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RIEMANNIAN MAPS OF CR-SUBMANIFOLDS OF KAEHLER MANIFOLDS

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ABSTRACT. In this paper, Riemannian maps from a CR-submanifold of an almost Hermitian manifold to another almost Hermitian manifold are studied. Such Riemannian maps include the special class of CR-submersions, which are well known in the literature. First, the notion of Riemannian map from CR-submanifold to an almost Hermitian manifold is presented and the invariant of the image spaces of these maps is shown. The harmonicity of these maps is investigated. Also, the relation between the holomorphic sectional curvature of the target manifold and the holomorphic sectional curvature of the CR-submanifold is obtained depending on the second fundamental form of the submanifold and the second fundamental form of the Riemannian map. Finally, the Ricci curvature of the ambient manifold of the CR-submanifold in the direction of the invariant distribution is established in terms of the Ricci curvature of the image space, the Ricci curvature of the anti-invariant distribution, the mean curvature vector fields of the holomorphic distribution and the anti-invariant distribution.

1. Introduction

Two important research areas of differential geometry are the theory of submanifolds and Riemannian submersions. Especially submanifolds of almost Hermitian manifolds have been and are still being studied intensively. There are many submanifold classes of these manifolds as holomorphic, anti-invariant, CR-submanifold, slant, semi-slant, hemi-slant, bi-slant, pointwise slant submanifolds. We will focus on a class of these submanifolds. This is the class of CR-submanifolds. This class is a generalization of holomorphic submanifolds and anti-invariant submanifolds. Essentially, a CR-submanifold is a submanifold that has a holomorphic and anti-invariant distribution on it. The anti-invariant distribution on a CR-submanifold of a Kaehler manifold is always integrable.

On the other hand, the geometry of a Riemannian submersion is determined by the behavior of vertical and horizontal distributions defined on the total manifold. Also, the O'Neill tensor fields $\mathcal{A}$ and T play an important role in the geometry of Riemannian submersions. The vertical distribution in a Riemannian submersion is always integrable. The integrability of the horizontal distribution is determined by the tensor field $\mathcal{A}$, see [2] and [9].

By observing the integrability of the anti-invariant distribution of a CR-submanifold of a Kaehler manifold and the integrability of the vertical distribution of a Riemannian submersion, Kobayashi [13] defined the concept of CR-submersion from a CR-submanifold of an almost Hermitian manifold to an almost Hermitian manifold. In this definition, the vertical distribution and the anti-invariant distribution are essentially taken as the same. Therefore, the horizontal distribution and holomorphic distribution were taken as the same and the holomorphic map was considered between the

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holomorphic distribution and the base manifold. Then, he showed that if the ambient manifold of the CR-submanifold is a Kaehler manifold, then the base manifold is also a Kaehler manifold. He also showed that there is a relation between the sectional curvatures.

Riemannian maps were defined by Fischer [12] as generalizations of Riemannian submersions and Riemannian immersions. The basic concepts of differential geometry of Riemannian maps were introduced by the present author [22]. Studies on Riemannian maps continue and the relationships between this concept and important concepts in differential geometry such as Ricci solitons, Clairaut theorems, warped product and helix curves are examined, see [14, 15, 18, 24–27]. We note that vertical distribution is always integrable in a Riemannian map.

In this paper, a holomorphic Riemannian map, which was defined in [21], defined from CR-submanifold of an almost Hermitian manifold to an almost Hermitian manifold is considered and the character of the target manifold is investigated. Harmonicity of the map is also explored. Additionally, relations between sectional curvatures are obtained. Obviously the Riemannian map of a CR-submanifold is a broader concept that includes the concept of CR-submersion. Moreover, a relation is found for the Ricci tensor field depending on the vector field of the horizontal distribution. This relation is obtained among the Ricci tensor fields of the ambient manifold, the integral manifold of the range distribution and the anti-invariant distribution depending on the second fundamental form of the Riemannian map, the second fundamental form of the immersion and the $\mathcal{A}$ tensor field of the Riemannian submersion.

2. Preliminaries

Let $\tilde{M}$ be a Kaehler manifold of complex dimension m, equipped with a Kaehler metric g. It should be noted that in this case, $\tilde{M}$ has a complex structure $\jmath$ satisfying the properties given by

$$\jmath^2 = -I, \quad g(\zeta_1, \zeta_2) = g(\jmath\zeta_1, \jmath\zeta_2), \quad (\tilde{\nabla}_{\zeta_1}\jmath)\zeta_2 = 0, \tag{2.1}$$

for each vector fields $\zeta_1, \zeta_2 \in \Gamma(T\tilde{M})$, where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$. Let $\tilde{R}$ be the Riemannian curvature tensor field of $\tilde{M}$. Consider a unit vector U at a point p of $\tilde{M}$. Then the pair $\{U, \jmath U\}$ determines a plane π called a holomorphic sectional whose curvature $K^{\tilde{M}}$ is given by

$$K^{\tilde{M}} = g(\tilde{R}(U, \jmath U)\jmath U, U),$$

and is called the holomorphic sectional curvature with respect to U [16].

