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ASYMPTOTIC BEHAVIOR OF FRACTIONAL SUPER-LINEAR DIFFERENTIAL EQUATIONS

Miroslav Bartušek — Zuzana Došlá[c]

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ABSTRACT. This work is devoted to the nonlinear fractional differential equations of arbitrary order, considering the Riemann-Liouville derivative. We extend well-known oscillation criteria for super-linear integer-order differential equations. Finally, we show some discrepancies between the asymptotic behavior of solutions for fractional and integer order case.

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1. Introduction

Consider the fractional differential equations

$$D_0^\alpha x(t) + q(t)f(x(t)) = 0, \qquad t > 0, \tag{1.1}$$

and

$$D_0^\alpha x(t) - q(t)f(x(t)) = 0, \qquad t > 0, \tag{1.2}$$

where $n-1 < \alpha \le n$ and $n \in \{3,4,\dots\}$, and D_0^α denotes the Riemann-Liouville fractional differential operator of the order α, that is, for $\alpha < n$,

$$D_0^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-\alpha-1} x(s)\,\mathrm{d}s,$$

and $D_0^n x(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} x(t) = x^{(n)}(t)$. As usually, Γ denotes the Gamma function.

Throughout the paper, we assume:

(i_1) the function q is a real-valued positive continuous defined on $(0,\infty)$ such that for some $\lambda_0 > 1$

$$\int_0^1 t^{\alpha-1-\lambda_0(n-\alpha)} q(t)\,\mathrm{d}t < \infty; \tag{1.3}$$

(i_2) the function $f \in C^0(\mathbb{R})$, $f(u)u > 0$ for $u \ne 0$ satisfies for some λ such that $1 < \lambda \le \lambda_0$

$$|f(u)| \ge |u|^\lambda \qquad \text{for } u \in \mathbb{R}, \tag{1.4}$$

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[c] Corresponding author.

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and

$$\limsup_{|u|\to\infty} \frac{f(u)}{|u|^{\lambda_0} \operatorname{sgn} u} < \infty. \tag{1.5}$$

In last few years, the study of fractional differential equations and their applications have been developed in various directions. The current research in the asymptotic and oscillation theory is devoted to the differential and difference equations with various types of fractional operators, see, e.g., [1, 4, 5, 7–9, 12, 13, 15–17] and references therein.

Our aim here is to study asymptotic and oscillatory behavior of solutions to equations (1.1) and (1.2) when the nonlinearity term f is super-linear. In particular, we extend well-known oscillation criteria for super-linear integer-order differential equations [14: Corollary 10.1] and we point out some similarities and discrepancies between fractional and integer-order case.

If $\alpha = n$, then (1.1) and (1.2) are integer-order differential equations, which can be considered on $[t_0, \infty)$ for some $t_0 > 0$. The asymptotic and oscillatory behavior of integer-order differential equations have been deeply studied in the literature, see, e.g., monographs [6, 11, 14] and recent contributions [2, 3]. The classification of integer-order differential equations as having Property A or Property B has been introduced by I. T. Kiguradze, see [14: Sections 1.1,10.2] and the prototypes of such classification are equations $x^{(n)} + kx = 0$, $k \in \mathbb{R}$. Roughly speaking, Property A and Property B state that *every solution which may oscillate, does oscillate* ([11: Section 8.5]). The natural question which arises whether it is possible to extend these properties for nonlinear fractional differential equations (1.1) and (1.2).

A partial answer was given in [4, 5, 15]. In [15], the fractional linear equations of the order $\alpha \in (2, 3]$ were investigated assuming q is positive continuous function on $(0, \infty)$ such that $q \in L^1(0, 1)$ and $\int_0^\infty q(t)\,\mathrm{d}t = \infty$. Under the same conditions on q, oscillatory properties of solutions of (1.1) were treated in [4] for any $\alpha > 2$ and f satisfying (1.5) for $\lambda_0 = 1$. In [5], asymptotic and oscillatory behavior for (1.1) and (1.2)was investigated under the conditions (1.3)–(1.5) for $0 < \lambda \le 1$, $\lambda \le \lambda_0$. This paper extend and completes [4, 5] for the super-linear case $\lambda > 1$.

