mosquitoes and so, the disease spreads. Some infected humans acquire immunity to the disease after a certain time, they then pass to class R_H of recovered humans of the disease. The interactions between humans and mosquitoes are summarized in the compartmental representation given in Figure 2, [32, 33, 36].

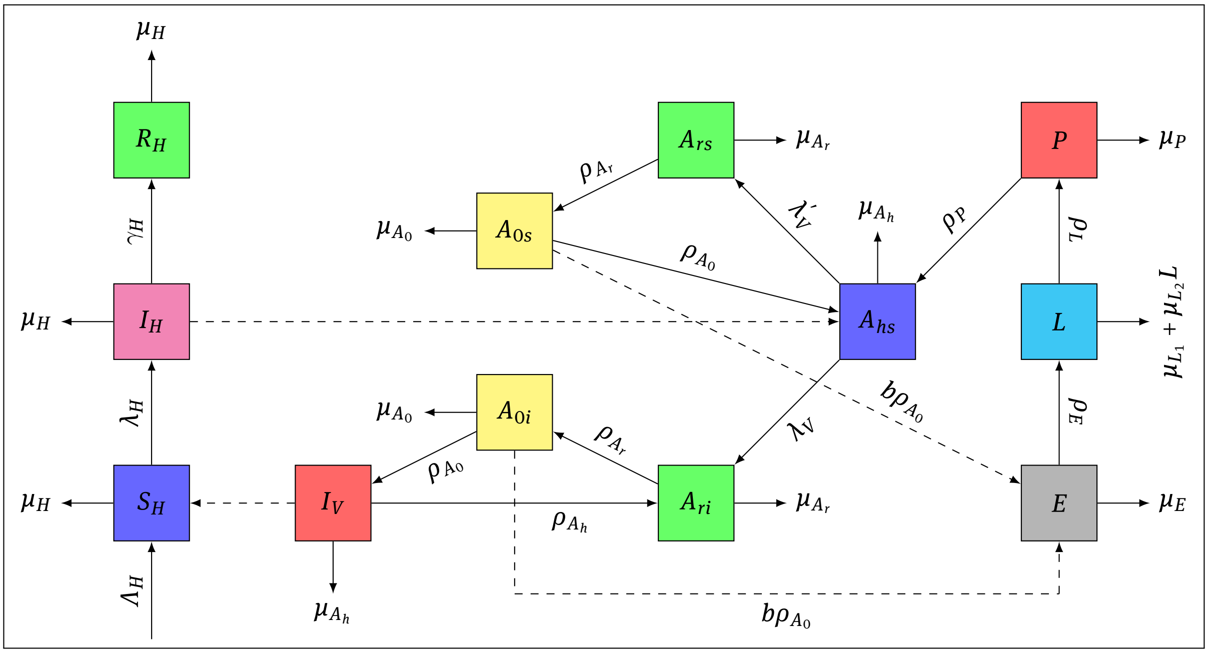

Figure 2: Compartmental representation of the interaction between humans and mosquitoes.

The solids arrows linking the compartments show the passage from one compartment to another, the dotted arrows between the compartments of humans and those of mosquitoes indicate the direction of infections, the arrow of Λ_H indicates recruitment at the human level and the other arrows represent natural mortalities.

From this diagram we obtain the following disease transmission model :

$$\begin{cases} \dfrac{dE}{dt} = b\rho_{A_0}(A_{0s} + A_{0i}) - (\mu_E + \rho_E)E, \\[2ex] \dfrac{dL}{dt} = \rho_E E - (\mu_{L_1} + \mu_{L_2} L + \rho_L)L, \\[2ex] \dfrac{dP}{dt} = \rho_L L - (\mu_P + \rho_P)P \\[2ex] \dfrac{dA_{hs}}{dt} = \rho_P P + \rho_{A_0} A_{0s} - \left(\mu_{A_h} + \lambda_V(I_H, I_V) + \lambda'_V(I_H, I_V)\right) A_{hs}, \\[2ex] \dfrac{dA_{rs}}{dt} = \lambda'_V(I_H, I_V) A_{hs} - (\mu_{A_r} + \rho_{A_r}) A_{rs}, \\[2ex] \dfrac{dA_{0s}}{dt} = \rho_{A_r} A_{rs} - (\mu_{A_0} + \rho_{A_0}) A_{0s}, \\[2ex] \dfrac{dA_{ri}}{dt} = \lambda_V(I_H, I_V) A_{hs} + \rho_{A_h} I_V - (\mu_{A_r} + \rho_{A_r}) A_{ri}, \\[2ex] \dfrac{dA_{0i}}{dt} = \rho_{A_r} A_{ri} - (\mu_{A_0} + \rho_{A_0}) A_{0i}, \\[2ex] \dfrac{dI_V}{dt} = \rho_{A_0} A_{0i} - (\mu_{A_h} + \rho_{A_h}) I_V, \\[2ex] \dfrac{dS_H}{dt} = \Lambda_H - \lambda_H(I_H, I_V) S_H - \mu_H S_H, \\[2ex] \dfrac{dI_H}{dt} = \lambda_H(I_H, I_V) S_H - (\gamma_H + \mu_H) I_H, \\[2ex] \dfrac{dR_H}{dt} = \gamma_H I_H - \mu_H R_H. \end{cases} \tag{7}$$

3.2 Mathematical analysis of transmission model

3.2.1 Existence, boundedness and uniqueness of the solutions

For the rest, the initial conditions being $(t_0, X_0) \in \mathbb{R}_+ \times \mathbb{R}_+^{12}$, we write system (7) as follows :

$$\begin{cases} X'(t) = h(X(t)), \\ (t_0, X_0) \in \mathbb{R}_+ \times \mathbb{R}_+^{12}. \end{cases} \tag{8}$$

With :

$$X(t) = (E,\ L,\ P,\ A_{hs},\ A_{rs},\ A_{0s},\ A_{ri},\ A_{0i},\ I_V,\ S_H,\ I_H,\ R_H)$$

Theorem 3.1. (***Existence and positivity of solutions***) *The Cauchy problem (8) admits a unique maximal solution* $([0, T], X)$ *and when* $X_0 > 0$, *then* $X > 0,\ \forall\, t \in [0, T]$.

Proof. Indeed,

- As h is of class $\mathscr{C}^\infty$ so it is of class $\mathscr{C}^1$, therefore locally Lipschitzian on $\mathbb{R}^{12}$, we deduce from it, the existence and uniqueness of a maximal solution of the Cauchy problem associated with system (7) and relative to the initial condition $(t_0, X_0) \in \mathbb{R} \times \mathbb{R}^{12}$. Moreover h being of class $\mathscr{C}^\infty$, we deduce that this solution is also of class $\mathscr{C}^\infty$.

- The proof of the positivity of $X \geq 0$ when $X_0 \geq 0$ is done in a similar way to that of Theorem 2.1.

☐

Consider

$$N = E + L + P + A_{hs} + A_{rs} + A_{0s} + A_{ri} + A_{0i} + I_V = E + L + P + Ah + A_r + A_0, \quad K = b\rho_{A_0} C$$

and

$$n = \min\left\{\mu_E, \mu_{L_1}, \mu_P, \mu_{A_h}, \mu_{A_r}, \mu_{A_0}\right\}.$$

C being the constant defined in (3) under the assumption **(H1)**.

Proposition 3.1. *System (7) admits a unique solution and this solution is bounded. Moreover the set*

$$\mathbb{D}_{VH} = \left\{ \begin{pmatrix} E \\ L \\ P \\ A_{hs} \\ A_{rs} \\ A_{0s} \\ A_{ri} \\ A_{0i} \\ I_V \\ S_H \\ I_H \\ R_H \end{pmatrix} \in \mathbb{R}_+^{12} \;\middle|\; \begin{array}{l} 0 \leq E + L + P + A_{hs} + A_{rs} + A_{0s} + A_{ri} + A_{0i} + I_V \leq \frac{K}{n}, \\[2ex] 0 \leq S_H + I_H + R_H \leq \frac{\Lambda_H}{\mu_H} \end{array} \right\}$$

is positively invariant and attracts all the positive orbits in $\mathbb{R}_+$.