Let N be a real submanifold of $\tilde{M}$ of real dimension n. Hence, the Riemannian metric induced on N by the Kaehlerian metric g is denoted by the same symbol g.
Denote the morphisms defined on the tangent bundle TN by P and F. Then for any vector field $\zeta_1 \in \Gamma(TN)$, we write

$$\jmath\zeta_1 = P\zeta_1 + F\zeta_1, \tag{2.2}$$

where $P\zeta_1$ and $F\zeta_1$ represent the tangential and normal components of $\jmath\zeta_1$, respectively. Similarly, for any vector field $\xi \in \Gamma(TN^{\perp})$, we put

$$\jmath\xi = t\xi + f\xi, \tag{2.3}$$

where t and f are the morphisms defined on the normal bundle. Let $\nabla^{^N}$ be the Levi-Civita connection on N. Then the Gauss formula and Weingarten formula are given, respectively, by

$$\tilde{\nabla}_{\zeta_1}\zeta_2 = \nabla^{^N}_{\zeta_1}\zeta_2 + h(\zeta_1, \zeta_2), \tag{2.4}$$

$$\tilde{\nabla}_{\zeta_1}\xi = -\mathcal{S}_{\xi}\zeta_1 + \nabla^{\perp}_{\zeta_1}\xi, \tag{2.5}$$

for any vector fields $\zeta_1, \zeta_2 \in \Gamma(TN)$ and $\xi \in \Gamma(TN^{\perp})$, where h represents the second fundamental form, $\nabla^{\perp}$ is the linear connection in the normal bundle $TN^{\perp}$ and $\mathcal{S}_{\xi}$ is the Weingarten map in the direction of ξ. Let $\overset{M}{R}$ be the curvature tensor field of N, respectively. Then the Gauss and Weingarten formulas imply

$$g(\tilde{R}(\zeta_1,\zeta_2)\zeta_3,\zeta_4) = g(\overset{M}{R}(\zeta_1,\zeta_2)\zeta_3,\zeta_4) - g(h(\zeta_2,\zeta_3),h(\zeta_1,\zeta_4)) + g(h(\zeta_1,\zeta_3),h(\zeta_2,\zeta_4)) \tag{2.6}$$

for $\zeta_1,\zeta_2,\zeta_3,\zeta_4 \in \chi(N)$.

Let $\tilde{M}$ be a Kaehler manifold and $\aleph$ a submanifold of $\tilde{M}$. $\aleph$ is called a CR-submanifold [6] if there exists a differentiable distribution $D\colon p \to D_p \subset T_p\aleph$ such that D is invariant with respect to $\jmath$, and the complementary distribution $\overset{\perp}{D}$ is anti-invariant with respect to $\jmath$ [6].

Let $(M^m, g_{_M})$ and $(N^n, g_{_N})$ be Riemannian manifolds, where $\dim(M) = m$, $\dim(N) = n$ and $m > n$. A Riemannian submersion [17] $\pi\colon M \longrightarrow N$ is a map of M onto N satisfying the following axioms:

(1) π has maximal rank.

(2) The differential π_* preserves the lenghts of horizontal vectors.

For each $q \in N$, $\pi^{-1}(q)$ is an $(m-n)$ dimensional submanifold of M. The submanifolds $\pi^{-1}(q)$, $q \in N$, are called fibers. A vector field on M is called vertical if it is always tangent to fibres. A vector field on M is called horizontal if it is always orthogonal to fibres. A vector field ζ_1 on M is called basic if ζ_1 is horizontal and π-related to a vector field ζ_{1*} on N. Note that we denote the projection morphisms on the distributions $\ker \pi_*$ and $(\ker \pi_*)^{\perp}$ by $\mathcal{V}$ and $\mathcal{H}$, respectively. For basic vector fields ξ_1, ξ_2 on M, we have

$$\overset{M}{\nabla}_{\xi_1}\xi_2 = \mathcal{A}_{\xi_1}\xi_2 + \mathcal{H}\overset{M}{\nabla}_{\xi_1}\xi_2, \tag{2.7}$$

where $\overset{M}{\nabla}$ is the Levi-Civita connection on M and $\mathcal{A}$ is the O'Neill tensor field which is anti-symmetric on the set of horizontal vector fields.

By observing the integrability of the anti-invariant distribution of a CR-submanifold and the vertical distribution of a Riemannian submersion, Kobayashi introduced the submersed CR-submanifolds as follows.

Definition 1 ([13]). Let $\mathfrak{V}$ be a CR-submanifold of an almost Hermitian manifold $(\bar{\mathfrak{V}}, \mathfrak{J})$ with distributions $\mathfrak{W}$ and $\mathfrak{W}^{\perp}$ and the normal bundle ν. By a submersion $\pi\colon \mathfrak{V} \to \mathfrak{N}$ of $\mathfrak{V}$ onto an almost Hermitian manifold $\mathfrak{N}$ we mean a Riemannian submersion $\pi : \mathfrak{V} \to \mathfrak{N}$ together with the following conditions:

(i) $\mathfrak{W}^{\perp}$ is the kernel of π_*, that is, $\pi_*\mathfrak{W}^{\perp} = \{0\}$;

(ii) $\pi_*\mathfrak{W}_p = T_{\pi(p)}\mathfrak{N}$ is complex isometry, where $p \in \mathfrak{V}$ and $T_{\pi(p)}\mathfrak{N}$ is the tangent space of $\mathfrak{N}$ at $\pi(p)$;

(iii) $\mathfrak{J}$ interchanges $\mathfrak{W}^{\perp}$ and ν, that is, $\mathfrak{J}\mathfrak{W}^{\perp} = \nu$.

Starting from this definition, Kobayashi showed that if the ambient manifold $\bar{\mathfrak{V}}$ is a Kaehler manifold, then the target manifold $\mathfrak{N}$ is also a Kaehler manifold. He also obtained the relation between holomorphic sectional curvatures. After Kobayashi's results, Deshmukh, Shahid and Husein studied the effects of such submersions on the geometry of the CR-submanifold [7]. Moreover, Deshmukh, Ghazal and Hashem studied such submersions in the case of the ambient space being quasi-Kaehler and nearly Kaehler manifold [8]. This concept is still an active research area in submanifold theory [10, 11].