By a solution x of (1.1) or (1.2), we mean a real-valued function in $C(0, \infty)$ such that

$$t^{n-\alpha} x(t) \in C[0, \infty), \tag{1.6}$$

$D_0^\alpha x$ exists on $(0, \infty)$ and x satisfies (1.1) or (1.2) for $t > 0$. A solution x is said to be *oscillatory* if it has arbitrary large zeros, otherwise is called *nonoscillatory.*

Let x be a solution of equation (1.1) or (1.2). Denote for $\alpha \in (n - 1, n)$

$$x_1(t) = J_0^{n-\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x(s)\,\mathrm{d}s, \qquad t > 0, \tag{1.7}$$

where $J_0^{n-\alpha}$ is the Riemann–Liouville fractional integral operator. For $\alpha = n$, we set $x_1(t) = x(t)$. Any solution x of (1.1) satisfies $x_1 \in C[0, \infty)$, and $x_1^{(n)} = D^n J_0^{n-\alpha} x = D_0^\alpha x$, see [5: Lemma 1 and Remark 1]. Thus from (1.1)

$$x_1^{(n)}(t) = -q(t) f(x(t)), \qquad t > 0, \tag{1.8}$$

and similarly, any solution x of (1.2) satisfies $x_1^{(n)}(t) = q(t) f(x(t))$ for $t > 0$.

To describe oscillatory properties of solutions of (1.1), (1.2), the following definitions were introduced in [5] as an extension of Property A and Property B for integer-order differential equations.

Definition 1. Equation (1.1) is said to have *weak Property A* if for n odd, every its nonoscillatory solution x satisfies for large t

$$(-1)^i x(t)x_1^{(i)}(t) > 0 \qquad (i = 0, \dots, n-1), \tag{1.9}$$

and for n even, every its nonoscillatory solution x satisfies either

$$(-1)^{i+1} x(t)x_1^{(i)}(t) > 0 \qquad (i = 0, \dots, n-1), \tag{1.10}$$

or

$$x(t)x_1(t) > 0, \quad (-1)^{i+1} x(t)x_1^{(i)}(t) > 0 \qquad (i = 1, \dots, n-1). \tag{1.11}$$

Moreover, solutions of type (1.9), (1.10) and (1.11) satisfy

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} x_1^{(i)}(t) = 0, \qquad (i = 2, \dots, n-1), \tag{1.12}$$

and solutions of type (1.11) satisfy $\lim_{t\to\infty} x_1(t) = c \neq 0$.

Equation (1.1) is said to have *Property A* if it has weak Property A, there exist no solutions of type (1.11), and solutions of type (1.9) and (1.10) satisfy $\lim_{t\to\infty} x_1(t) = 0$.

Definition 2. Equation (1.2) is said to have *weak Property B* if for n even, every its nonoscillatory solution x satisfies for large t either (1.9) or

$$x(t)x_1^{(i)}(t) > 0 \qquad (i = 0, \dots, n-1), \tag{1.13}$$

and for n odd, either (1.10) or (1.11) or (1.13). Moreover, solutions of type (1.9) and (1.10) satisfy (1.12), solutions of type (1.11) satisfy $\lim_{t\to\infty} x_1(t) = c \neq 0$ and solutions of type (1.13) satisfy

$$\lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |x_1^{(i)}(t)| = \infty \qquad (i = 0, 1, \dots, n-1). \tag{1.14}$$

Equation (1.2) is said to have *Property B* if it has weak Property B, there exist no solutions of type (1.11), and solutions of type (1.9) and (1.10) satisfy $\lim_{t\to\infty} x_1(t) = 0$.

Remark 1. If $\alpha = n$, then $x_1(t) = x(t)$ for $t > 0$ and solutions satisfying (1.10) as well as solutions satisfying (1.11) and (1.12) cannot exist. Thus weak Property A and weak Property B coincide with Property A and Property B for integer-order differential equations, see [14: Sections 1.1,10.2]. Moreover, solutions satisfying (1.9) are sometimes called *Kneser solutions* and are typical for (1.1) [(1.2)] when n is odd [even], while solutions satisfying (1.13) are called *strongly increasing solutions* and are typical for (1.2).