Proof. By taking the sum of the nine (9) first equations of system (7), we obtain :

$$N'(t) = b\rho_{A_0} A_0 - \left(\mu_E E + \left(\mu_{L_1} + \mu_{L_2} L\right) L + \mu_P P + \mu_{A_h} A_h + \mu_{A_r} A_r + \mu_{A_0} A_0\right).$$

So

$$N'(t) \leq b\rho_{A_0} A_0 - \left(\mu_E E + \mu_{L_1} L + \mu_P P + \mu_{A_h} A_h + \mu_{A_r} A_r + \mu_{A_0} A_0\right).$$

We then have

$$N'(t) \leq b\rho_{A_0} C - \min\left\{\mu_E, \mu_{L_1}, \mu_P, \mu_{A_h}, \mu_{A_r}, \mu_{A_0}\right\} N.$$

As

$$K = b\rho_{A_0} C \quad \text{and} \quad n = \min\left\{\mu_E, \mu_{L_1}, \mu_P, \mu_{A_h}, \mu_{A_r}, \mu_{A_0}\right\},$$

then

$$N'(t) \leq K - nN.$$

Let us solve the equation

$$N_2'(t) = K - nN_2.$$

Thanks to the constant variation formula between t and t_0, a particular solution is given as follows :

$$N_2(t) = \frac{K}{n} + \left(N_2(t_0) - \frac{K}{n}\right) e^{-n(t-t_0)}$$

Therefore, $N_2 \longrightarrow \frac{K}{n}$ when $t \longrightarrow +\infty$ and we obtain $N_2(t) \leq \frac{K}{n}$.
By applying the principle of comparison, we obtain :

$$\lim_{t \longrightarrow +\infty} \sup N(t) \leq \frac{K}{n}.$$

We therefore conclude that

$$0 \leq N(t) \leq \frac{K}{n},$$

so, N is bounded and $T = +\infty$.

Suppose now that

$$N(t) > \frac{K}{n},$$

we then have

$$N'(t) < K - n \times \frac{K}{n},$$

i.e

$$N'(t) < 0.$$

In this case the mosquito population is decreasing and as the domain $\mathbb{D}_{VH}$ is compact, we conclude that all the solutions remain there.

On the other hand, by taking the sum of the last three equations of system (7) and by simplifying, we obtain the differential equation

$$N_H' = \Lambda_H - \mu_H N_H.$$

Using the constant variation formula between t and t_0, a particular solution is given as follows :

$$N_H(t) = \frac{\Lambda_H}{\mu_H} + \left(N_H(t_0) - \frac{\Lambda_H}{\mu_H}\right) e^{-\mu_H(t-t_0)}.$$

Therefore, $N_H \longrightarrow \frac{\Lambda_H}{\mu_H}$ when $t \longrightarrow +\infty$ and we have :

$$0 \leq N_H(t) \leq \frac{\Lambda_H}{\mu_H}.$$

So N_H is bounded and $T = +\infty$.
Then $\mathbb{D}_{VH}$ is bounded and we therefore deduce the global existence of the solutions in $[0, +\infty[$.
On the other hand, if we assume that $N_H(t) > \frac{\Lambda_H}{\mu_H}$, we have $N_H'(t) < \Lambda_H - \mu_H \times \frac{\Lambda_H}{\mu_H}$ i.e $N_H'(t) < 0$.
In this case human population would be decreasing and as the domain is compact, thus all the solutions remain there. ☐

3.2.2 Existence and stability of equilibrium points

Theorem 3.2.
(i) *When* $\mathcal{R}_v \leq 1$ *then system (7) admits in domain* $\mathbb{D}_{VH}$ *a disease-free equilibrium point without mosquitoes* $X_0^\star = (0, 0, 0, 0, 0, 0, 0, 0, 0, S_H^\star, 0, 0)$, *with* $S_H^\star = \frac{\Lambda_H}{\mu_H}$.
(ii) *When* $\mathcal{R}_v > 1$ *then system (7) admits in domain* $\mathbb{D}_{VH}$ *a disease-free equilibrium point with mosquitoes* $X_1^\star = (E^\star, L^\star, P^\star, A_{hs}^\star, A_{rs}^\star, A_{0s}^\star, 0, 0, 0, S_H^\star, 0, 0)$,

where

$$E^\star = \frac{b\rho_{A_0}A_0^\star}{\mu_E + \rho_E}, \quad L^\star = \frac{(\mu_{L_1} + \rho_L)(\mathcal{R}_v - 1)}{\mu_{L_2}}, \quad P^\star = \frac{\rho_L L^\star}{\mu_P + \rho_P}, \quad A_{hs}^\star = \frac{\rho_P P^\star \mathcal{R}_v}{(\mu_{A_h} + \rho_{A_h})B_1},$$

$$A_{rs}^\star = \frac{\rho_{A_h}A_h^\star}{\mu_{A_r} + \rho_{A_r}}, \quad A_{0s}^\star = \frac{\rho_{A_r}A_r^\star}{\mu_{A_0} + \rho_{A_0}} \quad \textit{and} \quad S_H^\star = \frac{\Lambda_H}{\mu_H},$$

Proof. Since there is no disease then $\lambda_V(I_H, I_V) = \lambda_H(I_H, I_V) = 0$ and $\lambda_V'(I_H, I_V) = \rho_{A_h}$.

(i) Let pose $h(X_0^\star) = 0$, we then obtain a unique equation $\Lambda_H - \mu_H S_H^\star = 0$. We then have $S_H^\star = \dfrac{\Lambda_H}{\mu_H}$

(ii) Setting the right side of system (7) to zero and solving this equation gives us the values of $E^\star$, $L^\star$, $P^\star$, $A_{hs}^\star$, $A_{rs}^\star$, $A_{0s}^\star$ and $S_H^\star$.

□

Remark 3.1. *The disease-free equilibrium point $X_0^\star$ represents the case where there is no disease and the area is completely devoid of mosquitoes. This case is difficult to obtain in most areas where malaria is intensive due to the difficulties in completely eradicating the mosquito population. The disease-free equilibrium $X_1^\star$ represents the case where the mosquito population exists but there is no disease. Therefore, in this paper, we focus our study on the $X_1^\star$ equilibrium point because it is more biologically realistic [20, 32, 33, 36].*

In the following part of the section, we determine the basic reproduction number. This number noted $\mathcal{R}_0$ is defined "heuristically" as the average number of new cases of infection, generated by an average individual (during his period of infectivity), in a population entirely made up of susceptible. Mathematically $\mathcal{R}_0$ is defined as the spectral radius of next-generation matrix [37]. Consider the compartments carrying the disease A_{ri}, A_{0i}, I_V and I_H, and adopt the notation $x = (A_{ri}, A_{0i}, I_V, I_H)$. We have the following vector functions

$$\mathcal{F}(x) = \begin{pmatrix} \beta_{VH}\dfrac{I_H A_{hs}}{N_H} \\ 0 \\ 0 \\ \beta_{HV}\dfrac{I_V S_H}{N_H} \end{pmatrix} \text{ et } \mathcal{V}(x) = \begin{pmatrix} \rho_{A_h} I_V - (\mu_{A_r} + \rho_{A_r})A_{ri} \\ \rho_{A_r} A_{ri} - (\mu_{A_0} + \rho_{A_0})A_{0i} \\ \rho_{A_0} A_{0i} - (\mu_{A_h} + \rho_{A_h})I_V \\ -(\gamma_H + \mu_H)I_H \end{pmatrix}.$$

The next-generation matrix is $-Jac_{\mathcal{F}}(X_1^\star)(Jac_{\mathcal{V}}(X_1^\star))^{-1}$ where

$$Jac_{\mathcal{F}}(X_1^\star) = \begin{pmatrix} 0 & 0 & 0 & \dfrac{B_{VH}A_h^\star}{S_H^\star} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_{HV} & 0 \end{pmatrix} \text{ and}$$

$$Jac_{\mathcal{V}}(X_1^\star) = \begin{pmatrix} -(\mu_{A_r} + \rho_{A_r}) & 0 & \rho_{A_h} & 0 \\ \rho_{A_r} & -(\mu_{A_0} + \rho_{A_0}) & 0 & 0 \\ 0 & \rho_{A_0} & -(\mu_{A_h} + \rho_{A_h}) & 0 \\ 0 & 0 & 0 & -(\gamma_H + \mu_H) \end{pmatrix}.$$

We then have

$$-Jac_{\mathcal{F}}(X_1^\star)(Jac_{\mathcal{V}}(X_1^\star))^{-1} = \begin{pmatrix} 0 & 0 & 0 & \dfrac{\beta_{VH}A_h^\star}{K_1 S_H^\star} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_1 & M_2 & M_3 & 0 \end{pmatrix}.$$

With

$$M_1 = \frac{\rho_{A_r}\rho_{A_0}\mathcal{R}_v\beta_{HV}}{a_4a_5a_6B_1}, \quad M_2 = \frac{\rho_{A_0}a_5\mathcal{R}_v\beta_{HV}}{a_4a_5a_6B_1}, \quad M_3 = \frac{a_5a_6\mathcal{R}_v\beta_{HV}}{a_4a_5a_6B_1} \text{ and } K_1 = \gamma_H + \mu_H.$$

We recall that we previously posed :

$a_1 = \mu_E + \rho_E, \quad a_2 = \mu_{L_1} + \rho_L, \quad a_3 = \mu_P + \rho_P, \quad a_4 = \mu_{A_h} + \rho_{A_h}, \quad a_5 = \mu_{A_r} + \rho_{A_r}, \quad a_6 = \mu_{A_0} + \rho_{A_0}.$

The eigenvalues of $-Jac_{\mathcal{F}}(X_1^\star)(Jac_{\mathcal{V}}(X_1^\star))^{-1}$ are 0 and $\sqrt{\dfrac{\beta_{VH}M_1A_h^\star}{K_1S_H^\star}}$.