Now, a smooth map $\pi\colon (M^{^m}, g_{_M}) \longrightarrow (N^{^n}, g_{_N})$ is called Riemannian map at $p_1 \in M$ [12] if the horizontal restriction

$$\overset{h}{\pi}_{*p_1}\colon (\ker \pi_{*p_1})^{\perp} \longrightarrow (\operatorname{range} \pi_{*p_1})$$

is a linear isometry between the inner product spaces $((\ker\pi_{*p_1})^{\perp}, g_{M}(p_1)\mid_{(\ker\pi_{*p_1})^{\perp}})$ and $(\mathrm{range}\,\pi_{*p_1}, g_{N}(p_2)\mid_{(\mathrm{range}\,\pi_{*p_1})})$, $p_2=\pi(p_1)$. From now on, unless otherwise stated, in the Riemannian map $\pi\colon M\longrightarrow N$ between Riemannian manifolds (M^{m}, g_{M}) and (N^{n}, g_{N}), the Levi-Civita connection of a manifold M is denoted as ∇^{M} and the Levi-Civita connection of a manifold N is denoted as ∇^{N}. For the curvature tensor fields of manifolds, R^M and R^N will be used. In [20], we showed that $(\nabla\pi_*)(\zeta_1,\zeta_2)\in\Gamma((\ker\pi_*)^{\perp})$ for the second fundamental form $(\nabla\pi_*)$ of the Riemannian map π for $\zeta_1,\zeta_2\in\Gamma((\ker\pi_*)^{\perp})$. Thus at $p\in M$, we write

$$\nabla^{\pi}{}_{\zeta_1}\pi_*(\zeta_2)(p)=\pi_*(\nabla^{M}_{\zeta_1}\zeta_2)(p)+(\nabla\pi_*)(\zeta_1,\zeta_2)(p), \tag{2.8}$$

∇^{π} is the pull-back connection. Hence we have

$$\begin{aligned} g_{N}(R^{N}(\pi_*\zeta_1,\pi_*\zeta_2)\pi_*\zeta_3,\pi_*\zeta_4)&=g_{M}(R^{M}(\zeta_1,\zeta_2)\zeta_3,\zeta_4)+g_{N}((\nabla\pi_*)(\zeta_1,\zeta_3),(\nabla\pi_*)(\zeta_2,\zeta_4))\\ &\quad-g_{N}((\nabla\pi_*)(\zeta_2,\zeta_3),(\nabla\pi_*)(\zeta_1,\zeta_4)) \end{aligned} \tag{2.9}$$

for $\zeta_3,\zeta_4\in\Gamma((\ker\pi_*)^{\perp})$ [22]. We note that the equation (2.7) is valid for a Riemannian map, see [22: p. 188].

3. Riemannian mapped CR-submanifolds

In this section, the notion of Riemannian map from a CR-submanifold of a Hermitian manifold to another almost-Hermitian manifold is introduced and its effects on the properties of the Riemannian map and the geometry of the source manifold and the target manifold are investigated. We first give the definition as follows.

Definition 2. Let M be a CR-submanifold of an almost Hermitian manifold $(\bar{M},\bar{\jmath},\bar{g})$ with distributions D and $\overset{\perp}{D}$ and the normal bundle $TM^{\perp}$. By a Riemamian map $\pi\colon M\to B$ of M to an almost Hermitian manifold $(B,\jmath_{B},g_{B})$ we mean a Riemannian map $\pi\colon M\to B$ with the following conditions:

(i) $\overset{\perp}{D}$ is the kernel of π_*, that is, $\pi_*\Big(\overset{\perp}{D}\Big)=0$;

(ii) π_* is a holomorphic Riemannian map between D_p and range π_{*p} for $p\in M$, where (range π_{*p}) is the range of π_* at $p\in M$;

(iii) $\bar{\jmath}$ interchanges $\overset{\perp}{D}$ and $TM^{\perp}$, that is $\bar{\jmath}\Big(\overset{\perp}{D}\Big)=TM^{\perp}$.

In this case, we say that M is Riemannian mapped into B. Recall that a holomorphic Riemannian map is

$$\jmath_{B\,\pi(p)}\pi_{*p}=\pi_{*p}\,(\bar{\jmath}_p) \tag{3.1}$$

for any $p\in M$. By definition, we have the following result.

Lemma 3.1. *Let M be an CR-submanifold of an almost Hermitian manifold $(\bar{M},\bar{\jmath},\bar{g})$. Suppose M is Riemannian mapped into almost Hermitian manifold $(B,\jmath_{B},g_{B})$ by a Riemannian map $\pi\colon M\to B$. Then the* range π_* *is invariant with respect to $\jmath_{B}$.*

Proof. We denote the orthogonal complementary distribution to range π_{*p} in $T_{\pi(p)}{}^{B}$ by $(\mathrm{range}\,\pi_{*p})^{\perp}$. We have

$$g_B\,(\pi_*(\zeta_1),W)=0\qquad\text{for } W\in\Gamma\big((\mathrm{range}\,\pi_*)^{\perp}\big).$$

Then since π is Riemannian map between M and B, we have

$$g_B\left(\jmath_B\left(\pi_*(\zeta_1)\right), W\right) = g_B\left(\pi_*\left(\bar{\jmath}\zeta_1\right), W\right) = 0,$$

which shows that $\jmath_B\left(\pi_*(\zeta_1)\right) \in \Gamma\left(\operatorname{range}\pi_*\right)$. □

Lemma 3.2. *Let M be a CR-submanifold of a Kaehler manifold $(\bar{M}, \bar{\jmath}, \bar{g})$. Suppose that M is Riemannian mapped into an almost Hermitian manifold $(B, \jmath_B, g_B)$ by a Riemannian map $\pi\colon M \to B$. Then we have*

$$h(\zeta_1, \bar{\jmath}\zeta_2) = \bar{\jmath}\mathcal{A}_{\zeta_1}\zeta_2 \tag{3.2}$$

and

$$h(\zeta_1, \bar{\jmath}\zeta_1) = 0 \tag{3.3}$$

for $\zeta_1, \zeta_2 \in \Gamma(D)$.

