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THE UNIFORM EFFROS PROPERTY AND LOCAL HOMOGENEITY

Sergio Macías

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ABSTRACT. Kathryn F. Porter wrote a nice paper about several definitions of local homogeneity [*Local homogeneity*, JP Journal of Geometry and Topology **9** (2009), 129–136]. In this paper, she mentions that G. S. Ungar defined a uniformly locally homogeneous space [*Local homogeneity*, Duke Math. J. **34** (1967), 693–700]. We realized that this notion is very similar to what we call the uniform property of Effros [*On Jones' set function $\mathcal{T}$ and the property of Kelley for Hausdorff continua*, Topology Appl. **226** (2017), 51–65]. Here, we compare the uniform property of Effros with the uniform local homogeneity. We also consider other definitions of local homogeneity given in Porter's paper and compare them with the uniform property of Effros. We show that in the presence of compactness, the uniform property of Effros is equivalent to uniform local homogeneity and the local homogeneity according to Ho. With this result, we can change the hypothesis of the uniform property of Effros in Jones' and Prajs' decomposition theorems to uniform local homogeneity and local homogeneity according to Ho. We add to these two results the fact that the corresponding quotient space also has the uniform property of Effros.

1. Introduction

Kathryn F. Porter [11] wrote a nice paper about several definitions of local homogeneity. In particular, she mentions the uniform local homogeneity defined by Gerald S. Ungar [13]. We realized that this notion is very similar to the uniform property of Effros that we gave to present a nonmetric version of Jones' Aposyndetic Decomposition Theorem [7]. It is known that a metric continuum X is homogeneous if and only if X has the property of Effros [9: Theorems 4.2.31 and 4.2.38]. We were interested in proving a nonmetric version of Jones' Aposyndetic Decomposition Theorem [9: Theorem 5.1.18] and of Prajs' Mutual Aposyndetic Decomposition Theorem [12: Theorem 3.1]. We did that in [8: Theorem 4.3] and [8: Theorem 5.9], respectively.

In the present paper, we consider several definitions of local homogeneity given in Porter's paper and compare them with the uniform property of Effros. We show that in the presence of compactness, the uniform property of Effros is equivalent to the uniform local homogeneity and the local homogeneity according to Ho (Theorem 3.8). Using this theorem, we can change the hypothesis of the uniform property of Effros in Jones' [10: Theorem 3.3.8] and Prajs' [10: Theorem 3.4.9] decomposition theorems to uniform local homogeneity and local homogeneity according to Ho and, in both cases, we added here the fact that the corresponding quotient space also has the uniform property of Effros (Theorem 4.5 and Theorem 4.6, respectively).

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2. Preliminaries

Let Z be a Hausdorff space. If A is a subset of Z, then $\mathrm{Int}_Z(A)$ denotes the interior of A in Z. A *map* is a continuous function. If Z is a Hausdorff space, 1_Z denotes the identity map on Z. A *compactum* is a compact Hausdorff space, and a *continuum* is a connected compactum. A topological space Z is *homogeneous* provided that for each pair z_1 and z_2 of its points, there exists a homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(z_1) = z_2$.

Let Z be a Hausdorff space. If V and U are subsets of $Z \times Z$, then

$$-V = \{(z', z) \mid (z, z') \in V\}$$

and

$$V + U = \{(z, z'') \mid \text{there exists } z' \in Z \text{ such that } (z, z') \in V \text{ and } (z', z'') \in U\}.$$

We write $1V = V$ and for each positive integer n, $(n+1)V = nV + 1V$.

The *diagonal* of Z is the set $\Delta_Z = \{(z, z) \mid z \in Z\}$. An *entourage* of the diagonal of Z is a subset V of $Z \times Z$ such that $\Delta_Z \subset V$ and $V = -V$. The family of entourages of the diagonal of Z is denoted by $\mathfrak{D}_Z$. If $V \in \mathfrak{D}_Z$ and $(z, z') \in V$, then we write $\rho_Z(z, z') < V$. If $V \in \mathfrak{D}_Z$ and $z \in Z$, then $B(z, V) = \{z' \in Z \mid \rho_Z(z, z') < V\}$. We also have that if z, z' and z'' are points of Z, and V and U belong to $\mathfrak{D}_Z$, then the following hold [3: p. 426]:

(i) $\rho_Z(z, z) < V$.