The basic reproduction number is given by :

$$\mathcal{R}_0 = \rho(-Jac_{\mathcal{F}}(X_1^\star)(Jac_{\mathcal{V}}(X_1^\star))^{-1}) = \sqrt{\frac{\beta_{VH}M_1A_h^\star}{K_1S_H^\star}}.$$

By successively replacing M_1, $A_h^\star$, $P^\star$, $L^\star$ and $S_H^\star$ by their expressions, we obtain :

$$\mathcal{R}_0 = \sqrt{\frac{a_1a_2^2\mu_H\beta_{VH}\beta_{HV}\mathcal{R}_v^2(\mathcal{R}_v - 1)}{b\rho_E\rho_{A_h}\mu_{L_2}a_4B_1K_1\Lambda_H}}$$

Whether

$$\kappa = \frac{a_1a_2^2\mu_H\beta_{VH}\beta_{HV}\mathcal{R}_v^2}{b\rho_E\rho_{A_h}\mu_{L_2}a_4B_1K_1\Lambda_H},$$

We then have :

$$\mathcal{R}_0 = \sqrt{\kappa(\mathcal{R}_v - 1)}.$$

We find that this number $\mathcal{R}_0$ is closely related to the vector reproduction rate $\mathcal{R}_v$.

Theorem 3.3. *When $\mathcal{R}_v > 1$, system (7) admits in $\mathbb{D}_{VH}$ an endemic equilibrium $X_2^\star$ defined by :*

$$X_2^\star = (E^{**},\, L^{**},\, P^{**},\, A_{hs}^{**},\, A_{rs}^{**},\, A_{0s}^{**},\, A_{ri}^{**},\, A_{0i}^{**},\, I_V^{**},\, S_H^{**},\, I_H^{**},\, R_H^{**})$$

where :

$$E^{**} = \frac{b\rho_{A_0}(A_{0s}^{**} + A_{0i}^{**})}{a_1}, \quad L^{**} = \frac{a_2(\mathcal{R}_v - 1)}{\mu_{L_2}}, \quad P^{**} = \frac{\rho_L L^{**}}{a_3},$$

$$A_{hs}^{**} = \frac{a_5a_6\Lambda_H\mathcal{R}_v\rho_p P^{**}}{a_4a_5a_6B_1\Lambda_H + \rho_{A_r}\rho_{A_0}\mu_H\beta_{VH}\mathcal{R}_v I_H^{**}}, \quad A_{rs}^{**} = \frac{\rho_{A_h}A_{hs}^{**}\Lambda_H - \mu_H\beta_{VH}I_H^{**}A_{hs}^{**}}{a_5\Lambda_H}, \quad A_{0s}^{**} = \frac{\rho_{A_r}A_{rs}^{**}}{a_6},$$

$$A_{ri}^{**} = \frac{\rho_{A_h}I_V^{**}\Lambda_H + \mu_H\beta_{VH}I_H^{**}A_{hs}^{**}}{a_5\Lambda_H}, \quad A_{0i}^{**} = \frac{\rho_{A_r}A_{ri}^{**}}{a_6}, \quad I_V^{**} = \frac{\rho_{A_r}\rho_{A_0}\mu_H\beta_{VH}\mathcal{R}_v I_H^{**}A_h^{**}}{a_4a_5a_6B_1\Lambda_H},$$

$$S_H^{**} = \frac{\Lambda_H - K_1I_H^{**}}{\mu_H}, \quad I_H^{**} = \frac{a_4a_5a_6B_1\Lambda_H(\mathcal{R}_0^2 - 1)}{a_4a_5a_6B_1K_1\mathcal{R}_0^2 + \rho_{A_r}\rho_{A_0}\mu_H\beta_{VH}\mathcal{R}_v} \quad \textit{and} \quad R_H^{**} = \frac{\gamma_H I_H^{**}}{\mu_H}.$$

Proof. When we set the right side of system (7) to zero, the resolution gives the values of E^{**}, L^{**}, P^{**}, A_{hs}^{**}, A_{rs}^{**}, A_{0s}^{**}, A_{ri}^{**}, A_{0i}^{**}, I_V^{**}, S_H^{**}, I_H^{**} and R_H^{**} given. □

Theorem 3.4. *The disease-free equilibrium point with mosquitoes $X_1^\star$ is locally asymptotically stable when $\mathcal{R}_0 < 1$ and unstable when $\mathcal{R}_0 > 1$.*

Proof. Indeed, from Varga's theorem in [38], the disease-free equilibrium $X_1^\star$ is locally asymptotically stable when $\mathcal{R}_0 = \rho(-Jac_{\mathcal{F}}(X_1^\star)(Jac_{\mathcal{V}}(X_1^\star))^{-1}) < 1$. □

For the study of the global stability of our transmission model, we state the following fundamental theorem [8, 32, 33, 36, 39].

Theorem 3.5. *Consider the following ordinary differential equation :*

$$\left\{\begin{array}{l} X' = f(X);\ X \in \mathbb{R}^n, \\[2ex] Y' = g(X, Y);\ Y \in \mathbb{R}^m. \end{array}\right. \tag{9}$$

Let $(X^\star, Y^\star)$ be an equilibrium point for system (9) i.e.,

$$f(X^\star) = 0 \ \ and \ \ g(X^\star, Y^\star) = 0.$$

If system (9) satisfies the following assumptions :

- $X^\star$ *is globally asymptotically stable for the subsystem* $X' = f(X)$,
- $Y^\star$ *is globally asymptotically stable for the subsystem* $Y' = g(X^\star, Y)$,

then $(X^\star, Y^\star)$ is locally asymptotically stable for system (9).
Moreover, if there exists a positively invariant compact $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ containing $(X^\star, Y^\star)$ in its interior, then $(X^\star, Y^\star)$ is a globally asymptotically stable equilibrium for the restriction from system (9) to Ω.

For the application of Theorem 3.5, to our transmission model (7), we rewrite it in two subsystems as follows:

$$\left\{\begin{array}{l}
\left\{\begin{array}{l}
\dfrac{dE}{dt} = b\rho_{A_0}(A_{0s} + A_{0i}) - (\mu_E + \rho_E)E, \\[2ex]
\dfrac{dL}{dt} = \rho_E E - (\mu_{L_1} + \mu_{L_2}L + \rho_L)L, \\[2ex]
\dfrac{dP}{dt} = \rho_L L - (\mu_P + \rho_P)P \\[2ex]
\dfrac{dA_{hs}}{dt} = \rho_P P + \rho_{A_0}A_{0s} - \left(\mu_{A_h} + \lambda_V(I_H, I_V) + \lambda'_V(I_H, I_V)\right)A_{hs}, \qquad (S1) \\[2ex]
\dfrac{dA_{rs}}{dt} = \lambda'_V(I_H, I_V)A_{hs} - (\mu_{A_r} + \rho_{A_r})A_{rs}, \\[2ex]
\dfrac{dA_{0s}}{dt} = \rho_{A_r}A_{rs} - (\mu_{A_0} + \rho_{A_0})A_{0s}, \\[2ex]
\dfrac{dA_{ri}}{dt} = \lambda_V(I_H, I_V)A_{hs} + \rho_{A_h}I_V - (\mu_{A_r} + \rho_{A_r})A_{ri}, \\[2ex]
\dfrac{dA_{0i}}{dt} = \rho_{A_r}A_{ri} - (\mu_{A_0} + \rho_{A_0})A_{0i}, \\[2ex]
\dfrac{dI_V}{dt} = \rho_{A_0}A_{0i} - (\mu_{A_h} + \rho_{A_h})I_V,
\end{array}\right. \\[4ex]
\left\{\begin{array}{l}
\dfrac{dS_H}{dt} = \Lambda_H - \lambda_H(I_H, I_V)S_H - \mu_H S_H, \\[2ex]
\dfrac{dI_H}{dt} = \lambda_H(I_H, I_V)S_H - (\gamma_H + \mu_H)I_H, \qquad (S2) \\[2ex]
\dfrac{dR_H}{dt} = \gamma_H I_H - \mu_H R_H.
\end{array}\right.
\end{array}\right.$$

