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IRREDUCIBILITY OF STRONG SIZE LEVELS

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ABSTRACT. Given a continuum X, $C_n(X)$ denotes the hyperspace of nonempty closed subsets of X with at most n components. A strong size level is a subset of the form $\sigma^{-1}(t)$, where σ is a strong size map for $C_n(X)$ and $t \in (0,1]$. In this paper, answering a question by Capulín-Pérez, Fuentes-Montes de Oca, Lara-Mejía and Orozco-Zitli, we prove that for each $n \geq 2$, no strong size level for $C_n(X)$ is irreducible.

1. Introduction

A *continuum* is a compact connected metric space with more than one point, a *subcontinuum* of X is a compact connected subspace of X, so subcontinua include one-point sets.

In this paper we use the following hyperspaces of X:

$$2^X = \{A \subset X : A \text{ is a closed nonempty subset of } X\},$$
$$C_n(X) = \{A \in 2^X : A \text{ has at most n components}\},$$
$$F_n(X) = \{A \in 2^X : A \text{ has at most n points}\}.$$

All these hyperspaces are considered with the Hausdorff metric [6: Theorem 3.1]. We write $C(X)$ instead of $C_1(X)$. A *map* is a continuous function.

A continuum X is *irreducible* provided that there exist points $p, q \in X$ such that no proper subcontinuum of X contains both p and q.

A *Whitney map* for $C_n(X)$ is a map $\mu\colon C_n(X) \to [0,1]$ such that:

(a) for each $p \in X$, $\mu(\{p\}) = 0$;

(b) $\mu(X) = 1$; and

(c) if $A, B \in C_n(X)$ and $A \subsetneq B$, then $\mu(A) < \mu(B)$.

Sets of the form $\mu^{-1}(t)$ where $t \in (0,1)$ are *Whitney levels.*

A *strong size map* for $C_n(X)$ is a map $\sigma\colon C_n(X) \to [0,1]$ such that:

(a) $\sigma(A) = 0$ for each $A \in F_n(X)$;

(b) $\sigma(X) = 1$; and

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(c) if $A, B \in C_n(X)$, $A \subsetneq B$ and $B \notin F_n(X)$, then $\sigma(A) < \sigma(B)$.

Sets of the form $\sigma^{-1}(t)$ where $t \in [0,1)$ are *strong size levels*.

Notice that the restriction of a strong size map to $C(X)$ is a Whitney map. Whitney levels and strong size levels are continua ([6: Theorem 19.9] and [3: Theorem 2.10]). Hosokawa [3] introduced and studied topological properties of strong size levels. Topological properties of Whitney levels have been studied widely. In [6: Chapter VIII], there is a very complete account of what was known on Whitney levels up to 1998.

A very important line of research about Whitney and strong size levels is to determine for which topological properties $\mathcal{P}$ the following implications hold:

(a) if X has property $\mathcal{P}$, then Whitney (strong size) levels have property $\mathcal{P}$, and

(b) if Whitney (strong size) levels have property $\mathcal{P}$, then X has property $\mathcal{P}$.

Statements (a) and (b) have been studied for more than 40 years (see [6: Chapter VIII]). The study of strong size levels started with the paper [3] in 2011. Since then statements (a) and (b) have been studied for strong size levels in [1–5] and [7–9].

For the case that property $\mathcal{P}$ is irreducibility and $n = 1$, in [4], it was shown that if X is a continuum having a Whitney level irreducible with respect to a finite subset, then X is also irreducible with respect to a finite set; and in [9], it was shown an irreducible, hereditarily decomposable continuum such that none of its proper strong size levels is irreducible. In [1], the authors asked whether there exists a continuum with irreducible strong size levels. In this paper we give a complete solution to this question for the case $n \geq 2$ with the following theorem.

Theorem 1. *Let X be a continuum and $n \geq 2$. Then the strong size levels for $C_n(X)$ are not irreducible.*

2. Proof of Theorem 1

Given elements $A, B \in 2^X$ such that $A \subsetneq B$, an *order arc from A to B* is a map $\alpha\colon [0,1] \to 2^X$ such that $\alpha(0) = A$, $\alpha(1) = B$ and if $0 \leq s < t \leq 1$, then $\alpha(s) \subsetneq \alpha(t)$. The most important result about order arcs says that if $A, B \in 2^X$, then there exists an order arc from A to B if and only if $A \subsetneq B$ and each component of B intersects A [6: Theorem 15.3].

Proof of Theorem 1. Take a strong size level $\mathcal{A} = \sigma^{-1}(t_0)$, where $\sigma\colon C_n(X) \to [0,1]$ is a strong size map and $t_0 \in [0,1)$. We will use the following claim.