Proof. Since $\bar{M}$ is a Kaehler manifold, using Gauss formula (2.4) and (2.7) we derive

$$\nabla_{\zeta_1}\bar{\jmath}\zeta_2 + h(\zeta_1, \bar{\jmath}\zeta_2) = \bar{\jmath}\mathcal{A}_{\zeta_1}\zeta_2 + \bar{\jmath}\mathcal{H}\nabla^{M}{}_{\zeta_1}\zeta_2 + \bar{\jmath}h(\zeta_1, \zeta_2).$$

Taking the normal parts of the above equation, we get (3.2). Since $\mathcal{A}$ is anti-symmetric on the set of horizontal vector fields, (3.3) follows from (3.2). □

It is easily obtained that the image distribution $\operatorname{range}\pi_*$ determined by a Riemannian map π on the target manifold is integrable. We will now investigate whether the integral manifold of this image distribution $\operatorname{range}\pi_*$ is a Kaehler manifold with respect to ∇^{N}, i.e. $(\nabla^{N}{}_{\zeta_1}\jmath_N)\pi_*(\zeta_2) = 0$ along D for $\zeta_1, \zeta_2 \in \Gamma(D)$.

Theorem 3.1. *Let (M, g_M) be a CR-submanifold of a Kaehler manifold $(\bar{M}, \bar{\jmath}, g_{\bar{M}})$. Suppose that M is Riemannian mapped to an almost Hermitian manifold N by a Riemannian map $\pi\colon (M, g_M) \to (N, g_N, \jmath_N)$. Then the integral manifold of the distribution* $\operatorname{range}\pi_*$ *is a Kaehler manifold if and only if*

$$(\nabla\pi_*)(\zeta_1, \bar{\jmath}\zeta_2) = \jmath_N(\nabla\pi_*)(\zeta_1, \zeta_2) \tag{3.4}$$

for $\zeta_1, \zeta_2 \in \Gamma(D)$.

Proof. For $\zeta_1, \zeta_2 \in \Gamma(D)$ we have

$$(\nabla^{N}{}_{\zeta_1}\jmath_N)\pi_*(\zeta_2) = \nabla^{\pi}_{\zeta_1}\jmath_N\pi_*(\zeta_2) - \jmath_N\nabla^{N}{}_{\zeta_1}\pi_*(\zeta_2),$$

where ∇^{π} and ∇^{N} are the pull-back connection and the Levi-Civita connection of N, respectively. Using (3.1) we get

$$(\nabla^{N}{}_{\zeta_1}\jmath_N)\pi_*(\zeta_2) = (\nabla\pi_*)(\zeta_1, \bar{\jmath}\zeta_2) + \pi_*(\nabla^{M}{}_{\zeta_1}\bar{\jmath}\zeta_2) - \jmath_N\nabla^{N}{}_{\zeta_1}\pi_*(\zeta_2),$$

where ∇^{M} is the Levi-Civita connection of M. Using (2.4) we derive

$$(\nabla^{N}{}_{\zeta_1}\jmath_N)\pi_*(\zeta_2) = (\nabla\pi_*)(\zeta_1, \bar{\jmath}\zeta_2) + \pi_*(\bar{\nabla}_{\zeta_1}\bar{\jmath}\zeta_2 - h(\zeta_1, \bar{\jmath}\zeta_2)) - \jmath_N\nabla^{N}{}_{\zeta_1}\pi_*(\zeta_2),$$

where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{M}$. Since $\bar{M}$ is a Kaehler manifold, using (3.2) we have

$$(\nabla^{N}{}_{\zeta_1}\jmath_N)\pi_*(\zeta_2) = (\nabla\pi_*)(\zeta_1, \bar{\jmath}\zeta_2) + \pi_*(\bar{\jmath}\bar{\nabla}_{\zeta_1}\zeta_2 - \bar{\jmath}\mathcal{A}_{\zeta_1}\zeta_2) - \jmath_N\nabla^{N}{}_{\zeta_1}\pi_*(\zeta_2).$$

Then from (2.4) we obtain

$$(\nabla^{N}{}_{\zeta_1}\jmath_N)\pi_*(\zeta_2) = (\nabla\pi_*)(\zeta_1, \bar{\jmath}\zeta_2) + \pi_*(\bar{\jmath}\nabla^{M}{}_{\zeta_1}\zeta_2 + \bar{\jmath}h(\zeta_1, \zeta_2) - \bar{\jmath}\mathcal{A}_{\zeta_1}\zeta_2) - \jmath_N\nabla^{N}{}_{\zeta_1}\pi_*(\zeta_2).$$

On the other hand, from Definition 2(ii), we know that $\bar{\jmath}h(\zeta_1, \zeta_2) \in \Gamma(\overset{\perp}{D}) = \Gamma(\ker\pi_*)$. Then using (2.7) we derive

$$(\nabla^{N}{}_{\zeta_1}\jmath_N)\pi_*(\zeta_2) = (\nabla\pi_*)(\zeta_1, \bar{\jmath}\zeta_2) + \pi_*(\bar{\jmath}\mathcal{H}\nabla^{M}{}_{\zeta_1}\zeta_2) - \jmath_N\nabla^{N}{}_{\zeta_1}\pi_*(\zeta_2).$$

Thus (3.1) implies that

$$(\nabla^{N}{}_{\zeta_1} \jmath_{N})\pi_*(\zeta_2) = (\nabla\pi_*)(\zeta_1, \bar{\jmath}\zeta_2) + \jmath_{N}\pi_*(\nabla^{M}{}_{\zeta_1}\zeta_2) - \jmath_{N}\nabla^{N}{}_{\zeta_1}\pi_*(\zeta_2),$$

which is equaivalent to

$$(\nabla^{N}{}_{\zeta_1} \jmath_{N})\pi_*(\zeta_2) = (\nabla\pi_*)(\zeta_1, \bar{\jmath}\zeta_2) - \jmath_{N}(\nabla\pi_*)(\zeta_1, \zeta_2).$$

Thus proof is complete. □

From Theorem 3.1.1 we have the following result.

