

Research Article

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Mark O. Gluzman, Nataliia V. Gorban, and Pavlo O. Kasyanov*

Lyapunov Functions for Weak Solutions of Reaction-Diffusion Equations with Discontinuous Interaction Functions and its Applications

Abstract: In this paper we investigate additional regularity properties for global and trajectory attractors of all globally defined weak solutions of semi-linear parabolic differential reaction-diffusion equations with discontinuous nonlinearities, when initial data $u_\tau \in L^2(\Omega)$. The main contributions in this paper are: (i) sufficient conditions for the existence of a Lyapunov function for all weak solutions of autonomous differential reaction-diffusion equations with discontinuous and multivalued interaction functions; (ii) convergence results for all weak solutions in the strongest topologies; (iii) new structure and regularity properties for global and trajectory attractors. The obtained results allow investigating the long-time behavior of state functions for the following problems: (a) a model of combustion in porous media; (b) a model of conduction of electrical impulses in nerve axons; (c) a climate energy balance model; (d) a parabolic feedback control problem.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded and open subset with a smooth boundary $\partial\Omega$, $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function, $f_1, f_2 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are some real measurable functions such that $f_i(x, \cdot)$ is convex for a.e. $x \in \Omega$, $i = 1, 2$. We denote by $\partial f_i(x, u)$ the subdifferential of a function $f_i(x, \cdot)$ at a point u for a.e. $x \in \Omega$, for each $u \in \mathbb{R}$, $i = 1, 2$. Note that $u^* \in \partial f_i(x, u)$ if and only if $u^*(v - u) \leq f_i(x, v) - f_i(x, u) \forall v \in \mathbb{R}$.

We consider the semilinear reaction-diffusion equation with discontinuous/multivaluated nonlinearity:

$$u_t - \Delta u + \partial f_1(x, u) - \partial f_2(x, u) \ni g(x, t) \text{ in } \Omega \times (\tau, T), \quad (-\infty < \tau < T < +\infty), \quad (1.1)$$

with Dirichlet (or Neumann) boundary condition

$$u|_{\partial\Omega} = 0 \quad \left(\text{or } \frac{\partial u}{\partial \bar{n}}|_{\partial\Omega} = 0 \right). \quad (1.2)$$

In many applications in the climatology and Earth sciences (a climate energy balance model; see, for example, Díaz et al. [11–13]), biology and medicine (a model of conduction of electrical impulses in nerve axons;

***Corresponding Author: Pavlo O. Kasyanov:** Institute for Applied System Analysis, National Technical University of Ukraine “Kyiv Polytechnic Institute”, Peremogy ave., 37, build, 35, 03056, Kyiv, Ukraine, E-mail: kasyanov@i.ua

Mark O. Gluzman: Institute for Applied System Analysis, National Technical University of Ukraine “Kyiv Polytechnic Institute”, Peremogy ave., 37, build, 35, 03056, Kyiv, Ukraine, E-mail: gluzmanmark@gmail.com

Nataliia V. Gorban: Institute for Applied System Analysis, National Technical University of Ukraine “Kyiv Polytechnic Institute”, Peremogy ave., 37, build, 35, 03056, Kyiv, Ukraine, E-mail: nata_gorban@i.ua

see, for example, Terman [40, 41]), evolution models of mechanics (a model of combustion in porous media; see, for example, Feireisl and Norbury [15]) etc, if the exterior force g satisfies the following stabilization like assumption: there exists $g_\infty : \Omega \rightarrow \mathbb{R}$ such that all the integrals in the following formula exist and

$$\int_t^{t+1} \int_\Omega |g(x, t) - g_\infty(x)|^2 dx dt \rightarrow 0, \quad t \rightarrow +\infty, \quad (1.3)$$

then the long-time behaviour of solutions of problem (1.1)–(1.2) is described by the respective autonomous problem with the exterior force g_∞ ; see Remark 1.5 below and Diaz et al [12] (see also Ball [4, Section 6]). Of course even if all solutions on the non-autonomous problem converge to the same attracting set of the respective autonomous problem, the dynamical properties in the non-autonomous setting are inherently different from the autonomous one (such as for example the concept of invariance). Therefore, we focus our efforts in the direction of problem (1.1)–(1.2) with the exterior force g_∞ . Since the interaction functions ∂f_1 and ∂f_2 depend on x , further, to simplify the conclusions, we assume that $g_\infty \equiv 0$.

The main purpose of this paper are : (i) to prove the existence of a Lyapunov type function and justify energy equalities for all weak solutions of problem (1.1)–(1.2) in the autonomous case (see Theorem 2.2); and (ii) to investigate the long-time behavior of all weak solutions of problem (1.1)–(1.2), as time $t \rightarrow +\infty$, in strongest topologies (see Theorem 3.5).

We note that a large class of important models for distributed parameters control problems are also included in the formulation (1.1)–(1.2). In this sense, the set $\partial f_1(x, u) - \partial f_2(x, u)$ can be considered as an admissible control set. The obtained results are applied to the following problems: (i) stabilization of a parabolic feedback control problem; (ii) a model of combustion in porous media; (iii) a model of conduction of electrical impulses in nerve axons; and (iv) a climate energy balance model; see Section 5.

We shall use the following standard notations: $H = L^2(\Omega)$, $V = H_0^1(\Omega)$ for homogeneous Dirichlet boundary conditions ($V = H^1(\Omega)$ for homogeneous Neumann boundary conditions), V' is the dual space of V .