Claim 2. *Let $D, E \in \mathcal{A}$. Suppose that $E = F \cup G$, G is a component of E, $F \cap G = \emptyset$; L is a nonempty finite subset of F, with $|L|$ equal to the number of components of F; $\alpha\colon [0,1] \to C_n(X)$ is an order arc from L to D such that $F \in \alpha([0,1])$ and q is a point of G. Then there exists a subcontinuum $\mathcal{B}$ of $\mathcal{A}$ such that $\{D, E\} \subset \mathcal{B}$ and for each $K \in \mathcal{B}$, either each component of K intersects $L \cup \{q\}$ or $K = K_0 \cup \{p\}$, where each component of K_0 intersects L, and $p \in X$.*

Proof. Since $E \in C_n(X)$ and G is component of E, we have that E has more components than F. Then $|L| < n$. Since α is an order arc, for each $t \in (0,1]$, there exists an order arc from L to $\alpha(t)$, then each component of $\alpha(t)$ has a point of L, so $\alpha(t)$ has at most $|L|$ components. In particular, D has less components than E. Since α is an order arc from L to D, we have that L is a proper subset of D and each component of D intersects L. This implies that D is not a finite set. In particular, $\sigma(D) = t_0 > 0$. Let $t_F \in [0,1]$ be such that $\alpha(t_F) = F$. If $E \subset D$, since $\sigma(E) = \sigma(D)$, we have that $E = D$, a contradiction. Therefore E is not contained in D.

We construct two subcontinua $\mathcal{B}_1$ and $\mathcal{B}_2$. In the case $G = \{q\}$, we define $\mathcal{B}_1 = \{F \cup \{q\}\} = \{E\}$.

Now, suppose that $\{q\} \subsetneq G$. Take an order arc $\beta\colon [0,1] \to C(X)$, from $\{q\}$ to G.

Given $u \in [0,1]$, $\beta(u) \cup \alpha(0) = \beta(u) \cup L \subset G \cup F = E$, so $\sigma(\beta(u) \cup \alpha(0)) \leq \sigma(E) = t_0$. On the other hand, $\beta(u) \cup \alpha(1) = \beta(u) \cup D$. This implies that $\sigma(\beta(u) \cup \alpha(1)) \geq t_0$. This implies that there exists $v(u) \in [0,1]$ such that $\sigma(\beta(u) \cup \alpha(v(u))) = t_0$.

Define $g\colon [0,1] \to \mathcal{A}$ by $g(u) = \beta(u) \cup \alpha(v(u))$. First, we show that the definition of g does not depend on the choice of $v(u)$. Suppose that $v_1, v_2 \in [0,1]$ satisfy that $\sigma(\beta(u) \cup \alpha(v_1)) = \sigma(\beta(u) \cup \alpha(v_2))$. We may assume that $v_1 < v_2$. Since α is an order arc, we have that $\alpha(v_2)$ is not a finite set. Since $\beta(u) \cup \alpha(v_1) \subset \beta(u) \cup \alpha(v_2)$ and $\sigma(\beta(u) \cup \alpha(v_1)) = \sigma(\beta(u) \cup \alpha(v_2))$, we conclude that $\beta(u) \cup \alpha(v_1) = \beta(u) \cup \alpha(v_2)$. Therefore $g(u)$ does not depend on the choice of $v(u)$. We prove that g is continuous. Take a sequence $\{u_m\}_{m=1}^{\infty}$ converging to $u \in [0,1]$. We may suppose that $\lim_{m\to\infty} v(u_m) = v$ for some $v \in [0,1]$. Thus $\lim_{m\to\infty} \beta(u_m) \cup \alpha(v(u_m)) = \beta(u) \cup \alpha(v)$. By the continuity of σ, $\sigma(\beta(u) \cup \alpha(v)) = t_0$. By the independence of $v(u)$, $g(u) = \beta(u) \cup \alpha(v)$. Therefore $\lim_{m\to\infty} g(u_m) = g(u)$ and g is continuous.

Define $\mathcal{B}_1 = g([0,1])$. Then $\mathcal{B}_1$ is a subcontinuum of $\mathcal{A}$. Given $u \in [0,1]$, $\{q\} \subset \beta(u)$ and each component of $\alpha(v(u))$ intersects L, so each component of $g(u)$ intersects $L \cup \{q\}$. Since $\beta(1) = G$, we have that $g(1) = \beta(1) \cup \alpha(v(1)) = G \cup \alpha(v(1))$. In the case that $v(1) \leq t_F$, we have that $\alpha(v(1)) \cup G \subset \alpha(t_F) \cup G = F \cup G = E$. Then $g(1) \subset E$. In the case that $t_F \leq v(1)$, we have that $E \subset g(1)$. We have shown that one of the sets E and $g(1)$ is contained in the other. Since the value of σ on both sets is the equal to t_0 and $t_0 > 0$, we conclude that $E = g(1) \in \mathcal{B}_1$. Set $E_1 = g(0) = \{q\} \cup \alpha(v(0))$.