By adding the compartments A_{hs} and I_V, A_{rs} and A_{ri}, A_{0s} and A_{0i}, then taking into account the fact that $A_{hs} + I_V = A_h$, $A_{rs} + A_{ri} = A_r$, $A_{0s} + A_{0i} = A_0$ and $\lambda_V(I_H, I_V) + \lambda'_V(I_H, I_V) = \rho_{A_h}$, this model becomes the equivalent model (10):

$$\left\{\begin{array}{l}
\left\{\begin{array}{l}
\dfrac{dE}{dt} = b\rho_{A_0}A_0 - (\mu_E + \rho_E)E, \\[2ex]
\dfrac{dL}{dt} = \rho_E E - (\mu_{L_1} + \mu_{L_2}L + \rho_L)L, \\[2ex]
\dfrac{dP}{dt} = \rho_L L - (\mu_P + \rho_P)P \\[2ex]
\dfrac{dA_h}{dt} = \rho_P P + \rho_{A_0}A_0 - \left(\mu_{A_h} + \rho_{A_h}\right) A_h, \qquad (S1) \\[2ex]
\dfrac{dA_r}{dt} = \rho_{A_h}A_h - (\mu_{A_r} + \rho_{A_r})A_r, \\[2ex]
\dfrac{dA_0}{dt} = \rho_{A_r}A_r - (\mu_{A_0} + \rho_{A_0})A_0,
\end{array}\right. \\[2ex]
\left\{\begin{array}{l}
\dfrac{dS_H}{dt} = \Lambda_H - \lambda_H(I_H, I_V)S_H - \mu_H S_H, \\[2ex]
\dfrac{dI_H}{dt} = \lambda_H(I_H, I_V)S_H - (\gamma_H + \mu_H)I_H, \qquad (S2) \\[2ex]
\dfrac{dR_H}{dt} = \gamma_H I_H - \mu_H R_H.
\end{array}\right.
\end{array}\right. \tag{10}$$

If we replace $\lambda_H(I_H, I_V)$ by its expression in subsystem $(S2)$, it becomes :

$$\left\{\begin{array}{l}
\dfrac{dS_H}{dt} = \Lambda_H - \dfrac{\beta_{HV}I_V S_H}{N_H} - \mu_H S_H, \\[2ex]
\dfrac{dI_H}{dt} = \dfrac{\beta_{HV}I_V S_H}{N_H} - (\gamma_H + \mu_H)I_H, \qquad (S2) \\[2ex]
\dfrac{dR_H}{dt} = \gamma_H I_H - \mu_H R_H.
\end{array}\right.$$

Subsystem $(S1)$ is the same as system (1) which we wrote previously in the following way :

$$X' = f(X).$$

On the other hand, by posing :

$$Y = \begin{pmatrix} S_H \\ I_H \\ R_H \end{pmatrix} \quad \text{and} \quad g(X, Y) = \begin{pmatrix} g_1(X, Y) \\[2ex] g_2(X, Y) \\[2ex] g_3(X, Y) \end{pmatrix} = \begin{pmatrix} \Lambda_H - \dfrac{\beta_{HV}I_V S_H}{N_H} - \mu_H S_H, \\[2ex] \dfrac{\beta_{HV}I_V S_H}{N_H} - (\gamma_H + \mu_H)I_H, \\[2ex] \gamma_H I_H - \mu_H R_H \end{pmatrix},$$

subsystem $(S2)$ can be rewritten as :

$$Y' = g(X, Y).$$

Thus, system (10) can be rewritten:

$$\left\{\begin{array}{l} X' = f(X), \\[2ex] Y' = g(X, Y). \end{array}\right.$$

Consider

$$\mathbb{D}_H = \left\{ \begin{pmatrix} S_H \\ I_H \\ R_H \end{pmatrix} \in \mathbb{R}^3_+ \;\middle|\; 0 \leq S_H + I_H + R_H \leq \tfrac{\Lambda_H}{\mu_H} \right\}.$$

Let :

$$\mathcal{D}_{VH} = \mathbb{D}_V \times \mathbb{D}_H,$$

then

$$X_3^\star = (X^\star, Y^\star) \in \mathcal{D}_{VH}, \text{ with } X^\star = (E^\star, L^\star, P^\star, A_h^\star, A_r^\star, A_0^\star) \text{ and } Y^\star = (S_H^\star, 0, 0)$$

is a disease-free equilibrium with mosquitoes of system (10). By Theorem 2.5, $X^\star$ is globally asymptotically stable for $(S1)$ in $\mathbb{D}_V$ when $\mathcal{R}_v > 1$.
We obtain from subsystem $(S2)$, the following limit system $Y' = g(X^\star, Y)$:

$$\begin{cases} \dfrac{dS_H}{dt} = \Lambda_H - \dfrac{\beta_{HV} I_V^\star S_H}{N_H} - \mu_H S_H, \\[2ex] \dfrac{dI_H}{dt} = \dfrac{\beta_{HV} I_V^\star S_H}{N_H} - (\gamma_H + \mu_H) I_H, \\[2ex] \dfrac{dR_H}{dt} = \gamma_H I_H - \mu_H R_H. \end{cases} \tag{S3}$$

Thus, according to Theorem 3.5, to show that $X_3^\star$ is globally asymptotically stable for system (10) in $\mathcal{D}_{VH}$ when $\mathcal{R}_v > 1$ and $\mathcal{R}_0 < 1$, just show that $Y^\star$ is globally asymptotically stable for $(S3)$ in $\mathbb{D}_H$ when $\mathcal{R}_0 < 1$.

Theorem 3.6. *The equilibrium point $Y^\star$ is globally asymptotically stable for system $(S3)$ in $\mathbb{D}_H$ when $\mathcal{R}_0 < 1$.*

Proof. Consider the following function :

$$V(Y) = \left(S_H - S_H^\star - S_H^\star \ln \frac{S_H}{S_H^\star} \right) + I_H + R_H \text{ with } Y = (S_H, I_H, R_H)^T \in \mathbb{R}^3.$$

So, we have :

$$V(Y^\star) = 0 \;\text{ and }\; \forall\; Y \in \mathbb{R}^3 \setminus \left\{ Y^\star \right\}, \;\; V(X) > 0.$$

On the other hand

$$\frac{dV}{dt} = \left(1 - \frac{S_H^\star}{S_H} \right) \frac{dS_H}{dt} + \frac{dI_H}{dt} + \frac{dR_H}{dt}.$$

We replace $\dfrac{dS_H}{dt}$, $\dfrac{dI_H}{dt}$ and $\dfrac{dR_H}{dt}$ by their expressions to get

$$\frac{dV}{dt} = \left(1 - \frac{S_H^\star}{S_H} \right) \left(\Lambda_H - \frac{\beta_{HV} I_V^\star S_H}{N_H} - \mu_H S_H \right) + \frac{\beta_{HV} I_V^\star S_H}{N_H} - (\gamma_H + \mu_H) I_H + \gamma_H I_H - \mu_H R_H.$$

As $I_V^\star = 0$ therefore

$$\frac{dV}{dt} = \left(1 - \frac{S_H^\star}{S_H} \right) (\Lambda_H - \mu_H S_H) - \mu_H I_H - \mu_H R_H.$$

Taking into account that

$$\Lambda_H = \mu_H S_H^\star,$$

we obtain :

$$\frac{dV}{dt} = \left(1 - \frac{S_H^\star}{S_H} \right) \left(\mu_H S_H^\star - \mu_H S_H \right) - \mu_H I_H - \mu_H R_H.$$

By rearranging, we obtain :

$$\frac{dV}{dt} = -\mu_H \frac{(S_H - S_H^\star)^2}{S_H} - \mu_H I_H - \mu_H R_H.$$

From where

$$\forall \ Y \in \mathbb{R}^3 \setminus \left\{Y^\star\right\}, \quad \frac{dV}{dt}(Y) < 0.$$

So, V is a strict Lyapunov function when $R_0 \leq 1$.
Finally

$$\frac{dV}{dt}(Y) = 0 \Longleftrightarrow Y = Y_1^\star.$$

According to Lasalle's invariance principle [20, 21, 27, 29], we deduce that $Y^\star$ is globally asymptotically stable for system $(S3)$ in $\mathbb{D}_H$ when $\mathcal{R}_0 < 1$. ☐

We have just shown that $X_3^\star$ is globally asymptotically stable for system (10), which allows us to do the following conclusion:

Theorem 3.7. *The disease-free equilibrium with mosquitoes $X_1^\star$ is globally asymptotically stable for system (7) in $\mathbb{D}_{VH}$ when $\mathcal{R}_0 < 1$.*