2. Preliminaries

Define $\mathbb{R}^+ = [0, \infty)$. For $\beta > 0$, the operator J_0^β is defined on $L^1(0, b]$ by

$$J_0^\beta[y](t) = J_0^\beta y(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)\,\mathrm{d}s$$

for $0 < t \le b$, and is called the *Riemann-Liouville fractional integral operator of order β*. For $\beta = 0$, we set $J_0^0 := I$, the identity operator.

By D, we denote the operator that maps differentiable functions onto its derivative, i.e.,

$$Dy(t) = \frac{\mathrm{d}}{\mathrm{d}t} y(t) = y'(t).$$

For $\beta > 0$, the operator D_0^β is the *Riemann-Liouville fractional differential operator of order* β

$$D_0^\beta[y](t) = D_0^\beta y(t) := D^n J_0^{n-\beta} y(t),$$

where $n = \lceil \beta \rceil$ is the ceiling function and D^n denotes the n-fold iterates of D. For $\beta = n \in \mathbb{N}$, D_0^β coincides with the classical differential operator, it means that $D_0^\beta y(t) = y^{(n)}(t)$. It is worth to note that $y^{(n)}$ is local, while $D_0^\alpha y$ is non-local, depending on $(0, b]$. We refer [10] for the basic results of fractional calculus.

In the sequel, x is a solution of (1.1) or (1.2) and x_1 is defined by (1.7). We prove some properties for eventually positive solutions of (1.1); the analogous results hold for eventually negative solutions.

The next two lemmas can be viewed as an extension of the well-known Kiguradze lemma ([14: Lemma 1.1]) for integer-order differential equations.

Lemma 2.1. *Let x be an eventually positive solution of* (1.1). *Then there exists $k \in \{0, 1, \dots, n-1\}$ such that for large t either $n-k$ is odd and*

$$x_1^{(i)}(t) > 0 \ (i = 0, 1, \dots, k), \quad (-1)^{n+i+1} x_1^{(i)}(t) > 0 \qquad (i = k+1, \dots, n), \tag{2.1}$$

or n is even and

$$(-1)^{i+1} x_1^{(i)}(t) > 0 \quad (i = 0, 1, \dots, n), \quad \lim_{t\to\infty} x_1(t) = 0. \tag{2.2}$$

Proof. It follows from Lemmas 3 and 5, Corollary 1(a) and Remark 3 in [5]. □

Lemma 2.2. *If x is an eventually positive solution of* (1.2)*, then there exists $k \in \{0, 1, \dots, n\}$ such that for large t either $n-k$ is even and*

$$x_1^{(i)}(t) > 0 \quad (i = 0, 1, \dots, k), \quad (-1)^{n+i} x_1^{(i)}(t) > 0 \quad (i = k+1, \dots, n), \tag{2.3}$$

or n is odd and (2.2) *holds.*

Proof. It follows from Lemmas 4 and 5, Corollary 1(b), Remark 3 in [5]. □

Lemma 2.3. *Let $k \in \{1, \dots, n-1\}$ be fixed and*

$$\int_1^\infty t^{n-k-\lambda(n-\alpha-k+1)}\, q(t)\,\mathrm{d}t = \infty. \tag{2.4}$$

Assume that x is an eventually positive solution of (1.1) *satisfying* (2.1) *or solution of* (1.2) *satisfying* (2.3) *for $t \ge t_0 > 0$. In addition, for $k = 1$ assume that $\lim_{t\to\infty} x_1(t) = \infty$. Then there exists $T \ge t_0$ such that for $t \ge T$,*

$$x(t) \ge \frac{c_0}{2}\, t^{\alpha-n} x_1(t) \ge \frac{c_0}{2k!}\, t^{\alpha-n+k-1} x_1^{(k-1)}(t) \ge \frac{c_0}{2k!}\, t^{\alpha-n+k} x_1^{(k)}(t), \tag{2.5}$$

where $c_0 = 1/\Gamma(\alpha - n + 1)$, and

$$x_1^{(k-1)}(t) \ge x_1^{(k-1)}(T) + \frac{1}{2(n-k)!} \int_T^t s^{n-k}\, |x_1^{(n)}(s)|\,\mathrm{d}s. \tag{2.6}$$