Suppose that the following assumptions hold:

Assumption 1.1. (Growth condition) *there exist $c_0 \in L^1(\Omega)$, $c_0(x) \geq 0$ for a.e. $x \in \Omega$, and $c_1 \geq 0$ such that $|u_i^*|^2 \leq c_0(x) + c_1|u|^2$ for a.e. $x \in \Omega$, for each $u \in \mathbb{R}$, and $u_i^* \in \partial f_i(x, u)$, $i = 1, 2$;*

Assumption 1.2. (Sign condition) *there exists $\lambda < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, and $c_2 \in L^1(\Omega)$, $c_2(x) \geq 0$ for a.e. $x \in \Omega$, such that $(u_1^* - u_2^*)u \geq -\lambda u^2 - c_2(x)$ for a.e. $x \in \Omega$, for each $u \in \mathbb{R}$, and $u_i^* \in \partial f_i(x, u)$, $i = 1, 2$.*

Definition 1.3. *Let $-\infty < \tau < T < +\infty$. The function $u(\cdot) \in L^2(\tau, T; V)$ is called a weak solution of problem (1.1)–(1.2) on $[\tau, T]$, if there exists a measurable function $d : \Omega \times (\tau, T) \rightarrow \mathbb{R}$ such that*

$$d(x, t) \in \partial f_1(x, u(x, t)) - \partial f_2(x, u(x, t)) \quad \text{for a.e. } (x, t) \in \Omega \times (\tau, T); \quad (1.4)$$

and

$$-\int_\tau^T \left\langle u, \frac{\partial \xi}{\partial t} \right\rangle dt + \int_\tau^T \int_\Omega (\nabla u, \nabla \xi) dx dt + \int_\tau^T \int_\Omega (d, \xi) dx dt = \int_\tau^T \int_\Omega (g, \xi), \quad (1.5)$$

for all $\xi \in C_0^\infty(\Omega \times (\tau, T))$, where $\langle \cdot, \cdot \rangle$ denotes the pairing in space V .

Remark 1.4. Let Assumptions 1.1 and 1.2 hold. Let $-\infty < \tau < T < +\infty$. Then for each weak solution $u(\cdot)$ of problem (1.1)–(1.2) on $[\tau, T]$ there exist measurable functions $d_1, d_2 : \Omega \times (\tau, T) \rightarrow \mathbb{R}$ such that $d_i(x, t) \in \partial f_i(x, u(x, t))$ for a.e. $(x, t) \in \Omega \times (\tau, T)$, $i = 1, 2$; and $d(x, t) = d_1(x, t) - d_2(x, t)$ for a.e. $(x, t) \in \Omega \times (\tau, T)$. Indeed, according to Definition 1.3 there exists a measurable function $d : \Omega \times (\tau, T) \rightarrow \mathbb{R}$ satisfying (1.4) and (1.5). Assumption 1.1 implies that $d(\cdot) \in L^2(\tau, T; H)$. Moreover, $d(\cdot) \in \partial J_1(u(\cdot)) - \partial J_2(u(\cdot))$, where $J_i(v(\cdot)) = \int_\tau^T \int_\Omega f_i(x, v(x, t)) dx dt$, $v(\cdot) \in L^2(\tau, T; H)$, $i = 1, 2$, because $\partial J_i(u(\cdot)) = \{p(\cdot) \in L^2(\tau, T; H) : p(x, t) \in \partial f_i(x, u(x, t)) \text{ for a.e. } (x, t) \in \Omega \times (\tau, T)\}$, $i = 1, 2$; see, for example, Aubin-Clarke Theorem [10, Theorem 2.7.5,

p. 83] (see also Clarke [10, Proposition 2.3.6, p. 40]). Thus, there exist measurable functions $d_1, d_2 : \Omega \times (\tau, T) \rightarrow \mathbb{R}$ such that $d_i(x, t) \in \partial f_i(x, u(x, t))$ for a.e. $(x, t) \in \Omega \times (\tau, T)$, $i = 1, 2$; and $d(x, t) = d_1(x, t) - d_2(x, t)$ for a.e. $(x, t) \in \Omega \times (\tau, T)$.

Remark 1.5. Let $-\infty < \tau < T < +\infty$ and $g \in L^2(\tau, T; H)$. We note that each weak solution of problem (1.1)–(1.2) on $[\tau, T]$ is *regular*, that is, if $u(\cdot)$ is a weak solution of problem (1.1)–(1.2) on $[\tau, T]$, then $u(\cdot) \in C([\tau + \varepsilon, T]; V) \cap L^2(\tau + \varepsilon, T; H^2(\Omega) \cap V)$ and $u_t(\cdot) \in L^2(\tau + \varepsilon, T; H)$, for each $\varepsilon \in (0, T - \tau)$; cf. Kasyanov et al. [28, Theorem 1]. Moreover, each weak solution of problem (1.1)–(1.2) can be extended to a global one defined on $[0, +\infty)$; see Zgurovsky et al. [46, p. 62].

Sufficient conditions for the existence of a Lyapunov function for autonomous evolution inclusions of hyperbolic type were considered by Kasyanov et al. [31, 45, 48]. Arrieta et al. [1] constructed a Lyapunov function for the equation $u_t - u_{xx} = \omega u + H(u)$, where $0 \leq \omega \leq \pi^2$, H is a Heaviside function: $H(0) = [-1, 1]$, $H(s) = 1$ for $s > 0$, $H(s) = -1$ for $s < 0$. We remark that the existence of a Lyapunov function for a class of parabolic feedback control problems and its applications were already announced in Gluzman et al. [17].

The global attractors for such kind of systems were at first proved in Valero [42]. Kalita and Łukaszewicz [23, 24] also imply the existence of the global attractor for the problems under consideration. Regularity properties of global and trajectory attractors were provided by Gorban et al. [19–21, 26, 28].

In this article we also provide a Lyapunov function and the strongest convergence results for quasilinear parabolic PDEs with discontinuous and/or multivalued nonlinearities in a general setting.