We can write the endemic equilibrium point of system (10) as follows :

$$X_4^\star = (X^\star, Y^{\star\star}), \text{ with } X^\star = (E^\star, L^\star, P^\star, A_h^\star, A_r^\star, A_0^\star) \text{ and } Y^{\star\star} = (S_H^{\star\star}, I_H^{\star\star}, R_H^{\star\star}).$$

By Theorem 2.5, $X^\star$ is globally asymptotically stable for $(S1)$ in $\mathbb{D}_V$ when $\mathbb{R}_v > 1$.
We obtain from subsystem $(S2)$, the following limit system $Y' = g(X^\star, Y)$:

$$\begin{cases} \dfrac{dS_H}{dt} = \Lambda_H - \dfrac{\beta_{HV} I_V^{\star\star} S_H}{N_H} - \mu_H S_H, \\[2ex] \dfrac{dI_H}{dt} = \dfrac{\beta_{HV} I_V^{\star\star} S_H}{N_H} - K_1 I_H, \\[2ex] \dfrac{dR_H}{dt} = \gamma_H I_H - \mu_H R_H. \end{cases} \qquad (S4)$$

So, to show that $X_4^\star$ is globally asymptotically stable for system (7) in $\mathbb{D}_{VH}$ when $\mathcal{R}_0 > 1$, just show that $Y^{\star\star}$ is globally asymptotically stable for system $(S4)$ in $\mathbb{D}_H$ when $\mathcal{R}_0 > 1$.

Theorem 3.8. *The equilibrium point $Y^{\star\star}$ is globally asymptotically stable for system $(S4)$ in $\mathbb{D}_H$ when $\mathcal{R}_0 > 1$.*

Proof. Consider the following function :

$$V(Y) = N_H - N_H^{\star\star} - N_H^{\star\star} \ln \frac{N_H}{N_H^{\star\star}}, \text{ with } Y = (S_H, I_H, R_H)^T \in \mathbb{R}^3,$$

so we have

$$V(Y^{\star\star}) = 0 \ \text{ and } \ \forall \ Y \in \mathbb{R}^3 \setminus \left\{Y^{\star\star}\right\}, \quad V(Y) > 0.$$

On the other hand

$$\frac{dV}{dt} = \left(1 - \frac{N_H^{\star\star}}{N_H}\right) \frac{dN_H}{dt},$$

which gives :

$$\frac{dV}{dt}(Y) = \frac{N_H - N_H^{\star\star}}{N_H}(\Lambda_H - \mu_H N_H) \ \text{ because } \ \frac{dN_H}{dt} = \Lambda_H - \mu_H N_H.$$

We have

$$\Lambda_H = \mu_H N_H^{\star\star},$$

which allows us to obtain :

$$\frac{dV}{dt}(Y) = -\mu_H \frac{\left(N_H - N_H^{\star\star}\right)^2}{N_H}.$$

From where

$$\forall \;\; Y \in \mathbb{R}^3 \setminus \left\{Y^{**}\right\}, \quad \frac{dV}{dt}(Y) < 0.$$

Then, V is a strict Lyapunov function when $\mathcal{R}_0 > 1$.
Finally

$$\frac{dV}{dt}(Y) = 0 \Longleftrightarrow Y = Y^{**}.$$

According to Lasalle's invariance principle [20, 21, 27, 29], we deduce that Y^{**} is globally asymptotically stable for system $(S4)$ in $\mathbb{D}_H$ when $\mathcal{R}_0 > 1$. ☐

We have shown that $X_4^{\star}$ is globally asymptotically stable for system (10), which allows us to draw the following conclusion:

Theorem 3.9. *The endemic equilibrium point* $X_2^{\star}$ *is globally asymptotically stable for model (7) in* $\mathbb{D}_{VH}$ *when* $\mathcal{R}_0 > 1$.

4 Numerical simulations

The objective of this section is to corroborate by numerical results the mathematical theoretical results established during the study of our models. The MATLAB technical computing software with the fourth-order Runge-Kutta method is used to perform our numerical simulations.

4.1 Numerical simulations of vector model

In this part, we present some results of numerical simulations to predict the evolution of mosquitoes in the case of persistence and extinction. Our initial conditions are as follows: $E(0) = 150$, $L(0) = 25$, $P(0) = 10$, $A_h(0) = 55$, $A_r(0) = 19$ and $A_0(0) = 12$. The numerical values of the parameters used are summarized in Table 5.

Table 5: Numerical values of the mosquito model parameters.

Symbols	Values for persistence	Values for extinction	Range	Sources	Units
b	100	50	50–300	[23, 31]	–
ρ_E	0.50	0.33	0.33–1.0	[14, 31]	/day
ρ_L	0.16	0.08	0.08–0.17	[14, 31]	/day
ρ_P	0.50	0.33	0.33–1.0	[14, 31]	/day
μ_E	0.56	0.80	0.32–0.80	[23, 25]	/day
μ_{L_1}	0.44	0.58	0.30–0.58	[23, 25]	/day
μ_{L_2}	0.05	0.05	0.0–1.0	[21, 23]	/day/mosq
μ_P	0.37	0.52	0.22–0.52	[23, 25]	/day
ρ_{A_h}	0.46	0.322	0.322–0.598	[6, 23]	/day
ρ_{A_r}	0.43	0.30	0.30–0.56	[6, 23]	/day
ρ_{A_0}	3	3	3.0–4.0	[6, 23]	/day
μ_{A_h}	0.18	0.233	0.125–0.233	[6, 23]	/day
μ_{A_r}	0.0043	0.01	0.0034–0.01	[6, 23]	/day
μ_{A_0}	0.41	0.56	0.41–0.56	[6, 23]	/day

4.1.1 Dynamics of vector model for $\mathcal{R}_v > 1$

For the case of the persistence of mosquitoes, we obtain the curves of Figures 3, 4 and 5.

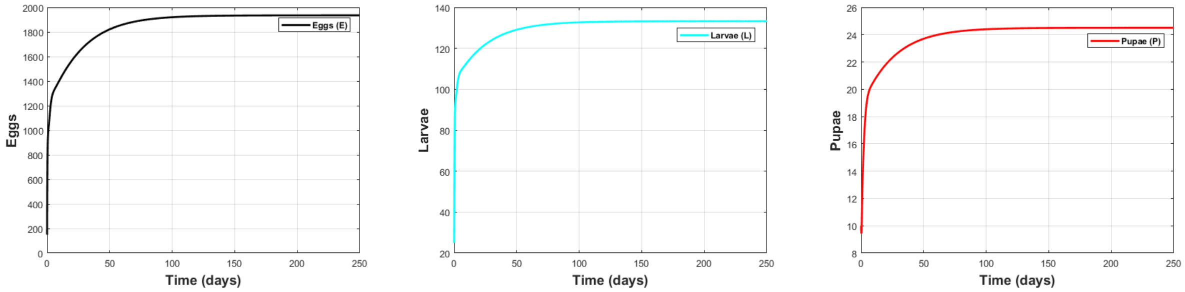

Figure 3: Distribution of population of eggs, larvae and pupae when $r_a \simeq 7.2291$, $r_g \simeq 0.6261 > \dfrac{1}{r_a + 1}$ and $\mathcal{R}_v \simeq 12.1037 > 1$.

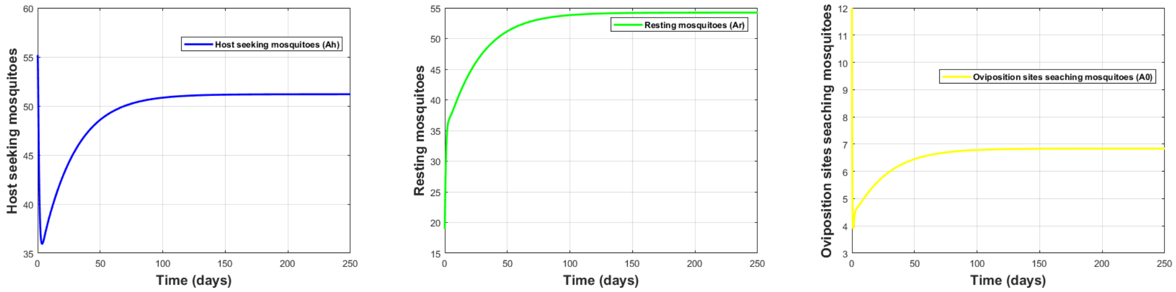

Figure 4: Distribution of A_h, A_r and A_0 mosquito population when $r_a \simeq 7.2291$, $r_g \simeq 0.6261 > \dfrac{1}{r_a + 1}$ and $\mathcal{R}_v \simeq 12.1037 > 1$.