Proof. Using [5: Lemma 6 and Remark 3], we get the first two inequalities in (2.5). Moreover, by Lemmas 2.1 and 2.2, we have $(-1)^{n-k} x_1^{(n)} \ge 0$ and (2.6) holds. By (2.4), we have

$$\int_T^\infty t^{n-k}\, |x_1^{(n)}(t)|\,\mathrm{d}t = \infty.$$

From here the last inequality in (2.5) follows, taking into account [14: Lemma 1.3], choosing $i = 1$ and $l = k$ in (1.20). □

Lemma 2.4. *Let x be an eventually positive solution of (1.1) [(1.2)] satisfying either (1.9) or (1.11) with $\lim_{t\to\infty} x_1(t) < \infty$. Then*

$$\int_1^\infty t^{n-1} q(t) f(x(t))\,\mathrm{d}t < \infty. \tag{2.7}$$

P r o o f. It is similar as the proof of [4: Lemma 9] for (1.1). Assume that x is an eventually positive solution of (1.1) [(1.2)] satisfying (1.11) such that for $t \ge t_0$,

$$x_1(t) > 0, \quad (-1)^{i+1} x_1^{(i)}(t) > 0, \qquad i = 1, \dots, n-1,$$

and $\lim_{t\to\infty} x_1(t) = d < \infty$. The Taylor series at t, where $t \ge t_0$, gives

$$-d \le x_1(t_0) - x_1(t) = \frac{x_1'(t)}{1!}(t_0 - t) + \frac{x_1''(t)}{2!}(t_0 - t)^2 + \dots + \frac{x_1^{(n-1)}(t)}{(n-1)!}(t_0 - t)^{n-1} + \int_t^{t_0} \frac{(t_0 - s)^{n-1}}{(n-1)!} x_1^{(n)}(s)\,\mathrm{d}s.$$

If x is a solution of (1.1) [(1.2)], then by Lemma 2.1 [Lemma 2.2] with $k = 1$, we have n is even and $x_1^{(n)}(t) < 0$ [n is odd and $x_1^{(n)}(t) > 0$] for $t \ge t_0$. Thus all members on the right-hand side are negative, and we get (2.7).

Similarly, if x is an eventually positive solution of (1.1) [(1.2)] satisfying (1.9), then x_1 is positive decreasing. Thus, there exists $\lim_{t\to\infty} x_1(t) = d < \infty$ and the rest of the proof is the same as above. □

Lemma 2.5. *Let x be an eventually positive solution of (1.1) or (1.2) satisfying (1.9) or (1.10) or (1.11) with bounded x_1. Then there exists $c > 0$ such that*

$$0 < x(t) \le \frac{c}{t^{n-\alpha}} \qquad \textit{for large } t. \tag{2.8}$$

P r o o f. By [5: Lemma 2 and Remark 3], for $T > 0$ fixed, there exists $M > 0$ such that

$$|x_1'(t)| \le M t^{n-\alpha-1} \qquad \text{for } t \in (0, T],$$

and

$$x(t) = \frac{1}{\Gamma(\alpha - n + 1)} \Big(\frac{x_1(0)}{t^{n-\alpha}} + \int_0^t \frac{x_1'(s)}{(t-s)^{n-\alpha}}\,\mathrm{d}s \Big) \qquad \text{for } t > 0\,.$$

Under these conditions, the estimation (2.8) has been proved in [4: Lemma 7]. □

We close this section by the following inequality.