2 A Lyapunov Function of All Weak Solutions and its Application to the Convergence

Let $g \equiv 0$. Let us consider the family $\mathcal{K}_+$ of all weak solutions of problem (1.1)–(1.2) defined on the semi-infinite interval $[0, +\infty)$. We note that $\mathcal{K}_+$ is *translation invariant*, that is $u(\cdot + h) \in \mathcal{K}_+$ for each $u(\cdot) \in \mathcal{K}_+$ and for each $h \geq 0$.

Let us consider problem (1.1)–(1.2) on the entire time axis. A function $u \in L^\infty(\mathbb{R}; H)$ is called a *complete trajectory* of problem (1.1)–(1.2), if $\Pi_+ u_h(\cdot) \in \mathcal{K}_+$ for each $h \in \mathbb{R}$, where Π_+ is the restriction operator to the interval $[0, +\infty)$ and $u_h(s) = u(s + h)$, $s \geq 0$.

Let $\mathcal{K}$ be the family of all complete trajectories of problem (1.1)–(1.2). According to Remark 1.5 and Theorem 2.3 below, each complete trajectory $u(\cdot)$ of problem (1.1)–(1.2) satisfies the following: $\Pi_{\tau, T} u(\cdot) \in C_{loc}([\tau, T]; V) \cap L^2(\tau, T; H^2(\Omega) \cap V)$ and $\Pi_{\tau, T} u_t(\cdot) \in L^2(\tau, T; H)$ for each $-\infty < \tau < T < +\infty$, where $\Pi_{\tau, T}$ is the restriction operator to the interval $[\tau, T]$; Chepyzhov and Vishik [7, p. 18]. Moreover, there exists $\tilde{C} > 0$ such that for each $u(\cdot) \in \mathcal{K}$ the following estimate holds:

$$\|u(t)\|_V^2 \leq \tilde{C}(1 + \|u(t - 1)\|_H^2) \text{ for each } t \in \mathbb{R}.$$

Therefore, in particular, each bounded (in H) complete trajectory is bounded in V .

A complete trajectory $u(\cdot) \in \mathcal{K}$ is *stationary* if there is $z \in H^2(\Omega) \cap V$ such that $u(t) = z$ for all $t \in \mathbb{R}$. Each such z is called a *rest point*. We denote the set of all rest points by Z .

Definition 2.1. $E : V \rightarrow \mathbb{R}$ is a *Lyapunov function* for $\mathcal{K}_+$, if the following conditions hold: (a) E is continuous on V ; (b) $E(u(t)) \leq E(u(s))$ whenever $u \in \mathcal{K}_+$ and $t \geq s > 0$; (c) if $E(u(\cdot)) \equiv \text{const}$, for some $u \in \mathcal{K}$, then u is stationary.

Let us set

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u(x)|^2 dx + J_1(u) - J_2(u), \quad u \in V, \quad (2.1)$$

where $J_i(u) = \int_{\Omega} f_i(x, u(x)) dx$, $u \in H$, $i = 1, 2$.

Assumption 1.1 implies that there exist $c_3 \in L^1(\Omega)$, $c_3(x) \geq 0$ for a.e. $x \in \Omega$, and $c_4 \geq 0$ such that $|f_i(x, u)| \leq c_3(x) + c_4|u|^2$ for a.e. $x \in \Omega$, and for each $u \in \mathbb{R}$, $i = 1, 2$. Therefore, the functions $J_i(u) = \int_{\Omega} f_i(x, u(x))dx$, $u \in H$, $i = 1, 2$, are correctly defined.

Theorem 2.2. *Let $g \equiv 0$. Let Assumptions 1.1–1.2 hold. Then, the function $E : V \rightarrow \mathbb{R}$, defined in (2.1), is a Lyapunov function for $\mathcal{K}_+$. Moreover, for each $u \in \mathcal{K}_+$ and all τ and T , $0 < \tau < T < \infty$, the energy equality holds*

$$E(u(T)) - E(u(\tau)) = - \int_{\tau}^T \|u_t(s)\|_H^2 ds. \quad (2.2)$$

Proof. By the definition the function E is continuous on V , that is statement (a) of Definition 2.1 holds.

Let us prove statement (b) of Definition 2.1. Let $u(\cdot) \in \mathcal{K}_+$ be arbitrary and fixed and let $0 < \tau < T < \infty$. We denote the restriction of $u(\cdot)$ on $[\tau, T]$ by the same symbol $u(\cdot)$. Note that $u(\cdot) \in C([\tau, T]; V) \cap L^2(\tau, T; H^2(\Omega) \cap V)$ and $u_t(\cdot) \in L^2(\tau, T; H)$, because $\tau > 0$; cf. Kasyanov et al. [28, Theorem 1]. Then the mapping $t \rightarrow \|u(t)\|_V^2 = \int_{\Omega} |\nabla u(x, t)|^2 dx$ is absolutely continuous on $[\tau, T]$ and for a.e. $t \in (\tau, T)$ the equality holds:

$$\frac{d}{dt} \|u(t)\|_V^2 = -2 \int_{\Omega} \frac{\partial u(x, t)}{\partial t} \Delta u(x, t) dx; \quad (2.3)$$

Gajewski et al. [16, Chapter IV].