This finishes the construction of $\mathcal{B}_1$.

Given $p \in X$, the set $\alpha(0) \cup \{p\} = L \cup \{p\}$ is finite, so $\sigma(\alpha(0) \cup \{p\}) = 0$. On the other hand, $\alpha(1) \cup \{p\}$ contains the set D, so $\sigma(\alpha(1) \cup \{p\}) \geq t_0$. Thus there exists $w(p) \in [0,1]$ such that $\sigma(\alpha(w(p)) \cup \{p\}) = t_0$. Define $h\colon X \to \mathcal{A}$ by $h(p) = \alpha(w(p)) \cup \{p\}$. Proceeding as we did with g, it is possible to prove that h does not depend on the choice of $w(p)$ and that h is continuous.

Define $\mathcal{B}_2 = h(X)$. Then $\mathcal{B}_2$ is a subcontinuum of $\mathcal{A}$. Since $h(q) = \alpha(w(q)) \cup \{q\}$, h does not depend on $w(q)$ and $E_1 = \{q\} \cup \alpha(v(0)) \in \mathcal{A}$, we conclude that $h(q) = E_1$. Therefore, $E_1 \in \mathcal{B}_2$.

Fix a point $p_0 \in L$. Then $h(p_0) = \{p_0\} \cup \alpha(w(p_0)) \subset \alpha(w(p_0)) \subset D$. Since $\sigma(h(p_0)) = t_0 = \sigma(D)$ and $t_0 > 0$, we conclude that $h(p_0) = D$. Hence $D \in \mathcal{B}_2$.

Define $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$.

In the case that $G = \{q\}$, $h(q) = \alpha(w(q)) \cup \{q\} = \alpha(w(q)) \cup G$. Since $F \in \alpha([0,1])$, one of the sets $F \cup G$ or $\alpha(w(q)) \cup G$ is contained in the other. Since σ takes the value t_0 on both sets and $t_0 > 0$, we conclude that $F \cup G = \alpha(w(q)) \cup G = h(q)$. Thus $E \in \mathcal{B}_2$, $\mathcal{B} = \mathcal{B}_2$ is a subcontinuum of $\mathcal{A}$ containing the set $\{D, E\}$ and each element of $\mathcal{B}$ is of the form $K = K_0 \cup \{p\}$, where $K_0 \in \alpha([0,1])$ and $p \in X$. Notice that each component of K_0 intersects L.

In the case that $G \neq \{q\}$, since $E_1 \in \mathcal{B}_1 \cap \mathcal{B}_2$, we have that $\mathcal{B}$ is a subcontinuum of $\mathcal{A}$ that contains the elements D and E and for each $K \in \mathcal{B}$, if $K \in \mathcal{B}_1$ each component of K intersects $L \cup \{q\}$ and if $K \in \mathcal{B}_2$, then $K = K_0 \cup \{p\}$ for some $K_0 \in \alpha([0,1])$ and $p \in X$, so each component of K_0 intersects L. This ends the proof of the claim. □

In order to prove that $\mathcal{A}$ is not irreducible, take $A, B \in \mathcal{A}$. Suppose that $A_1, \ldots, A_r$ are the components of A and $B_1, \ldots, B_s$ are the components of B. Fix points $p_1 \in A_1, \ldots, p_r \in A_r$ and $q_1 \in B_1, \ldots, q_s \in B_s$. We consider two cases.

Case 1. $t_0 = 0$. For each $i \in \{1, \ldots, r\}$, $A_i = \{p_i\}$ and for each $j \in \{1, \ldots, s\}$, $B_j = \{q_j\}$. Fix two points $p_0 \neq q_0$ in $X \setminus (A \cup B)$. Let $\mathcal{C} = \{\{p_1\} \cup E : E \in F_{n-1}(X)\}$. Then $\mathcal{C}$ is a subcontinuum of $\mathcal{A}$ such that $A, \{p_1\} \in \mathcal{C}$ and $\{p_0, q_0\} \notin \mathcal{C}$. Similarly, there exists a subcontinuum $\mathcal{D}$ of $\mathcal{A}$ such that $B, \{q_1\} \in \mathcal{D}$ and $\{p_0, q_0\} \notin \mathcal{D}$. Then $\mathcal{C} \cup \mathcal{D} \cup F_1(X)$ is a subcontinuum of $\mathcal{A}$ containing $\{A, B\}$ and not having the element $\{p_0, q_0\}$. Therefore, in this case, $\mathcal{A}$ is not irreducible.