Lemma 2.6 ([14: Lemma 10.3])**.** *Let $\varphi \in C([a,\infty) \times \mathbb{R})$ be such that*

$$\varphi(t,x) \ge \varphi(t,y) \qquad \textit{for } t \ge a,\ x \ge y \ge 0$$

and the differential equation

$$x'(t) = \varphi(t,x) \tag{2.9}$$

has no positive solution. Let $c_0 \in (0,\infty)$, $t_0 \in [a,\infty)$ and $h\colon [t_0,\infty) \to (0,\infty)$ be continuous nondecreasing function. Then there exists no continuous function $y\colon [t_0,\infty) \to (0,\infty)$ satisfying the inequality

$$y(t) \ge c_0 + \int_{t_0}^t \frac{1}{h(s)} \varphi\big(s, h(s)y(s)\big)\,\mathrm{d}s, \qquad t \ge t_0. \tag{2.10}$$

3. Property A and Property B

Our main results are the following.

THEOREM 3.1. *Let $n-1<\alpha\le n$, $\lambda>1$ and*

$$\int_1^\infty t^{n-1-\lambda(n-\alpha)}q(t)\,\mathrm{d}t=\infty. \tag{3.1}$$

Then equation (1.1) *has weak Property A and equation* (1.2) *has weak Property B.*

Proof. Let $n-1<\alpha<n$, $k\in\{1,\dots,n-1\}$ and x be an eventually positive solution of (1.1). When $k=1$, let $\lim\limits_{t\to\infty}x_1(t)=\infty$. By Lemma 2.1, x is either of type (2.1) or (2.2). First, we prove that solutions satisfying (2.1) do not exist.

Assume that x satisfies (2.1) for $t\ge t_0>0$. By Lemma 2.1, $n-k$ is odd. The condition (3.1) implies the validity of (2.4), and Lemma 2.3 can be applied. From (1.4), (1.8), (2.5) and (2.6), we have that there exists $T\ge t_0$ such that

$$x_1^{(k-1)}(t)\ge x_1^{(k-1)}(T)+c_1\int_T^t s^{n-k+\lambda(k-1+\alpha-n)}\,q(s)\left(x_1^{(k-1)}(s)\right)^\lambda\,\mathrm{d}s, \tag{3.2}$$

where $c_1=\frac{1}{2(n-k)!}\left(2\,k!\,\Gamma(\alpha-n+1)\right)^{-\lambda}$. Put for $t\ge T$

$$y(t)=x_1^{(k-1)}(t),\quad \varphi(t,x)=t^{n-1-\lambda(n-\alpha)}\,q(t)|x|^\lambda,\quad h(t)=c_1^{1/(\lambda-1)}t^{k-1}.$$

Since h is nondecreasing, from (3.2) we have for $t\ge T$

$$y(t)\ge y(T)+\int_T^t \frac{1}{h(s)}s^{n-1-\lambda(n-\alpha)}\,q(s)\,(h(s)y(s))^\lambda\,\mathrm{d}s,$$

thus

$$y(t)\ge y(T)+\int_T^t \frac{1}{h(s)}\varphi\Big(s,h(s)y(s)\Big)\,\mathrm{d}s. \tag{3.3}$$

Consider equation (2.9) where $a=T$. If there exists a solution y of (2.9) such that $y(t)>0$ for $t\ge\sigma\ge T$, then from (2.9) and the fact that $\lambda>1$, we get

$$\int_\sigma^t s^{n-1-\lambda(n-\alpha)}q(s)\,\mathrm{d}s=\int_\sigma^t \frac{y'(s)}{y^\lambda(s)}\,\mathrm{d}s=\int_{y(\sigma)}^{y(t)}\frac{1}{\tau^\lambda}\,\mathrm{d}\tau\le\int_{y(\sigma)}^{\infty}\frac{1}{\tau^\lambda}\,\mathrm{d}\tau<\infty,$$

which is a contradiction with (3.1). Hence, (2.9) has no positive solution. Applying Lemma 2.6 with $t_0=T$, there exists no function y satisfying (2.10), which is a contradiction with (3.3). Thus every nonoscillatory solution of (1.1) is one of types (1.9)–(1.11) and their limit properties (1.12) follow from [4: Lemma 2.10].

By the similar way, we get that equation (1.2) does not have solution of type (2.3) for $k\in\{1,\dots,n-1\}$ with $\lim\limits_{t\to\infty}x_1(t)=\infty$ in case $k=1$. Therefore, every eventually positive solution is of type either (1.9) or (1.10) or (1.11) or (1.13).