Let $d : \Omega \times (\tau, T) \rightarrow \mathbb{R}$ be the function from (1.4)–(1.5) and $d_1, d_2 \in L^2(\tau, T; H)$ be the functions from Remark 1.4. Barbu [6, Lemma 2.1, p. 189] yields that the function $J_i(u(\cdot))$ is absolutely continuous on $[\tau, T]$ and for a.e. $t \in (\tau, T)$ the following equality holds:

$$\frac{d}{dt} J_i(u(t)) = \int_{\Omega} h_i(x, t) \frac{\partial u(x, t)}{\partial t} dx, \quad (2.4)$$

for all measurable h_i , $h_i(\cdot, t) \in \partial J_i(s)|_{s=u(t)}$ for a.e. $t \in (\tau, T)$, $i = 1, 2$.

Thus, the function $E(u(\cdot))$ is absolutely continuous on $[\tau, T]$ as the linear combination of absolutely continuous on $[\tau, T]$ functions. According to formulae (2.3) and (2.4), $\frac{d}{dt} E(u(t)) = -\|u_t(t)\|_H^2$ for a.e. $t \in (\tau, T)$. The last statement implies (2.2). In particular, $E(u(t)) \leq E(u(s))$ whenever $T \geq t \geq s \geq \tau > 0$. Since $u(\cdot) \in \mathcal{K}_+$ and $0 < \tau < T < \infty$ are arbitrary, statement (b) of Definition 2.1 and the energy equality (2.2) hold.

To finish the proof we note that if $E(u(\cdot)) \equiv \text{const}$, for some $u \in \mathcal{K}$, then, according to energy equality (2.2), u is stationary. $\square$

Zgurovsky et al. [46, p. 56] and Kasyanov et al. [28, p.274] proved that for $\tau < T$ and for each weak solution $u(\cdot)$ of problem (1.1)–(1.2) on $[\tau, T]$ the following inequality holds:

$$\|u(t)\|_H^2 \leq \|u(s)\|_H^2 e^{-2\varepsilon^*(t-s)} + \frac{a}{\varepsilon^*} \quad \forall \tau \leq s \leq t \leq T, \quad (2.5)$$

where $\varepsilon^* = \lambda_1 - \lambda$ and $a = \int_{\Omega} c_2(x) dx$.

Further on $V \cap H^2(\Omega)$ we define the equivalent norm $v \rightarrow \|\Delta v\|_H$; Temam [39, Chapter III].

Before the proof of convergence results for all weak solutions in the strongest topologies, we need to provide some additional estimates for weak solutions of problem (1.1)–(1.2).

Theorem 2.3. *Let $g \equiv 0$. Let Assumptions 1.1–1.2 hold. Then, there exists $C > 0$ such that for any $\tau < T$ and for each weak solution $u(\cdot)$ of problem (1.1)–(1.2) on $[\tau, T]$ the inequality holds*

$$(t - \tau) \|u(t)\|_V^2 + \int_{\tau}^t (s - \tau) \|u(s)\|_{H^2(\Omega) \cap V}^2 ds \leq C(1 + \|u(\tau)\|_H^2 + (t - \tau)^2) \quad \forall t \in (\tau, T).$$

Remark 2.4. The proof of Theorem 2.3 is similar to the proof of Theorem 2 from Kasyanov et al. [28], however it was proved under another assumptions on the interaction function.

Proof. Let $\tau < T$ and $u(\cdot)$ be an arbitrary weak solution of problem (1.1)–(1.2) on $[\tau, T]$. We fix $\varepsilon \in (0, T - \tau)$. Since $u(\cdot) \in C([\tau + \varepsilon, T]; V) \cap L^2(\tau + \varepsilon, T; H^2(\Omega) \cap V)$ and $u_t \in L^2(\tau + \varepsilon, T; H)$, then $\|u(\cdot)\|_V^2$ and $\|u(\cdot)\|_H^2$ are absolutely continuous on $[\tau + \varepsilon, T]$ and the following equalities hold: $\frac{d}{dt}\|u(t)\|_V^2 = -2 \int_{\Omega} \frac{\partial u(x, t)}{\partial t} \Delta u(x, t) dx$ and $\frac{d}{dt}\|u(t)\|_H^2 = 2 \int_{\Omega} \frac{\partial u(x, t)}{\partial t} u(x, t) dx$, for a.e. $s \in (\tau + \varepsilon, T)$; Gajewski et al. [16, Chapter IV]. Let $c^* = \int_{\Omega} c_0(x) dx$ and $c_{\max} = 2 \max\{c^*, c_1\}$, where $c_0(x)$ and c_1 are parameters from Assumption 1.1.

Assumptions 1.1, 1.2 imply that the following inequalities hold

$$\begin{aligned} & \frac{d}{ds} \left[(s - \tau - \varepsilon) \|u(s)\|_V^2 + \frac{1}{2} \|u(s)\|_H^2 \right] + (s - \tau - \varepsilon) \|u(s)\|_{H^2(\Omega) \cap V}^2 \\ & \leq \|u(s)\|_H^2 \left(2c_1 + \frac{1}{2} + 4c_1(s - \tau - \varepsilon) \right) + (2c^* + 4c^*(s - \tau - \varepsilon)) \\ & \leq \left(c_{\max} + \frac{1}{2} + 2c_{\max}(s - \tau - \varepsilon) \right) (\|u(s)\|_H^2 + 1) \\ & \leq \left(c_{\max} + \frac{1}{2} + 2c_{\max}(s - \tau - \varepsilon) \right) \left(\|u(\tau)\|_H^2 e^{-2\varepsilon^*(s-\tau)} + \frac{a}{\varepsilon^*} + 1 \right), \end{aligned} \quad (2.6)$$

for a.e. $s \in (\tau + \varepsilon, T)$, where the last inequality follows from (2.5). The inequality (2.6) and Kasyanov et al. [28, p. 275] yield

$$\|u(t)\|_V^2(t - \tau) + \int_{\tau}^t (s - \tau) \|u(s)\|_{H^2(\Omega) \cap V}^2 ds \leq C((t - \tau)^2 + \|u(\tau)\|_H^2 + 1), \quad \forall t \in (\tau, T]$$

where $C > 0$ is a constant that does not depend on τ , T , ε , and $u(\cdot)$. □

For any $u_{\tau} \in H$ we set

$$\mathcal{D}_{\tau, T}(u_{\tau}) = \{u(\cdot) \in L^2(\tau, T; V) \mid u(\cdot) \text{ is a weak solution of problem (1.1)–(1.2) and } u(\tau) = u_{\tau}\}.$$

The main convergence result for all weak solutions of problem (1.1)–(1.2) in the strongest topologies has the following formulation (see also Example 4.1).