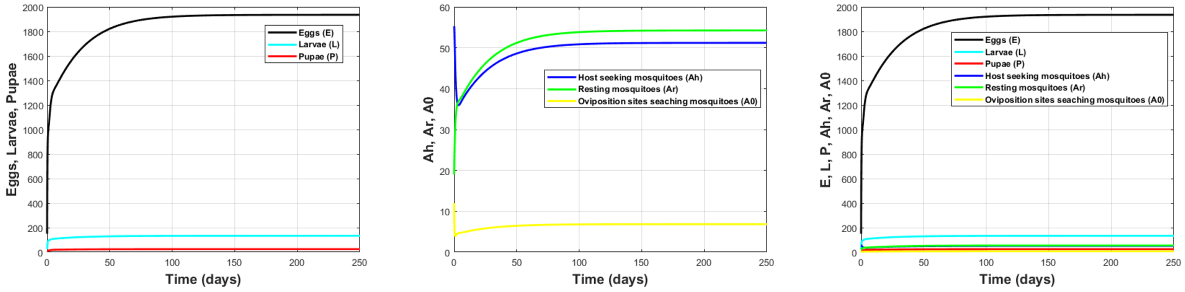

Figure 5: Combined curves of different populations for $r_a \simeq 7.2291$, $r_g \simeq 0.6261 > \dfrac{1}{r_a + 1}$ and $\mathcal{R}_v \simeq 12.1037 > 1$.

These curves describe the evolution of the different classes of the female mosquito population for $r_a \simeq 7.2291$, $r_g \simeq 0.6261 > \dfrac{1}{r_a + 1}$ and $\mathcal{R}_v \simeq 12.1037 > 1$. We observe a persistence of the population of the different classes and the solution $X = (E, L, P, A_h, A_r, A_0)$ of system (1) converges towards the equilibrium point of persistence of mosquitoes $X_1^\star = (1981, 133, 24, 50, 53, 7)$, which is globally asymptotically stable, which confirms the result obtained in Theorem 2.5.

4.1.2 Dynamics of vector model for $\mathcal{R}_v < 1$

By using the initial conditions given previously and the values of parameters for the case of extinction stated in Table 5, we obtained the curves of Figures 6, 7 and 8.

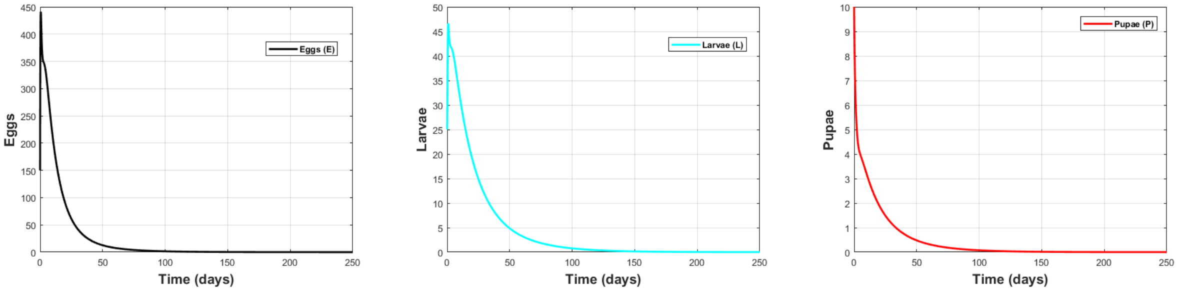

Figure 6: Distribution of population of eggs, larvae and pupae when $r_a \simeq 0.6871,\ r_g \simeq 0.4731 < \dfrac{1}{r_a + 1}$ and $\mathcal{R}_v \simeq 0.6171 < 1$.

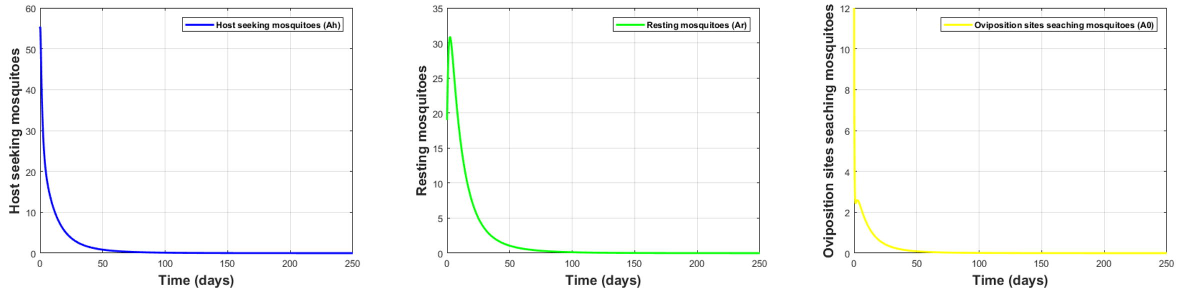

Figure 7: Distribution of A_h, A_r and A_0 mosquito population when $r_a \simeq 0.6871,\ r_g \simeq 0.4731 < \dfrac{1}{r_a + 1}$ and $\mathcal{R}_v \simeq 0.6171 < 1$.

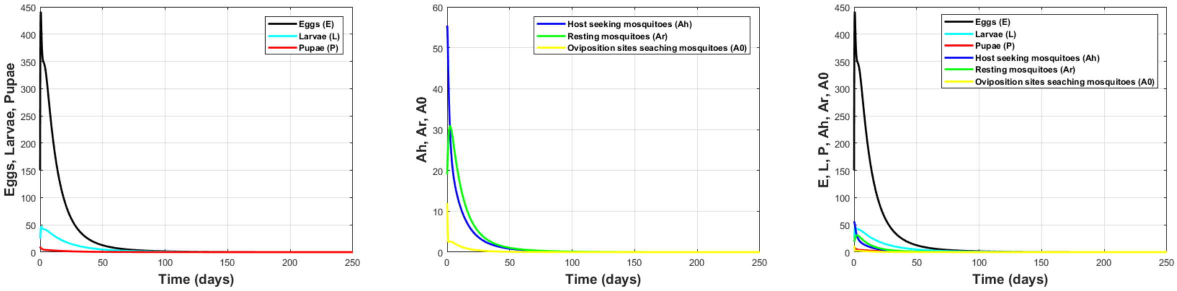

Figure 8: Combined curves of different populations for $r_a \simeq 0.6871,\ r_g \simeq 0.4731 < \dfrac{1}{r_a + 1}$ and $\mathcal{R}_v \simeq 0.6171 < 1$.

The curves obtained here describe the evolution of the different classes of the female mosquito population for $r_a \simeq 0.6871,\ r_g \simeq 0.4731 < \dfrac{1}{r_a + 1}$ and $\mathcal{R}_v \simeq 0.6171 < 1$. We observe an extinction of the different populations of each class from a certain time. In addition, we note that the equilibrium point $X_0^\star = (0, 0, 0, 0, 0, 0)$ is globally asymptotically stable, which confirms the result obtained in Theorem 2.4.

Remark 4.1. *We note that the threshold $\mathcal{R}_v$ has a considerable influence on the evolution of the mosquito population and this, at each stage of their evolution. In fact, the more this threshold increases, the more the number of eggs, larvae and pupae increases, which leads to the growth of adult mosquito populations A_h, A_r and A_0. Moreover, if this parameter is less than or equal to one, then the mosquito population converges towards a state of extinction, while if it is strictly greater than one, there is a proliferation of the vector population. So, it is a good control parameter of vector density. It will then be a question of finding suitable control methods which will minimize this threshold as much as possible in order to reduce the population of adult mosquitoes or to eradicate them. Some of these methods would then aim at reducing the parameter r_g, which would lead at the same time to reduce the threshold parameter R_v.*

4.2 Numerical simulations of the transmission dynamics model

Here we are interested in the numerical simulation of our transmission model under the influence of the mosquito model through the threshold $\mathcal{R}_v$. In other words, our objective is to obtain the curves of the evolution of the disease in the human population in the case of persistence of the female mosquito populations.

4.2.1 Dynamics of the transmission model for $\mathcal{R}_v > 1$ and $\mathcal{R}_0 < 1$

The numerical values of the parameters that we use in this part are those of persistence in Table 5, except μ_{L_2} which we replace by 0.0005 and those of the following Table 6 :

Table 6: Numerical values of other parameters in the case of extinction of the disease.