Corollary 3.1.1. *Let (M, g_{M}) be a CR-submanifold of a Kaehler manifold $(\bar{M}, \bar{\jmath}, g_{\bar{M}})$. Suppose that M is Riemannian mapped to into the Kaehler manifold $(B, \jmath_{B}, g_B)$ by a Riemannian map $\pi\colon (M, g_{M}) \to (B, g_{B}, \jmath_{B})$. Then we have*

$$(\nabla\pi_*)(\bar{\jmath}\zeta_1, \bar{\jmath}\zeta_2) = -(\nabla\pi_*)(\zeta_1, \zeta_2) \tag{3.5}$$

for $\zeta_1, \zeta_2 \in \Gamma(D)$.

We now check the harmonicity of the Riemannian map between a CR-submanifold M of a Kaehler manifold $(\bar{M}, \bar{\jmath}, \bar{g})$ to a Kaehler manifold $(B, \jmath_B, g_B)$.

Theorem 3.2. *Let M be a CR-submanifold of a Kaehler manifold $(\bar{M}, \bar{\jmath}, \bar{g})$. Suppose that M is Riemannian mapped into the Kaehler manifold $(B, \jmath_{B}, g_B)$. Then Riemannian map $\pi\colon M \to B$ is a harmonic map if and only if the fibers are minimal submanifolds in M.*

P r o o f. We choose an orthonormal frame of M such that $\{e_1, \ldots, e_p, \bar{\jmath}e_1, \ldots, \bar{\jmath}e_p\}$ is an orthonormal basis of D and $\{e_p^* \ldots e_r^*\}$ is an orthonormal basis of $\overset{\perp}{D}$. Then we write

$$\tau = \operatorname{trace}(\nabla\pi_*) = \sum_{i=1}^{q} (\nabla\pi_*)(e_i^*, e_i^*) + \sum_{j=1}^{r} (\nabla\pi_*)(e_j, e_j) + (\nabla\pi_*)(\bar{\jmath}e_j, \bar{\jmath}e_j). \tag{3.6}$$

Using (3.5) in (3.6) we arrive at

$$\tau = -\sum_{i=1}^{q} \pi_* \left(\nabla^{M}{}_{e_i^*} e_i^*\right),$$

which completes proof. □

Remark 1. Note here that holomorphic Riemannian maps between two Kaehler manifolds are always harmonic [21] as for arbitrary holomorphic maps between two Kaehler manifolds [4]. However, Riemannian mapping of a CR-submanifold of a Kaehler manifold to other Kaehler manifolds are harmonic subject to the extra condition.

We will now investigate the relation between the holomorphic sectional curvature defined on the holomorphic distribution of the CR-submanifold and the holomorphic sectional curvature of a Kaehler manifold in the case of a Riemannian mapping of a CR-submanifold to a Kaehler manifold. We first give the following lemma.

Lemma 3.3. *Let M be a CR-submanifold of a Kaehler manifold $(\bar{M}, \bar{\jmath}, \bar{g})$. Suppose that M is Riemannian mapped into the Kaehler manifold $(B, \jmath_{B}, g_B)$ by a Riemannian map $\pi\colon M \to B$. Then we have*

$$h(\bar{\jmath}\zeta_1, \bar{\jmath}\zeta_2) = h(\zeta_1, \zeta_2) \tag{3.7}$$

for $\zeta_1, \zeta_2 \in \Gamma(D)$.

P r o o f. From (3.2) we have

$$h(\zeta_1, \jmath\zeta_2) = \jmath\mathcal{A}_{\zeta_1}\zeta_2.$$

Since $\mathcal{A}$ is anti-symmetric for horizontal vector fields ζ_1 and ζ_2, we arrive at

$$h(\zeta_1, \jmath\zeta_2) = -\jmath\mathcal{A}_{\zeta_2}\zeta_1.$$

Thus using again (3.2) we get

$$h(\zeta_1, \jmath\zeta_2) = -h(\zeta_2, \jmath\zeta_1). \tag{3.8}$$

Since h is symmetric, from (2.1) we derive (3.7). □

We note that the ambient manifold $(\bar{M}, \jmath, g_{\bar{M}})$ reduces an almost Hermitian structure on the holomorphic distribution. Therefore, a holomorphic sectional curvature $H^{M}(\zeta_1) = g_{M}(R^{M}(\zeta_1, \jmath\zeta_1)\jmath\zeta_1), \zeta_1)$ is defined for a vector field ζ_1 on the holomorphic distribution.

Theorem 3.3. *Let (M, g_{M}) be a CR-submanifold of a Kaehler manifold $(\bar{M}, \jmath, g_{\bar{M}})$. Suppose that M is Riemannian mapped to a Kaehler manifold B by a Riemannian map $\pi\colon (M, g_{M}) \to (B, \jmath_{B}, g_{B})$. Then we have*

$$H^{M}(\zeta_1) = H^{B}(\zeta_1) - 2\,\|(\nabla\pi_*)(\zeta_1, \zeta_1)\|_{B}^2 - \|h(\zeta_1, \zeta_1)\|_{\bar{M}}^2$$

for $\zeta_1 \in \Gamma(D)$, where H^{M} and H^{B} are the holomorphic sectional curvatures of M and B, respectively.