It remains to prove that solution of (1.2) given by (1.13) satisfies (1.14). Let $k=n-1$. From (2.5) and the fact that $x_1^{(n-1)}$ is increasing, we have

$$x(t)\ge c_2t^{\alpha-1},\qquad t\ge T, \tag{3.4}$$

where $c_2 = c_0 x_1^{(n-1)}(T)/(2(n-1)!)$ and T, c_0 are given by Lemma 2.3. From (3.1) and (3.4), we have for $t \geq T$,

$$x_1^{(n-1)}(t) \geq \int_T^t x_1^{(n)}(s)\,\mathrm{d}s \geq c_2^\lambda \int_T^t q(s)\,s^{\lambda(\alpha-1)}\,\mathrm{d}s \geq c_2^\lambda \int_T^t q(s)\,s^{n-1+\lambda(\alpha-n)}\,\mathrm{d}s \ \to \infty \qquad \text{as } t \to \infty.$$

Thus, $\lim_{t\to\infty} x_1^{(n-1)}(t) = \infty$, and from (3.4), we have $\lim_{t\to\infty} x(t) = \infty$, so (1.14) holds.

If $\alpha = n$, then the result follows from [14: Corollary 10.1]. □

As claimed above, for $\alpha = n$, weak Property A for (1.1) coincides with Property A and weak Property B for (1.2) coincides with Property B introduced in [14]. In the fractional case, the following holds.

Theorem 3.2. *Let $n-1 < \alpha < n$, $\lambda > 1$ and (3.1) hold. If there exists $\varepsilon \in (0,1)$ such that*

$$\liminf_{t\to\infty} t^{-\varepsilon(\alpha-n+1)} \left(\int_1^t s^{-\frac{n-1}{\lambda-1}} q(s)^{-\frac{1}{\lambda-1}} \right)^{\frac{\lambda-1}{\lambda}} \mathrm{d}s = 0, \tag{3.5}$$

then equation (1.1) has Property A and equation (1.2) has Property B.

P r o o f. First observe that the solution of type (1.10) satisfies $\lim_{t\to\infty} x_1(t) = 0$, see Lemmas 2.1, 2.2. We have to prove that there exists no solution of type (1.11) and no solution of type (1.9) such that both solutions satisfy

$$\lim_{t\to\infty} x_1(t) = c, \qquad c \in \mathbb{R},\ c \neq 0. \tag{3.6}$$

Assume by contradiction that there exists an eventually positive solution x either of type (1.11) or (1.9) satisfying (3.6) for $c > 0$. We proceed following the idea as in the proof of in [4: Theorem 1.2]. Let $t_0 \geq 1$ such that $x(t) > 0$ for $t \geq t_0$ and ε be such that $0 < \varepsilon < 1$. By Lemma 2.5, there exists $c_0 > 0$ such that

$$0 < x(t) \leq 1, \quad x(t) \leq \frac{c_0}{t^{n-\alpha}} \qquad \text{for } t \geq t_0, \tag{3.7}$$

and for $t - t^\varepsilon \geq t_0$,

$$x_1(t) = \frac{1}{\Gamma(n-\alpha)} \left(\int_0^{t_0} \frac{x(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s + \int_{t_0}^{t-t^\varepsilon} \frac{x(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s + \int_{t-t^\varepsilon}^{t} \frac{x(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s \right).$$

By (1.6), there exists $c = \max_{0\leq s\leq t_0} |s^{n-\alpha} x(s)|$. Thus,

$$\left| \int_0^{t_0} \frac{x(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s \right| \leq \frac{c}{(t-t_0)^{\alpha-n+1}} \int_0^{t_0} \frac{\mathrm{d}s}{s^{n-\alpha}} \to 0$$

as $t \to \infty$. By Lemma 2.4 and (1.4), we can set

$$\mathcal{L} = \int_{t_0}^{\infty} s^{n-1} q(s) x^\lambda(s)\,\mathrm{d}s < \infty.$$