Parameters	Values	Sources	Units
β_{VH}	0.27	[20, 21]	/day
β_{HV}	0.15	[20, 35]	/day
Λ_H	900	[20, 21]	humans/day
μ_H	0.025	[20, 34]	/day
γ_H	0.42	[20, 21]	/day

The initial conditions are : $E(0) = 150$, $L(0) = 25$, $P(0) = 10$, $A_{hs}(0) = 55$, $A_{rs}(0) = 19$, $A_{0s}(0) = 12$, $A_{ri}(0) = 20$, $A_{0i}(0) = 10$, $I_v(0) = 15$, $S_H(0) = 50$, $I_H(0) = 10$ and $R_H(0) = 45$.
We thus obtain the curves of the following figures : Figures 9, 10, 11 and 12.

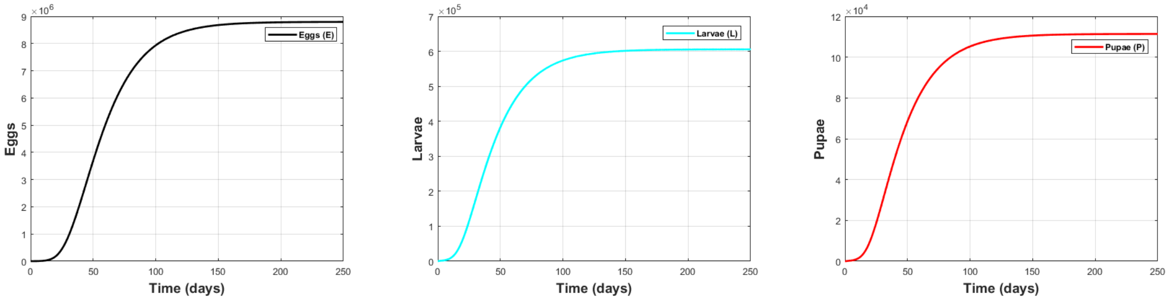

Figure 9: Distribution of population of eggs, larvae and pupae when $\mathcal{R}_v \simeq 12.1037 > 1$ and $\mathcal{R}_0 \simeq 0.2170 < 1$.

In these figures, we observe a growth until stabilization of the population of eggs, larvae, pupae,

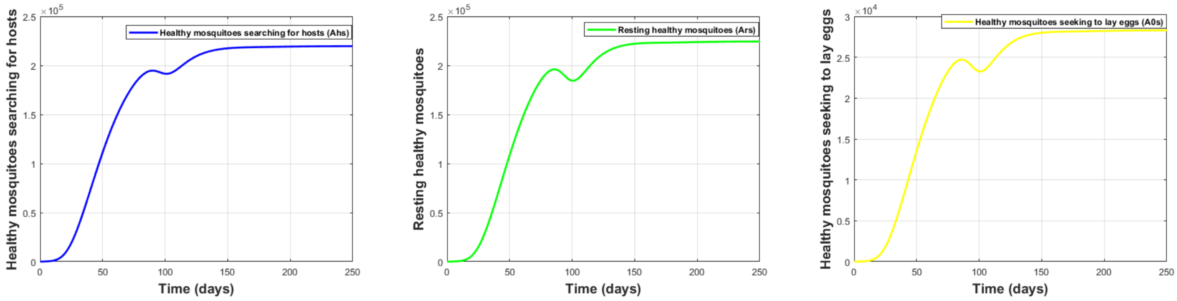

Figure 10: Distribution of A_{hs}, A_{rs} and A_{0s} mosquito population when $\mathcal{R}_v \simeq 12.1037 > 1$ and $\mathcal{R}_0 \simeq 0.2170 < 1$.

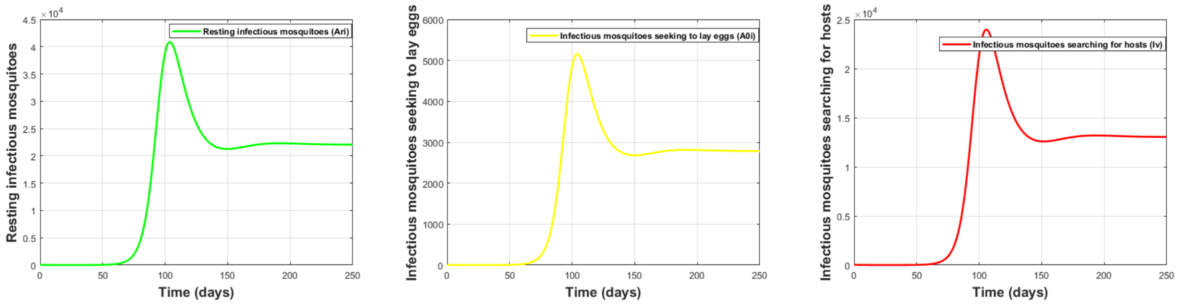

Figure 11: Distribution of A_{ri}, A_{0i} and I_V mosquito population when $\mathcal{R}_v \simeq 12.1037 > 1$ and $\mathcal{R}_0 \simeq 0.2170 < 1$.

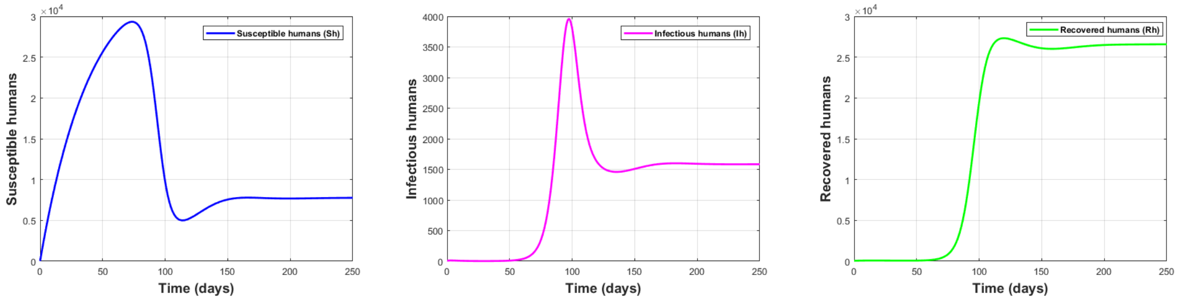

Figure 12: Distribution of the population of susceptible humans, infectious humans and recovered humans when $\mathcal{R}_v \simeq 12.1037 > 1$ and $\mathcal{R}_0 \simeq 0.2170 < 1$.

A_{hs}, A_{rs} and A_{0s} mosquitoes and susceptible humans S_H while that of A_{ri}, A_{0i} and I_V mosquitoes, infectious humans I_H and recovered humans R_H decreases until canceled. The disease therefore disappears after a certain time. The parameters used allow us to obtain $\mathcal{R}_v \simeq 12.1037 > 1$ and $\mathcal{R}_0 \simeq 0.2170 < 1$ and the solution of transmission model (7) converges towards the disease-free equilibrium $X_1^\star = (2091509, 14404, 2649, 5534, 5861, 739, 0, 0, 0, 36000, 0, 0)$ which is globally asymptotically stable. This result clearly illustrates our theoretical result obtained in Theorem 3.7.

4.2.2 Dynamics of the transmission model for $\mathcal{R}_v > 1$ and $\mathcal{R}_0 > 1$

The initial conditions in this part are : $E(0) = 150$, $L(0) = 25$, $P(0) = 10$, $A_{hs}(0) = 55$, $A_{rs}(0) = 19$, $A_{0s}(0) = 12$, $A_{ri}(0) = 40$, $A_{0i}(0) = 10$, $I_v(0) = 25$, $S_H(0) = 25$, $I_H(0) = 10$, $R_H(0) = 55$.
The numerical values of the parameters that we use in this part are those of persistence in Table 5, except μ_{L_2} which we replace by 0.000011 and those of the following table : Table 7.
Thus, we obtain the curves of Figures 13, 14, 15 and 16. On these figures obtained for $\mathcal{R}_v$ =

Table 7: Numerical values of other parameters of the model in the case of disease persistence.

Parameters	Value	Sources	Unity
β_{VH}	0.37	[20, 21]	/day
β_{HV}	0.25	[20, 35]	/day
Λ_H	900	[20, 21]	humans/day
μ_H	0.025	[20, 34]	/day
γ_H	0.42	[20, 21]	/day

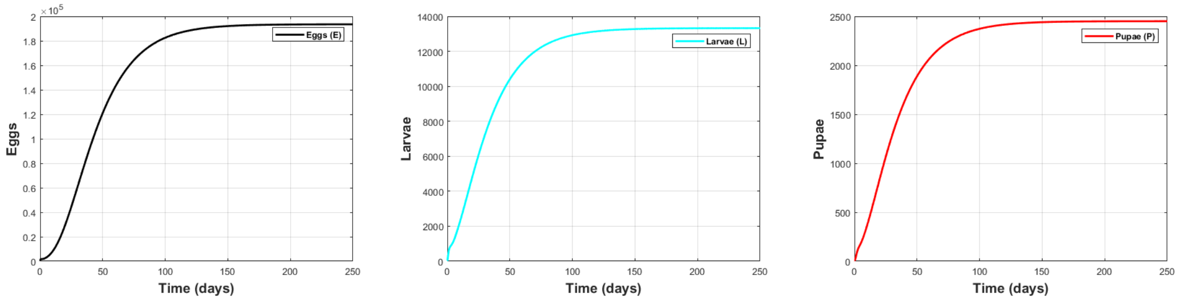

Figure 13: Distribution of population of eggs, larvae and pupae when $\mathcal{R}_v \simeq 12.1037 > 1$ and $\mathcal{R}_0 \simeq 2.2115 > 1$.