P r o o f. From (2.6) we have

$$H^{M}(\zeta_1) = g_{M}(R^{M}(\zeta_1, \jmath\zeta_1)\jmath\zeta_1, \zeta_1) - g_{\bar{M}}(h(\zeta_1, \zeta_1), h(\jmath\zeta_1, \jmath\zeta_1)) + \parallel h(\zeta_1, \jmath\zeta_1) \parallel_{\bar{M}}^2$$

for $\zeta_1 \in \Gamma(D)$. Then (3.3) implies that

$$H^{M}(\zeta_1) = g_{M}(R^{M}(\zeta_1, \jmath\zeta_1)\jmath\zeta_1, \zeta_1) - g_{\bar{M}}(h(\zeta_1, \zeta_1), h(\jmath\zeta_1, \jmath\zeta_1)).$$

Using (3.7) we derive

$$H^{M}(\zeta_1) = g_{M}(R^{M}(\zeta_1, \jmath\zeta_1)\jmath\zeta_1, \zeta_1) - \parallel h(\zeta_1, \zeta_1) \parallel_{\bar{M}}^2 .$$

Then from (2.9) we have

$$\begin{aligned} H^{M}(\zeta_1) = {} & g_{B}(R^{B}(\pi_*(\zeta_1), \pi_*(\jmath\zeta_1))\pi_*(\jmath\zeta_1), \pi_*(\zeta_1)) - g_{B}((\nabla\pi_*)(\zeta_1, \jmath\zeta_1), (\nabla\pi_*)(\jmath\zeta_1, \zeta_1)) \\ & + g_{B}((\nabla\pi_*)(\jmath\zeta_1, \jmath\zeta_1), (\nabla\pi_*)(\zeta_1, \zeta_1)) - \parallel h(\zeta_1, \zeta_1) \parallel_{\bar{M}}^2 . \end{aligned}$$

Applying (3.4) and taking into account the fact that manifold B is a Kaehler manifold and Definition 2(iii), we obtain

$$H^{M}(\zeta_1) = g_{B}(R^{B}(\pi_*(\zeta_1), \jmath_{B}\pi_*(\zeta_1))\jmath_{B}\pi_*(\zeta_1), \pi_*(\zeta_1)) - 2 \parallel (\nabla\pi_*)(\zeta_1, \zeta_1) \parallel_{B}^2 - \parallel h(\zeta_1, \zeta_1) \parallel_{\bar{M}}^2,$$

which proves our assertion. □

Finally, we will obtain the Ricci tensor through a Riemannian map from a CR-submanifold of a Kaehler manifold to an almost Hermitian manifold, depending on the distributions, the source manifold and the target manifold. But we first recall the notion of the mean curvature vector field from [5] for the holomorphic distribution D. Let $\{e_1, \ldots, e_{2p}\}$ be an orthonormal frame for D, we the mean curvature vector field $\overset{D}{H}$ is defined by

$$H^{D} = \frac{1}{2p}\sum_{i=1}^{2p} h(e_i, e_i) \tag{3.9}$$

for D. In a similar way, let $\{E_1,\dots,E_r\}$ be an orthonormal frame for $\overset{\perp}{D}$, we define the mean curvature vector field $H^{D^\perp}$ by

$$H^{D^\perp}=\frac{1}{r}\sum_{\alpha=1}^{r}h(E_\alpha,E_\alpha) \tag{3.10}$$

for $\overset{\perp}{D}$. We also define Ricci tensor fields of range π_* and $\overset{\perp}{D}$ for $\zeta_1\in\Gamma(D)$ by

$$\mathrm{Ric}(\pi_*(\zeta_1),\pi_*(\zeta_1))\mid_{\mathrm{range}\,\pi_*}=\sum_{i=1}^{2p}g_N(R^N(\pi_*e_i,\pi_*\zeta_1)\pi_*\zeta_1,\pi_*e_i)$$

and

$$\mathrm{Ric}(\zeta_1,\zeta_1)\mid_{\overset{\perp}{D}}=\sum_{\alpha=1}^{r}g_M(R^M(E_\alpha,\zeta_1)\zeta_1,E_\alpha),$$

respectively.

Theorem 3.4. *Let (M,g_M) be a CR-submanifold of a Kaehler manifold $(\bar M,\jmath,g_{\bar M})$. Suppose that M is Riemannian mapped to an almost Hermitian manifold N by a Riemannian map $\pi\colon(M,g_M)\to(N,g_N,\jmath_N)$ such that the integral manifold of the distribution* range π_* *is a Kaehler manifold. Then we have*

$$\begin{aligned}\mathrm{Ric}(\zeta_1,\zeta_1)\mid_{\bar M}&=\mathrm{Ric}\mid_{\mathrm{range}\,\pi_*}(\zeta_1,\zeta_1)+\mathrm{Ric}\mid_{\overset{\perp}{D}}(\zeta_1,\zeta_1)+\mathrm{Ric}\mid_{\overset{\perp}{D}}(\jmath\zeta_1,\jmath\zeta_1)\\&+\sum_{i=1}^{p}\parallel h(\zeta_1,e_i)\parallel^2_{\bar M}+\parallel\mathcal{A}_{\zeta_1}e_i\parallel^2_M-2\parallel(\nabla\pi_*)(e_i,\zeta_1)\parallel^2_N\\&+\sum_{\alpha=1}^{r}\parallel h(\jmath\zeta_1,E_\alpha)\parallel^2_{\bar M}+\parallel h(\zeta_1,E_\alpha)\parallel^2_{\bar M}-g_{\bar M}\Big(h(\zeta_1,\zeta_1),pH^D+2rH^{D^\perp}\Big)\end{aligned} \tag{3.11}$$

for $\zeta_1\in\Gamma(D)$, where $\mathrm{Ric}(\zeta_1,\zeta_1)\mid_{\bar M}$ denotes the Ricci tensor of $\bar M$ with respect to $\zeta_1\in\Gamma(D)$.