Theorem 2.5. *Let $g \equiv 0$. Let Assumptions 1.1–1.2 hold, $\tau < T$, $u_{\tau, n} \rightarrow u_{\tau}$ weakly in H , $u_n(\cdot) \in \mathcal{D}_{\tau, T}(u_{\tau, n})$ for each $n \geq 1$. Then there exists an increasing sequence $\{n_k\}_{k \geq 1}$ of natural numbers and $u(\cdot) \in \mathcal{D}_{\tau, T}(u_{\tau})$ such that*

$$\sup_{t \in [\tau + \varepsilon, T]} \|u_{n_k}(t) - u(t)\|_V \rightarrow 0, \quad (2.7)$$

$$\int_{\tau + \varepsilon}^T \|u_{n_k, t}(t) - u_t(t)\|_H^2 dt \rightarrow 0, \quad (2.8)$$

as $k \rightarrow +\infty$, for all $\varepsilon \in (0, T - \tau)$.

Proof. Theorem 2.3, Kasyanov et al. [28, Theorem 3], Banach-Alaoglu theorem, and Cantor diagonal arguments yield that there exist an increasing sequence $\{n_k\}_{k \geq 1}$ of natural numbers and $u(\cdot) \in \mathcal{D}_{\tau, T}(u_{\tau})$ such that the following statements hold: a) the restrictions of $u_{n_k}(\cdot)$ and $u(\cdot)$ on $[\tau + \varepsilon, T]$ belong to $C([\tau + \varepsilon, T]; V) \cap L^2(\tau + \varepsilon, T; H^2(\Omega) \cap V)$ and $u_{n_k, t}(\cdot), u_t(\cdot) \in L^2(\tau + \varepsilon, T; H)$; b) the following convergence hold:

$$\begin{aligned} & u_{n_k}(\cdot) \rightarrow u(\cdot) \text{ weakly in } L^2(\tau + \varepsilon, T; H^2(\Omega) \cap V), \\ & u_{n_k}(\cdot) \rightarrow u(\cdot) \text{ strongly in } C([\tau + \varepsilon, T]; V), \\ & u_{n_k, t}(\cdot) \rightarrow u_t(\cdot) \text{ weakly in } L^2(\tau + \varepsilon, T; H), \end{aligned} \quad (2.9)$$

as $k \rightarrow \infty$, for each $\varepsilon \in (0, T - \tau)$, that imply statement (2.7). Let us prove (2.8). Theorem 2.2 yields the following energy equalities

$$\int_{\tau + \varepsilon}^T \|u_t(t)\|_H^2 dt = E(u(\tau + \varepsilon)) - E(u(T)), \quad (2.10)$$

$$\int_{\tau+\varepsilon}^T \|u_{n_k,t}(t)\|_H^2 dt = E(u_{n_k}(\tau+\varepsilon)) - E(u_{n_k}(T)), \quad (2.11)$$

$k \geq 1$, $\varepsilon \in (0, T - \tau)$. The continuity of E on V and (2.7) imply

$$E(u_{n_k}(\tau+\varepsilon)) - E(u_{n_k}(T)) \rightarrow E(u(\tau+\varepsilon)) - E(u(T)), \quad m \rightarrow \infty. \quad (2.12)$$

Therefore, formulae (2.10)–(2.12) yield

$$\int_{\tau+\varepsilon}^T \|u_{n_k,t}(t)\|_H^2 dt \rightarrow \int_{\tau+\varepsilon}^T \|u_t(t)\|_H^2 dt, \quad (2.13)$$

as $k \rightarrow \infty$, for each $\varepsilon \in (0, T - \tau)$. Since, $L^2(\tau + \varepsilon; T)$ is a Hilbert space, (2.9) and (2.13) imply (2.8). $\square$

Define real Banach space $W(M_1, M_2) = \{u(\cdot) \in C([M_1, M_2]; V) : u_t(\cdot) \in L^2(M_1, M_2; H)\}$ with the norm $\|u(\cdot)\|_{W(M_1, M_2)} = \|u(\cdot)\|_{C([M_1, M_2]; V)} + \|u_t(\cdot)\|_{L^2(M_1, M_2; H)}$, $u(\cdot) \in W(M_1, M_2)$, $-\infty < M_1 < M_2 < +\infty$. Note that the existence of a Lyapunov type function allowed us to obtain the strongest convergence results in the space $W(\tau + \varepsilon, T)$ for all weak solutions of problem (1.1)–(1.2) on $[\tau, T]$, where $-\infty < \tau < T < +\infty$ (see also Example 4.1 below).

3 Structure Properties and Regularity of Global and Trajectory Attractors

In this section we prove the existence of trajectory and global attractors for all weak solutions of problem (1.1)–(1.2) and investigate their structure and regularity properties.

We denote the set of all nonempty (nonempty bounded) subsets of H by $P(H)$ ($\mathcal{B}(H)$).