Case 2. $t_0 > 0$. Taking an order arc from a one-point set to X, it is possible to show that there exists a connected element W in $\mathcal{A}$. Since $t_0 < 1$, the set $X \setminus W$ is infinite. Fix a point $z \in X \setminus (W \cup \{p_1, \dots, p_r\} \cup \{q_1, \dots, q_s\})$. Using an order arc from $\{z\}$ to X, it is possible to construct a subcontinuum M_0 of X with the properties: M_0 is non-degenerate, $M_0 \subset X \setminus (W \cup \{p_1, \dots, p_r\} \cup \{q_1, \dots, q_s\})$ and $\sigma(M_0) < t_0$. Since $\sigma(W \cup M_0) > t_0$ ($n \geq 2$) using again an order arc from a one-point set to W it is possible to obtain a subcontinuum N_0 of W such that the set $T = M_0 \cup N_0$ belongs to $\mathcal{A}$. Notice that $M_0 \cap N_0 = \emptyset$, so T has a component (the continuum M_0) not intersecting $\{p_1, \dots, p_r\} \cup \{q_1, \dots, q_s\}$.

We are going to show that there exist a subcontinuum S of X and a subcontinuum $\mathcal{C}$ of $\mathcal{A}$ such that $\{A, S\} \subset \mathcal{C}$ and $T \notin \mathcal{C}$. In the case that A is connected, set $S = A$ and $\mathcal{C} = \{A\}$. Since all the components of A intersect $\{p_1, \dots, p_r\}$, by the previous paragraph, we infer that $A \neq T$, so $T \notin \mathcal{C}$. Suppose then that A is not connected. Then $r > 1$.

We apply the Claim 2 to $F = A_1 \cup \dots \cup A_{r-1}$, $G = A_r$, $E = A$, $L = \{p_1, \dots, p_{r-1}\}$ and $q = p_r$. Taking an order arc from F to X, it is possible to find an element $S_1 \in \mathcal{A}$ such that each component of S_1 intersects F. Since each component of F intersects L, it is possible to find an order arc $\alpha\colon [0,1] \to C_n(X)$ such that $\alpha(0) = L$, $\alpha(1) = S_1$ and $F \in \alpha([0,1])$. Then there exists a subcontinuum $\mathcal{B}$ of $\mathcal{A}$ such that $\{S_1, A\} \subset \mathcal{B}$ and for each $K \in \mathcal{B}$, either each component of K intersects $\{p_1, \dots, p_r\}$ or $K = K_0 \cup \{p\}$, where each component of K_0 intersects L and $p \in X$.

Since the component M_0 of T is not degenerate and it does not intersect $\{p_1, \dots, p_r\}$, we have that $T \notin \mathcal{B}$. Since each component of S_1 intersects $\{p_1, \dots, p_{r-1}\}$, we obtain that S_1 has at most $r-1$ components. Suppose that $D_1, \dots, D_k$ are the components of S_1. Then we can choose points $p_{l_1} \in D_1, \dots, p_{l_k} \in D_k$.

We have shown that A can be connected by a continuum contained in $\mathcal{A} \setminus \{T\}$ with an element $S_1 \in \mathcal{A}$ having less components than A. In the case that S_1 is connected, let $S = S_1$ and $\mathcal{C} = \mathcal{B}$. In the case that S_1 is not connected, we can repeat the procedure to obtain an element $S_2 \in \mathcal{A}$ and a continuum $\mathcal{B}_1$ contained in $\mathcal{A} \setminus \{T\}$ such that $\{S_1, S_2\} \subset \mathcal{B}_1$, S_2 has less components than S_1 and each component of S_2 intersects $\{p_{l_1}, \dots, p_{l_k}\}$. If S_2 is connected, let $S = S_2$ and $\mathcal{C} = \mathcal{B}_1 \cup \mathcal{B}_2$. In the case that S_2 is not connected, we repeat this process until we obtain a connected set $S_m = S$ and a continuum $\mathcal{C} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_m$ contained in $\mathcal{A} \setminus \{T\}$ such that $\{A, S\} \subset \mathcal{C}$.

In a similar way, there exist a connected element $S_0 \in \mathcal{A}$ and a continuum $\mathcal{C}_0$ contained in $\mathcal{A} \setminus \{T\}$ such that $\{S_0, B\} \in \mathcal{C}_0$. Define

$$\mathcal{C}^* = \mathcal{C} \cup \mathcal{C}_0 \cup (\sigma|C(X))^{-1}(t_0).$$

Since $\sigma|C(X)$ is a Whitney map for $C(X)$, we have that $\sigma|C(X))^{-1}(t_0)$ is a Whitney level for $C(X)$ and then this set is connected. Thus $\mathcal{C}^*$ is a subcontinuum of $\mathcal{A}$. Since T has two components, $T \notin \mathcal{C}^*$. Therefore $\mathcal{C}^*$ is a proper subcontinuum of $\mathcal{A}$ containing A and B. Hence $\mathcal{A}$ is not irreducible. This finishes the proof of Theorem 1.

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