Proof. From the decomposition of the ambient manifold we have

$$\mathrm{Ric}(\zeta_1,\zeta_1)\mid_{\bar M}=\sum_{i=1}^{2p}g_{\bar M}(\overset{\bar M}{R}(e_i,\zeta_1)\zeta_1,e_i)+\sum_{\alpha=1}^{r}g_{\bar M}(\overset{\bar M}{R}(E_\alpha,\zeta_1)\zeta_1,E_\alpha)+g_{\bar M}(\overset{\bar M}{R}(\jmath(E_\alpha),\zeta_1)\zeta_1,\jmath(E_\alpha)) \tag{3.12}$$

for $\zeta_1\in\Gamma(D)$. Since $(\bar M,\jmath,g_{\bar M})$ is Kaehler manifold, we have

$$g_{\bar M}(\overset{\bar M}{R}(\jmath(E_\alpha),\zeta_1)\zeta_1,\jmath(E_\alpha))=g_{\bar M}(\overset{\bar M}{R}((E_\alpha),\jmath\zeta_1)\jmath\zeta_1,(E_\alpha)). \tag{3.13}$$

From (2.6) and (3.13) we get

$$\begin{aligned}\mathrm{Ric}(\zeta_1,\zeta_1)\mid_{\bar M}=&\sum_{i=1}^{2p}\{g_{\bar M}(R^M(e_i,\zeta_1)\zeta_1,e_i)-g_{\bar M}(h(\zeta_1,\zeta_1),h(e_i,e_i))\\&\quad+g_{\bar M}(h(\zeta_1,e_i),h(\zeta_1,e_i))\}\\&+\sum_{\alpha=1}^{r}\{g_{\bar M}(R^M(E_\alpha,\zeta_1)\zeta_1,E_\alpha)-g_{\bar M}(h(\zeta_1,\zeta_1),h(E_\alpha,E_\alpha))\\&\quad+g_{\bar M}(h(\zeta_1,E_\alpha),h(\zeta_1,E_\alpha)\\&\quad+g_{\bar M}(\overset{M}{R}((E_\alpha),\jmath\zeta_1)\jmath\zeta_1,(E_\alpha))-g_{\bar M}(h(\jmath\zeta_1,\jmath\zeta_1),h(E_\alpha,E_\alpha))\}\\&+g_{\bar M}(h(\jmath\zeta_1,E_\alpha),h(\jmath\zeta_1,E_\alpha).\end{aligned} \tag{3.14}$$

On the other hand, from (3.2) we have

$$h(\jmath\zeta_1, \jmath\zeta_1) = \jmath\mathcal{A}_{\jmath\zeta_1}\zeta_1 = -\jmath\mathcal{A}_{\zeta_1}\jmath\zeta_1 = h(\zeta_1, \zeta_1). \tag{3.15}$$

Using (3.15), we have

$$\sum_{i=1}^{2p} h(e_i, e_i) = pH^{D}. \tag{3.16}$$

Moreover, using again (3.2), we get

$$\begin{aligned}\sum_{i=1}^{2p} g(h(\zeta_1, e_i), h(\zeta_1, e_i)) &= \sum_{i=1}^{p} g(h(\zeta_1, e_i), h(\zeta_1, e_i)) + g(h(\zeta_1, \jmath e_i), h(\zeta_1, \jmath e_i)) \\ &= \sum_{i=1}^{p} g(h(\zeta_1, e_i), h(\zeta_1, e_i)) + g(\jmath\mathcal{A}_{\zeta_1} e_i, \jmath\mathcal{A}_{\zeta_1} e_i) \\ &= \sum_{i=1}^{p} g(h(\zeta_1, e_i), h(\zeta_1, e_i)) + g(\mathcal{A}_{\zeta_1} e_i, \mathcal{A}_{\zeta_1} e_i) \\ &= \sum_{i=1}^{p} \| h(\zeta_1, e_i) \|^2_{\bar{M}} + \| \mathcal{A}_{\zeta_1} e_i \|^2_{M} .\end{aligned} \tag{3.17}$$

Inserting (3.15), (3.16), (3.17) in (3.14) and using (3.10) we obtain

$$\begin{aligned}\mathrm{Ric}(\zeta_1, \zeta_1) \mid_{\bar{M}} = &\sum_{i=1}^{2p} g_{M}(R^{M}(e_i, \zeta_1)\zeta_1, e_i) - g_{\bar{M}}(h(\zeta_1, \zeta_1), pH^{D} + 2rH^{D^\perp}) \\ &+ \sum_{i=1}^{p} \| h(\zeta_1, e_i) \|^2_{\bar{M}} + \| \mathcal{A}_{\zeta_1} e_i \|^2_{M} + \sum_{\alpha=1}^{r} g_{M}(R^{M}(E_\alpha, \zeta_1)\zeta_1, E_\alpha) \\ &+ \| h(\zeta_1, E_\alpha \|^2_{\bar{M}} + g_{M}(R^{M}(E_\alpha, \jmath\zeta_1)\jmath\zeta_1, E_\alpha) + \| h(\jmath\zeta_1, E_\alpha \|^2_{\bar{M}} .\end{aligned}$$

Then from (2.9) we arrive at

$$\begin{aligned}\mathrm{Ric}(\zeta_1, \zeta_1) \mid_{\bar{M}} = &\sum_{i=1}^{2p} \{ g_{N}(R^{N}(\pi_* e_i, \pi_*\zeta_1)\pi_*\zeta_1, \pi_* e_i) - g_{N}((\nabla\pi_*)(e_i, \zeta_1), (\nabla\pi_*)(\zeta_1, e_i)) \\ &+ g_{N}((\nabla\pi_*)(\zeta_1, \zeta_1), (\nabla\pi_*)(e_i, e_i)) \} - g_{\bar{M}}(h(\zeta_1, \zeta_1), pH^{D} + 2rH^{D^\perp}) \\ &+ \sum_{i=1}^{p} \| h(\zeta_1, e_i) \|^2_{\bar{M}} + \| \mathcal{A}_{\zeta_1} e_i \|^2_{M} + \sum_{\alpha=1}^{r} g_{M}(R^{M}(E_\alpha, \zeta_1)\zeta_1, E_\alpha) \\ &+ \| h(\zeta_1, E_\alpha \|^2_{\bar{M}} + g_{M}(R^{M}(E_\alpha, \jmath\zeta_1)\jmath\zeta_1, E_\alpha) + \| h(\jmath\zeta_1, E_\alpha \|^2_{\bar{M}} .\end{aligned} \tag{3.18}$$