Definition 3.1. The multivalued map $G : \mathbb{R}_+ \times H \rightarrow P(H)$ is called a strict multivalued semiflow (strict m -semiflow) if:

- a) $G(0, \cdot) = \text{Id}$ (the identity map);
- b) $G(t + s, x) = G(t, G(s, x)) \forall x \in H, t, s \in \mathbb{R}_+$.

Define the multivalued map $G : \mathbb{R}_+ \times H \rightarrow P(H)$ in the following way:

$$G(t, u_0) = \{u(t) \mid u(\cdot) \in \mathcal{K}_+, u(0) = u_0\}. \quad (3.1)$$

Theorem 3.2. Let $g \equiv 0$. Let Assumptions 1.1–1.2 hold. Then, the multivalued map $G : \mathbb{R}_+ \times H \rightarrow P(H)$ is a strict m -semiflow.

Proof. The proof repeats several lines from Zgurovsky et al. [46, Lemma 2.7, p. 55] (see also Morillas and Valero [35, Lemma 7]). $\square$

Let $\text{dist}_X(C, D) = \sup_{c \in C} \inf_{d \in D} \|c - d\|_X$ denote the Hausdorff semidistance between nonempty subsets C and D of a Banach space X .

Definition 3.3. The set $\mathcal{A} \subseteq H$ is called an invariant global attractor of G if the following three conditions hold:

- 1) $\mathcal{A}$ is invariant, that is $\mathcal{A} = G(t, \mathcal{A})$ for each $t \geq 0$;
- 2) $\mathcal{A}$ be an attracting set, that is for each nonempty bounded subset $B \subset H$,

$$\text{dist}_H(G(t, B), \mathcal{A}) \rightarrow 0, \quad t \rightarrow +\infty; \quad (3.2)$$

3) for any closed set $Y \subseteq H$ satisfying (3.2), we have $\mathcal{A} \subseteq Y$ (minimality).

Let $\{T(h)\}_{h \geq 0}$ be the translation semigroup acting on $\mathcal{K}_+$, that is $T(h)u(\cdot) = u(\cdot + h)$, $h \geq 0$, $u(\cdot) \in \mathcal{K}_+$. On $\mathcal{K}_+$ we consider the topology induced from the Fréchet space $C_{loc}(\mathbb{R}_+; H)$. Note that $f_n(\cdot) \rightarrow f(\cdot)$ in $C_{loc}(\mathbb{R}_+; H)$ if and only if $\forall M > 0$ $\Pi_{0,M}f_n(\cdot) \rightarrow \Pi_{0,M}f(\cdot)$ in $C([0, M]; H)$.

Definition 3.4. A set $\mathcal{U} \subset \mathcal{K}_+$ is called a trajectory attractor in the trajectory space $\mathcal{K}_+$ with respect to the topology of $C_{loc}(\mathbb{R}_+; H)$, if $\mathcal{U} \subset \mathcal{K}_+$ is a global attractor for the translation semigroup $\{T(h)\}_{h \geq 0}$ acting on $\mathcal{K}_+$; Kasyanov et al. [28, Section 3].

The following theorem yields new structure and regularity properties for global and trajectory attractors for all weak solutions of problem (1.1)–(1.2).

Theorem 3.5. Let $g \equiv 0$. Let Assumptions 1.1–1.2 hold. Then the following statements hold:

- (i) the strict m -semiflow $G : \mathbb{R}_+ \times H \rightarrow P(H)$ has the invariant global attractor $\mathcal{A}$;
- (ii) there exists the trajectory attractor $\mathcal{U} \subset \mathcal{K}_+$ in the space $\mathcal{K}_+$;
- (iii) the following equalities hold:

$$\mathcal{U} = \Pi_+ \mathcal{K} = \{u(\cdot) \in \mathcal{K}_+ \mid u(t) \in \mathcal{A} \forall t \in \mathbb{R}_+\} = \{u(\cdot) \in \mathcal{K}_+ \mid u(0) \in \mathcal{A}\}; \quad (3.3)$$

- (iv) $\mathcal{A}$ is a compact subset of V ;
- (v) for each $B \in \mathcal{B}(H)$ $\text{dist}_V(G(t, B), \mathcal{A}) \rightarrow 0$, $t \rightarrow \infty$;
- (vi) $\mathcal{U}$ is a bounded subset of $L^\infty(\mathbb{R}_+; V)$ and $\Pi_{0,M}\mathcal{U}$ is compact in $W(0, M)$ for each $M > 0$;
- (vii) for any bounded in $L^\infty(\mathbb{R}_+; H)$ set $\mathbf{B} \subset \mathcal{K}_+$ and any $M \geq 0$ the following relation holds:

$$\text{dist}_{W(0,M)}(\Pi_{0,M}T(t)\mathbf{B}, \Pi_{0,M}\mathcal{U}) \rightarrow 0, \quad t \rightarrow +\infty;$$
- (viii) $\mathcal{K}$ is a bounded subset of $L^\infty(\mathbb{R}; V)$ and $\Pi_{0,M}\mathcal{U}$ is compact in $W(0, M)$ for each $M > 0$;
- (ix) for each $u \in \mathcal{K}$ the limit sets

$$\alpha(u) = \{z \in V \mid u(t_j) \rightarrow z \text{ in } V \text{ for some sequence } t_j \rightarrow -\infty\},$$

$$\omega(u) = \{z \in V \mid u(t_j) \rightarrow z \text{ in } V \text{ for some sequence } t_j \rightarrow +\infty\}$$

are connected subsets of Z on which E is constant. If Z is totally disconnected (in particular, if Z is countable) the limits in V

$$z_- = \lim_{t \rightarrow -\infty} u(t), \quad z_+ = \lim_{t \rightarrow +\infty} u(t)$$

exist and z_-, z_+ are rest points; furthermore, $u(t)$ tends in V to a rest point as $t \rightarrow +\infty$ for every $u \in \mathcal{K}_+$.