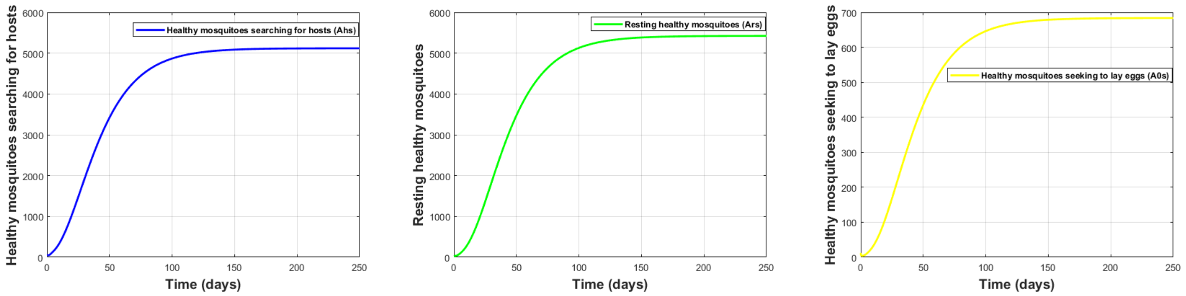

Figure 14: Distribution of A_{hs}, A_{rs} and A_{0s} mosquito population when $\mathcal{R}_v \simeq 12.1037 > 1$ and $\mathcal{R}_0 \simeq 2.2115 > 1$.

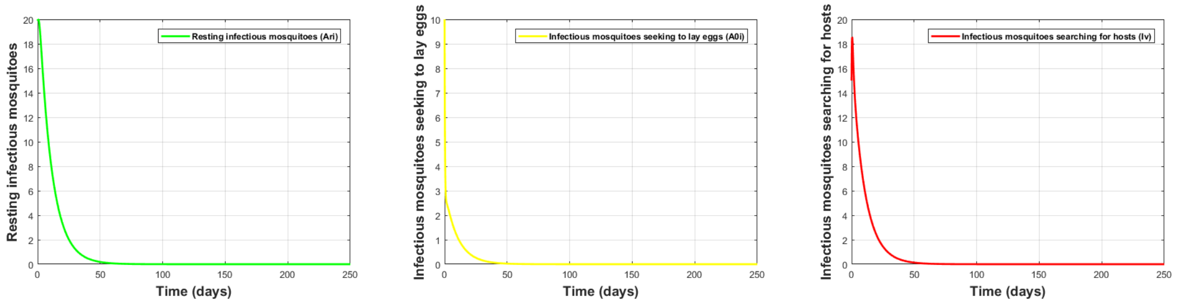

Figure 15: Distribution of A_{ri}, A_{0i} and I_V mosquito population when $\mathcal{R}_v \simeq 12.1037 > 1$ and $\mathcal{R}_0 \simeq 2.2115 > 1$.

$12.1037 > 1$ and $\mathcal{R}_0 \simeq 2,2115 > 1$, we observe a growth until complete stabilization of the populations of different classes of our transmission model (7). We are indeed noticing a persistence of mosquito and human populations with persistence of the disease. The solution $(E, L, P, A_{hs}, A_{rs}, A_{0s}, A_{ri}, A_{0i}, I_V, S_H, I_H, R_H)$ of this transmission model (7) converges globally towards endemic equilibrium $X_2^\star = (87175472, 600184, 110379, 217745, 222488$
$, 28056, 21773, 2746, 12870, 7876, 1580, 26544)$, which illustrates the result obtained in Theorem 3.9.

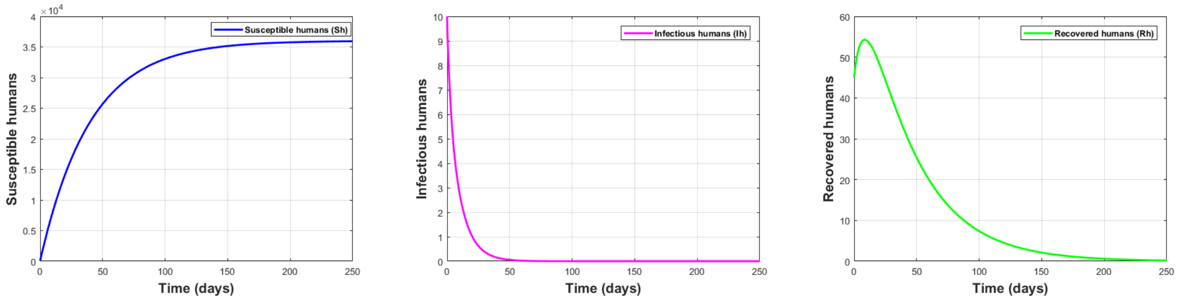

Figure 16: Distribution of the population of susceptible humans, infectious humans and recovered humans when $\mathcal{R}_v \simeq 12.1037 > 1$ and $\mathcal{R}_0 \simeq 2.2115 > 1$.

Remark 4.2. *We retain from these results that the behavior of the disease in the different populations is governed by the threshold parameter $\mathcal{R}_0$. Indeed, when this is less than or equal to one, the disease disappears and when it is strictly greater than one, there is persistence of the disease.Thus, the control methods against vector-borne diseases due to mosquitoes must aim at influencing the parameters related to this rate in order to minimize it. One of the approaches must then consist in reducing to the maximum by suitable methods of control the rate of passage of the gonotrophic cycle r_g in order to reduce at the same time the rate $\mathcal{R}_v$, then the rate $\mathcal{R}_0$. We can therefore affirm that the gonotrophic cycle has a great influence on the transmission mechanism of vector-borne diseases and it is therefore very important to give an interest to it in the context of the fight against these epidemics.*

5 Conclusion

In this work, we were interested in the modeling and the mathematical analysis of the dynamics of vector-borne diseases transmitted by mosquitoes, which are endemic diseases that plague the world's poverty belt, particularly in Africa. We started with the presentation of a model describing the evolution of the mosquito population, in particular the aquatic phase and the gonotrophic cycle due to A. M. Lutambi et al. [23]. After this step, we presented the mathematical results they obtained and our mathematical contribution to this model. Indeed, this mathematical analysis revealed the existence of a threshold parameter $\mathcal{R}_v$ which makes it possible to predict the evolution of the vector density in the area. Indeed, the extinction of the mosquito population is governed by the condition $\mathcal{R}_v < 1$ while mosquitoes invade the area when $\mathcal{R}_v > 1$. This first study highlighted the need to take into account the immature stages and the gonotrophic cycle in strategies to fight against these diseases because they constitute the manufacturing plant of anopheles and therefore one of the sources of the problem.

Then, in order to know the impact of the stages constitutive of the mosquito's life on the transmission dynamics of the vectorial disease, we developed a mathematical model of the transmission dynamics of the disease by coupling the two models previously studied. The mathematical study of this model revealed the existence of another threshold parameter $\mathcal{R}_0$ which, when it is greater than one ($\mathcal{R}_0 > 1$), the disease persists giving an endemic equilibrium point with mosquitoes which is generally stable and when it is less than one ($\mathcal{R}_0 < 1$), the disease disappears giving a disease-free equilibrium point with an overall stable mosquitoes population. We were also able to establish a relationship between $\mathcal{R}_v$ and $\mathcal{R}_0$ which shows that the spread of the disease is governed by these two threshold parameters. The fundamental result of this study is the following: *the vector density regulation threshold is a very important parameter that can intervene in control strategies for vector-borne diseases transmitted by mosquitoes because it influences considerably the basic reproduction rate*. These results therefore demonstrate the great need to consider the stages that make up the aquatic stage and the gonotrophic cycle of mosquitoes in plans for the eradication of vector-borne diseases.

The last stage of our work consisted in presenting some results of numerical simulations, in order to illustrate our theoretical results. The observations are that these results are in conformity with our previously established theoretical results.

In our future work, we would like to take into account control strategies in these two studied models in order to highlight the most suitable control strategy to the fight against mosquitoes borne diseases, taking into account the different classes of adult mosquitoes.

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