Since the integral manifold of the distribution range π_* is a Kaehler manifold, from Theorem 3.1.1 and symmetric $(\nabla\pi_*)$ we have

$$\sum_{i=1}^{2p} (\nabla\pi_*)(e_i, e_i) = \sum_{i=1}^{p} (\nabla\pi_*)(e_i, e_i) + (\nabla\pi_*)(\jmath e_i, \jmath e_i) = 0 \tag{3.19}$$

and

$$\begin{aligned}\sum_{i=1}^{2p}(\nabla\pi_*)(\zeta_1,e_i) &= \sum_{i=1}^{p}(\nabla\pi_*)(\zeta_1,e_i)+(\nabla\pi_*)(\zeta_1,\jmath e_i)\\ &= \sum_{i=1}^{p}(\nabla\pi_*)(\zeta_1,e_i)+\jmath_N(\nabla\pi_*)(\zeta_1,e_i).\end{aligned}\tag{3.20}$$

Putting (3.19) and (3.20) into (3.18) we derive

$$\begin{aligned}\mathrm{Ric}(\zeta_1,\zeta_1)\mid_{\bar M} = &\sum_{i=1}^{2p} g_N(R^N(\pi_*e_i,\pi_*\zeta_1)\pi_*\zeta_1,\pi_*e_i)-pg_{\bar M}(h(\zeta_1,\zeta_1),H^D)\\ &+\sum_{i=1}^{p}\parallel h(\zeta_1,e_i)\parallel^2_{\bar M}+\parallel \mathcal{A}_{\zeta_1}e_i\parallel^2_M-2\parallel(\nabla\pi_*)(e_i,\zeta_1)\parallel^2_N\\ &+\sum_{\alpha=1}^{r}\{g_M(R^M(E_\alpha,\zeta_1)\zeta_1,E_\alpha)+\parallel h(\zeta_1,E_\alpha\parallel^2_{\bar M}\\ &\quad+g_M(R^M(E_\alpha,\jmath\zeta_1)\jmath\zeta_1,E_\alpha)+\parallel h(\jmath\zeta_1,E_\alpha\parallel^2_{\bar M}\}\\ &-2rg_{\bar M}(h(\zeta_1,\zeta_1),H^{D^\perp})\end{aligned}$$

which gives (3.11). □

From Theorem 3.4, we have the following corollary.

Corollary 3.4.1. *Let (M,g_M) be a CR-submanifold of a Kaehler manifold $(\bar M,\jmath,g_{\bar M})$. Suppose that M is Riemannian mapped to an almost Hermitian manifold N by a Riemannian map $\pi\colon(M,g_M)\to(N,g_N,\jmath_N)$ such that the integral manifold of the distribution* range π_* *is a Kaehler manifold. Three of the following assertions determine the fourth:*

(i) *The Ricci curvature of $\bar M$ in the direction of $\zeta_1\in\Gamma(D)$ vanishes.*

(ii) $\mathrm{Ric}\mid_{\overset{\perp}{D}}(\zeta_1,\zeta_1)+\mathrm{Ric}\mid_{\overset{\perp}{D}}(\jmath\zeta_1,\jmath\zeta_1)=0.$

(iii) *The Ricci curvature of* range π_* *in the direction of $\zeta_1\in\Gamma(D)$ vanishes.*

(iv) *For $\zeta_1\in\Gamma(D)$, we have*

$$\begin{aligned}g_{\bar M}(h(\zeta_1,\zeta_1),pH^D+2rH^{D^\perp}) = &\sum_{i=1}^{p}\parallel h(\zeta_1,e_i)\parallel^2_{\bar M}+\parallel\mathcal{A}_{\zeta_1}e_i\parallel^2_M-2\parallel(\nabla\pi_*)(e_i,\zeta_1)\parallel^2_N\\ &+\sum_{\alpha=1}^{r}\parallel h(\jmath\zeta_1,E_\alpha)\parallel^2_{\bar M}+\parallel h(\zeta_1,E_\alpha)\parallel^2_{\bar M},\end{aligned}$$

where $\{e_1,\dots,e_{2p}\}$ and $\{E_1,\dots,E_r\}$ are orthonormal frames for D and $\overset{\perp}{D}$, respectively.

4. Conclusion remarks

In this paper, a relation is established between a CR-submanifold of an almost Hermitian manifold and a Riemannian map, and results are obtained in terms of the concepts of both research areas. Thus, this paper may further stimulate the investigation of the interactions between submanifolds and Riemannian maps. In this context, we suggest the following three problems to the interested reader.

Problem 1. In articles [19] and [1], Riemannian maps were studied on almost contact manifolds and nearly Kaehler manifolds. Therefore, the problem can be studied for Riemannian maps from contact CR and CR-submanifolds of these manifolds to a Riemannian manifold.

Problem 2. Recently, Riemannian submersions on generic submanifolds [10], CR-Warped product submanifolds [11] and hemi-slant submanifolds of a Kaehler manifold [23] were studied. Therefore, this problem can be studied for Riemannian maps as well. Thus, the results obtained in those articles will be generalized.

Problem 3. This problem is studied for a general Riemannian map in this paper. The problem can also be studied by selecting certain special Riemannian maps (semi-invariant, hemi-slant, semi-slant, generic, see [22]). This will make it possible to obtain sharp results for these special maps.

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