Proof. Statements (i)–(iii) follow from Kasyanov et al. [28, Theorems 4–6] (see also references therein). According to Theorem 2.3 and the third equality from formula (3.3), since $T(h)\mathcal{K} = \mathcal{K}$ for each $h \in \mathbb{R}$, then $\mathcal{K} \subset C_{loc}(\mathbb{R}; V)$ is a bounded subset of $L^\infty(\mathbb{R}; V)$. Therefore, the first equality from (3.3) yields that $\mathcal{U} \subset C_{loc}(\mathbb{R}_+; V)$ is a bounded subset of $L^\infty(\mathbb{R}_+; V)$ and $\mathcal{A}$ is a bounded subset of V .

Let us fix a positive constant M and a bounded in $L^\infty(\mathbb{R}_+; H)$ set $\mathbf{B} \subset \mathcal{K}_+$. If $t_n \rightarrow +\infty$ be arbitrary, then for each $n \geq 1$ there exists $z_n(\cdot) \in \Pi_{0,M}T(t_n)\mathbf{B}$ such that

$$\text{dist}_{W(0,M)}(\Pi_{0,M}T(t_n)\mathbf{B}, \Pi_{0,M}\mathcal{U}) < \inf_{y(\cdot) \in \Pi_{0,M}\mathcal{U}} \|z_n(\cdot) - y(\cdot)\|_{W(0,M)} + \frac{1}{n}. \quad (3.4)$$

On the other hand, for each $n \geq 1$ there exists $y_n(\cdot) \in \Pi_{0,M}\mathcal{U}$ such that

$$\begin{aligned} \|z_n(\cdot) - y_n(\cdot)\|_{C([0,M];H)} &< \inf_{y(\cdot) \in \Pi_{0,M}\mathcal{U}} \|z_n(\cdot) - y(\cdot)\|_{C([0,M];H)} + \frac{1}{n} \\ &\leq \text{dist}_{C([0,M];H)}(\Pi_{0,M}T(t_n)\mathbf{B}, \Pi_{0,M}\mathcal{U}) + \frac{1}{n} \rightarrow 0, \quad n \rightarrow \infty, \end{aligned} \quad (3.5)$$

because of statement (ii). Since the subsets $\mathcal{U}$ and $\mathbf{B}$ of $\mathcal{K}_+ \subset C_{loc}(\mathbb{R}_+; H)$ are bounded in $L^\infty(\mathbb{R}_+; H)$, then the sequence $\{y_n(0), z_n(0)\}_{n \geq 1}$ is bounded in H . Therefore, Theorem 2.5 yields that the set $\{z_n(\cdot) - y_n(\cdot) : n \geq 1\}$ is precompact in $W(0, M)$. According to (3.5) the following convergence holds: $\|z_n(\cdot) - y_n(\cdot)\|_{W(0, M)} \rightarrow 0$ as $n \rightarrow \infty$. Thus, inequality (3.4) imply statement (vii).

Statement (v) follows from statements (iii), (vii), and the definition of G (see formula (3.1)). Statements (iv), (vi), and (viii) follow from Theorem 2.5 and the boundedness of $\mathcal{A}$ in $V \subset H$, $\mathcal{U}$ in $L^\infty(\mathbb{R}_+; V) \subset L^\infty(\mathbb{R}_+; H)$, and $\mathcal{K}$ in $L^\infty(\mathbb{R}; V) \subset L^\infty(\mathbb{R}; H)$ respectively. Finally, statement (ix) follows from Theorem 2.2 and Ball [5, Theorem 2.7]. $\square$

4 Counterexample

In the following example we provide that the family Z of the rest points of problem (1.1)–(1.2) is not a precompact subset in $H^2(\Omega) \cap H_0^1(\Omega)$. Therefore, since $Z \subset H^2(\Omega) \cap H_0^1(\Omega)$ is a bounded subset, then (i) the global attractor $\mathcal{A}$ from Theorem 3.5 is not a compact subset of $H^2(\Omega) \cap H_0^1(\Omega)$; (ii) the trajectory attractor $\mathcal{U}$ from Theorem 3.5 is not a compact subset of $L_{loc}^2(\mathbb{R}_+; H^2(\Omega) \cap H_0^1(\Omega))$; (iii) strong convergence results (see Theorem 2.5) do not hold in $L_{loc}^2(\mathbb{R}_+; H^2(\Omega) \cap H_0^1(\Omega))$.

Example 4.1. Let $N = 1$, $\Omega = (0, 1)$, $f_1(x, u) = 0$, $f_2(x, u) = f(u) = |u|$, $x \in \Omega$, $u \in \mathbb{R}$. Consider the problem:

$$\Delta u + \partial f(u) \ni 0 \text{ in } \Omega, \quad u|_{x=0,1} = 0. \quad (4.1)$$

Let us set

$$\mathbf{I}\{x \in A\} := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise,} \end{cases}$$

$A \subseteq \mathbb{R}$, $x \in \mathbb{R}$. Problem (4.1) has a countable number of solutions:

$$u_0 \equiv 0, \quad u_n^\pm(x) = \pm \sum_{k=0}^{n-1} (-1)^k \left(-\frac{(x - \frac{k}{n})^2}{2} + \frac{(x - \frac{k}{n})}{2n} \right) \mathbf{I}\left\{x \in \left[\frac{k}{n}, \frac{k+1}{n}\right)\right\},$$

$x \in \Omega$, $n = 1, 2, \dots$; cf. Díaz and Tello [14]. Therefore, the subsequence $\{u_{2^k}^+\}_{k=1,2,\dots} \subset \{u_n^+\}_{n=1,2,\dots}$ is orthonormal in $H^2(\Omega) \cap H_0^1(\Omega)$. Thus, the set Z of solutions of problem (4.1) is not precompact in $H^2(\Omega) \cap H_0^1(\Omega)$.