Using Hölder inequality, we have

$$\int_{t_0}^{t-t^\varepsilon} \frac{x(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s \le \frac{1}{t^{\varepsilon(\alpha-n+1)}} \int_{t_0}^{t} \frac{(s^{n-1}q(s))^{1/\lambda}\,x(s)}{(s^{n-1}q(s))^{1/\lambda}}\,\mathrm{d}s$$

$$\le \frac{1}{t^{\varepsilon(\alpha-n+1)}} \left(\int_{t_0}^{t} s^{n-1}q(s)x^\lambda(s)\mathrm{d}s\right)^{1/\lambda} \left(\int_{t_0}^{t} s^{-(n-1)/(\lambda-1)}q^{-1/(\lambda-1)}(s)\mathrm{d}s\right)^{(\lambda-1)/\lambda}$$

$$\le \mathcal{L}^{1/\lambda}\, t^{-\varepsilon(\alpha-n+1)} \left(\int_{t_0}^{t} s^{-\frac{n-1}{\lambda-1}}\, q^{-\frac{1}{\lambda-1}}\,\mathrm{d}s\right)^{(\lambda-1)/\lambda}.$$

Thus, from (3.5), we have

$$\liminf_{t\to\infty} \int_{t_0}^{t-t^\varepsilon} \frac{x(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s = 0,$$

and using (3.7)

$$\int_{t-t^\varepsilon}^{t} \frac{x(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s \le \int_{t-t^\varepsilon}^{t} \frac{c_0}{(t-s)^{\alpha-n+1}s^{n-\alpha}}\,\mathrm{d}s \le \frac{c_0}{(t-t^\varepsilon)^{n-\alpha}}\,\frac{1}{n-\alpha}t^{\varepsilon(n-\alpha)} \to 0$$

as $t\to\infty$. Therefore, $\liminf_{t\to\infty} x_1(t) = 0$, so $\lim_{t\to\infty} x_1(t) = 0$, which is a contradiction with (3.6). □

4. Examples

The following example illustrates our results.

***Example* 1.** Consider equation

$$D_0^\alpha x(t) + q(t)|x(t)|^\lambda \operatorname{sgn} x = 0, \qquad t > 0, \tag{4.1}$$

where $n-1<\alpha<n$, $n\ge 3$, $\lambda>1$ and

$$q(t) = \begin{cases} t^{\lambda(n-\alpha)} & \text{for } t\in[0,1],\\ t^\mu & \text{for } t>1,\ \mu\in\mathbb{R}. \end{cases}$$

Obviously, (1.3) is satisfied. Moreover, (3.1) holds for

$$\mu \ge -n + \lambda(n-\alpha). \tag{4.2}$$

Applying Theorem 3.1, equation (4.1) has weak Property A for μ satisfying (4.2). Similarly, one can check that (3.5) is satisfied if the strict inequality in (4.2) holds. Thus, by Theorem 3.2, equation (4.1) has Property A for $\mu > -n+\lambda(n-\alpha)$.

Observe that this result is in accordance with the oscillation criterion for the integer-order equation

$$x^{(n)}(t) + t^\mu |x(t)|^\lambda \operatorname{sgn} x = 0, \qquad t\ge t_0>0,$$

where $\lambda>1$. Using [14: Corollary 10.1], this equation has Property A for $\mu\ge -n$, i.e., for n even all solutions are oscillatory and for n odd any solution is either oscillatory or Kneser solution tending to zero as $t\to\infty$.

Next example illustrates that (1.1) can have "purely fractional" solution satisfying (1.11) and $\lim_{t\to\infty} x_1(t) = \infty$. Moreover, it shows that conditions in Theorem 3.1 are optimal in a certain sense.

***Example* 2.** Consider equation

$$D_0^\alpha x(t) - q(t)|x(t)|^\lambda \operatorname{sgn} x(t) = 0, \qquad t > 0, \tag{4.3}$$

where $5/2 < \alpha < 3$, $\lambda > 1$,

$$q(t) = 3t^{-5/2}\left(c_1 t^{\alpha-3} + c_2 t^{\alpha-5/2}\right)^{-\lambda}$$

and $c_1 = 1/\Gamma(\alpha-2)$, $c_2 = 8\Gamma(\frac{3}{2})/\Gamma(\alpha-\frac{3}{2})$. The function

$$x(t) = c_1 t^{\alpha-3} + c_2 t^{\alpha-5/2}$$

satisfies (1.6), i.e., $t^{3-\alpha}x(t) \in C[0,\infty)$ and, using (1.7)

$$x_1(t) = 1 + 8\sqrt{t}, \qquad D_0^\alpha x(t) = x_1'''(t) = 3t^{-5/2},$$

and x is a solution of (4.3). Hence (4.3) does not have weak Property B. Observe that condition (1.3) is satisfied for $\lambda = \lambda_0$. Indeed, we have

$$q(t) \le c_3 t^{(3-\alpha)\lambda-5/2} \qquad \text{for } t \in (0,1],$$

where $c_3 = 3(c_1+c_2)^{-\lambda}$, and for $\lambda > 1$,

$$\int_0^1 t^{\alpha-1-\lambda_0(3-\alpha)} q(t)\,\mathrm{d}t \le c_3 \int_0^1 t^{\alpha-1-\lambda(3-\alpha)} t^{(3-\alpha)\lambda-5/2}\,\mathrm{d}t = \int_0^1 t^{\alpha-7/2}\,\mathrm{d}t < \infty.$$

On the other hand, if $\varepsilon > 0$ and $1 < \lambda \le 1+2\varepsilon$, then

$$\int_1^\infty t^{2-\lambda(3-\alpha)} q(t)\,\mathrm{d}t < \infty, \qquad \int_1^\infty t^{2-\lambda(3-\alpha)+\varepsilon} q(t)\,\mathrm{d}t = \infty.$$

Thus, the condition (3.1) in Theorem 3.1 is not satisfied and cannot be replaced by the weaker condition

$$\int_1^\infty t^{n-1-\lambda(n-\alpha)+\varepsilon} q(t)\,\mathrm{d}t = \infty, \qquad \varepsilon > 0.$$

5. Concluding remarks

We close the paper with some suggestions for possible further progress.

(1) If $\alpha = n$ and $f(u) = |u|^\lambda \operatorname{sgn} u$, $\lambda > 1$, then the condition (3.1) is necessary and sufficient in order to have (1.1) [(1.2)] Property A [Property B], see [14: Corollary 10.1 and Theorem 15.1].

If $n-1 < \alpha < n$ and $f(u) = |u|^\lambda \operatorname{sgn} u$, $\lambda > 1$, it is open problem whether the condition (3.1) is also necessary in order to have (1.1) [(1.2)] weak Property A [weak Property B].

(2) As claimed above, Property A for even-order equations means that all solutions are oscillatory. If $n-1 < \alpha < n$ and n is even, it is an open problem whether solutions of (1.1) satisfying (1.10) exist, i.e., whether property A means oscillation of all solutions of (1.1).

(3) The typical phenomena for the super-linear integer-order equation

$$x^{(n)} - q(t)\,|x|^\lambda \operatorname{sgn} x = 0, \qquad t \ge t_0, \tag{5.1}$$

where $\lambda > 1$, $n \ge 2$, is the existence of noncontinuable to infinity nonoscillatory solutions (blow-up solutions). Such solutions are defined on $[t_0, \tau)$, $\tau < \infty$, and different from zero in a left neighbourhood of τ such that $\lim_{t\to\tau-} |x(t)| = \infty$. Their existence can be proved by considering the Cauchy problem with the initial conditions $x^{(i)}(t_0) > 0$, $i = 0, 1, \dots, n-1$. If (5.1) has Property B and no solutions satisfying (1.13) exist, then solutions of the Cauchy problem must be noncontinuable to infinity, see [14: Theorem 11.4].

Hence, it is natural to study the existence of blow-up solutions for (1.2). In view of Theorem 2, this problem leads to the problem of the nonexistence of solutions satisfying (1.13).

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Department of Mathematics and Statistics
Masaryk University
611 37 Brno
CZECH REPUBLIC
E-mail: dosla@math.muni.cz
bartusek@math.muni.cz