5 Applications

Let us concentrate on the following four types of applications: (i) a parabolic feedback control problem; (ii) a model of combustion in porous media; (iii) a model of conduction of electrical impulses in nerve axons; and (iv) a climate energy balance model.

Example 5.1. (*A parabolic feedback control problem*). In a subset Ω of $\mathbb{R}^3$, we consider the nonstationary heat conduction equation

$$\frac{\partial y}{\partial t} - \Delta y = g(x, t) \text{ in } \Omega \times (0, +\infty)$$

with initial conditions and suitable boundary ones. Here $y = y(x, t)$ represents the temperature at the point $x \in \Omega$ and time $t > 0$. It is supposed that $g = g_1 + g_2$, where $g_2 \in H$ is given and g_1 is a known function of the temperature of the form

$$-g_1(x, t) \in \partial f_1(x, y(x, t)) - \partial f_2(x, y(x, t)) \text{ a.e. } (x, t) \in \Omega \times (0, +\infty).$$

In a physicist's language it means that the law is characterized by the generalized gradient of a nonsmooth potential $j = f_1 - f_2$.

If Assumptions 1.1 and 1.2 hold, then all statements of Theorems 2.2 and 3.5 hold. Statement (ix) of Theorem 3.5 provides sufficient conditions for stabilization of the considered feedback control problem.

Example 5.2. (*A model of combustion in porous media*). We consider the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - f(u) \in \lambda H(u - 1), & (t, x) \in \mathbb{R}_+ \times (0, \pi), \\ u(0, t) = u(\pi, t) = 0, & t \in \mathbb{R}_+, \end{cases} \quad (5.1)$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous and nondecreasing function satisfying growth and sign assumptions, $\lambda > 0$, and $H(0) = [0, 1]$, $H(s) = \mathbf{I}\{s > 0\}$, $s \neq 0$; Feireisl and Norbury [15]. Then all statements of Theorems 2.2 and 3.5 hold.

Example 5.3. (*A model of conduction of electrical impulses in nerve axons*). Consider the problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u \in \lambda H(u - a), & (t, x) \in (0, T) \times (0, \pi), \\ u(0, t) = u(\pi, t) = 0, & t \in \mathbb{R}_+, \end{cases} \quad (5.2)$$

where $a \in (0, \frac{1}{2})$; Terman [40, 41]. All statements of Theorems 2.2 and 3.5 hold for problem (5.2).

Example 5.4. (*A climate energy balance model*). Formulate the problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + Bu \in QS(x)\beta(u) + h(x, t), & (t, x) \in \mathbb{R}_+ \times (-1, 1), \\ u_x(-1, t) = u_x(1, t) = 0, & t \in \mathbb{R}_+, \end{cases} \quad (5.3)$$

where $B, Q > 0$ are constants, $S \in L^\infty(-1, 1)$, $h \in L^\infty((-1, 1) \times \mathbb{R}_+)$, $u_0 \in L^2(-1, 1)$ and β is a maximal monotone graph in $\mathbb{R}^2$. Assume that: (a) there exist $m, M \in \mathbb{R}$ such that $\forall s \in \mathbb{R}, \forall z \in \beta(s) \ m \leq z \leq M$; (b) for a.e. $x \in (-1, 1)$ $0 < S_0 \leq S(x) \leq S_1$. This energy balance climate model was proposed in Budyko [3] and researched also in Díaz et al. [11–13]. The unknown $u(t, x)$ represents the average temperature of the Earth's surface, Q is a solar constant, $S(x)$ is an insolation function, given the distribution of solar radiation falling on upper atmosphere, β represents the ratio between absorbed and incident solar energy at the point x of the Earth's surface (so-called co-albedo function). If $h \equiv 0$, then all the statements of Theorems 2.2 and 3.5 hold for problem (5.3).

6 Conclusions

Recent developments in the long-time dynamics (as time $t \rightarrow +\infty$) of solutions for various nonlinear evolution autonomous problems are based on the global and trajectory attractors theory for multivalued (in the general situations) semi-groups in the natural phase and extended phase spaces. An important class of problems under investigation is the so-called reaction-diffusion equation with discontinuous and multi-valued interaction functions.

This class includes feedback control problems for diffusion processes, the evolution models of mechanics (a model of combustion in porous media), biology and medicine (a model of conduction of electrical impulses in nerve axons), climatology and Earth sciences (a climate energy balance model) etc. There are many publications, where authors investigate the regularity properties and long-time behavior of solutions (as time $t \rightarrow +\infty$) for such mathematical models (see the Introduction and references). J.M. Ball's additional assumption (see, for example, [4, 5]) on the existence of a Lyapunov type function for all weak solutions partially allows providing a solution to the problem stated in papers of J. Díaz et al. (see [11–14]) on the connection between the ω -limit set of each trajectory for evolution problem under consideration and the set of “rest points” (stationary solutions). In the paper the authors clarify these questions. In particular: (i) they give sufficient conditions for the existence of a Lyapunov function for all weak solutions of autonomous differential reaction-diffusion equations with discontinuous and multivalued interaction functions; (ii) they prove

convergence results for all weak solutions in the strongest topologies; (iii) they obtain new structure and regularity properties for global and trajectory attractors. The results allow investigating the long-time behavior of state functions for the following problems: (a) a model of combustion in porous media; (b) a model of conduction of electrical impulses in nerve axons; (c) a climate energy balance model; (d) a parabolic feedback control problem.

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