(ii) $\rho_Z(z, z') < V$ if and only if $\rho_Z(z', z) < V$.

(iii) If $\rho_Z(z, z') < V$ and $\rho_Z(z', z'') < U$, then $\rho_Z(z, z'') < V + U$.

Let Z be a nonempty set. A *uniformity* on Z is a subfamily $\mathfrak{U}$ of $\mathfrak{D}_Z \setminus \{\Delta_Z\}$ such that:

(1) if $V \in \mathfrak{U}$, $U \in \mathfrak{D}_Z$ and $V \subset U$, then $U \in \mathfrak{U}$;

(2) if V and U belong to $\mathfrak{U}$, then $V \cap U \in \mathfrak{U}$;

(3) for every $V \in \mathfrak{U}$, there exists $U \in \mathfrak{U}$ such that $2U \subset V$;

(4) $\bigcap\{V \mid V \in \mathfrak{U}\} = \Delta_Z$.

A *uniform space* is a pair $(Z, \mathfrak{U})$ consisting of a nonempty set Z and a uniformity on the set Z. For any uniformity $\mathfrak{U}$ on a set Z, the family $\mathfrak{G} = \{G \subset Z \mid \text{for every } z \in G, \text{ there exists } V \in \mathfrak{U}$ such that $B(z, V) \subset G\}$ is a topology on the set Z [3: 8.1.1]. The topology $\mathfrak{G}$ is called the *topology induced by the uniformity* $\mathfrak{U}$. It is well known that a topology is induced by a uniformity if and only if it is Tychonoff [3: 8.1.20]. Note that, if the topology of Z is induced by a uniformity $\mathfrak{U}$ and $V \in \mathfrak{U}$, then, by [3: 8.1.3], $\mathrm{Int}_Z(B(z, V))$ is an open neighbourhood of z.

Remark 2.1. Let Z be a Tychonoff space and let $\mathfrak{U}$ be a uniformity of Z that induces its topology. If $V \in \mathfrak{U}$, then we define the *cover* of Z, $\mathfrak{C}(V) = \{B(z, V) \mid z \in Z\}$.

Remark 2.2. Note that by [3: 8.3.13], for every compactum Z, there exists a unique uniformity $\mathfrak{U}_Z$ on Z that induces the original topology of Z.

We need the following result [3: 8.3.G]:

Theorem 2.3. *Let Z be a compactum and let $\mathfrak{U}_Z$* (Remark 2.2) *be the unique uniformity of Z that induces its topology. Then for every open cover $\mathcal{W}$ of Z, there exists $V \in \mathfrak{U}_Z$ such that $\mathfrak{C}(V)$ refines $\mathcal{W}$.*

To know more about uniformities see [3: Chapter 8].

Let Z be a topological space and let $g\colon Z \to Z$ be a map. Then the *graph* of g is the set $\Gamma(g) = \{(z, g(z)) \mid z \in Z\}$.

Remark 2.4. Let Z be a Tychonoff space, let $\mathfrak{U}$ be a uniformity that induces the topology of Z, let $U \in \mathfrak{U}$ and let $g\colon Z \to Z$ be a map. Then the fact that $\Gamma(g) \subset U$ is equivalent to the fact that $\rho_Z(z, g(z)) < U$ for all points z of Z.

Notation 2.5. Let Z be a continuum and let $\mathcal{H}(Z)$ be its group of homeomorphisms with compact-open topology. If z is an element of Z, then define $\gamma_z\colon \mathcal{H}(Z) \to Z$ by $\gamma_z(h) = h(z)$.

A homogeneous continuum Z is said to be an *Effros continuum* if γ_z is an open map for all points z of Z (Notation 2.5).

Let Z be a Tychonoff space and let $\mathfrak{U}$ be a uniformity that induces the topology of Z. Then Z has the *uniform property of Effros with respect to* $\mathfrak{U}$, provided that for each $U \in \mathfrak{U}$, there exists $V \in \mathfrak{U}$ such that if z_1 and z_2 are two points of Z with $\rho_Z(z_1, z_2) < V$, there exists a homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(z_1) = z_2$ and $\rho_Z(z, h(z)) < U$ for all elements z of Z. Note that this last statement is equivalent to the fact that $\Gamma(h) \subset U$ (Remark 2.4). The entourage V is called an *Effros entourage for* U. A homeomorphism $h\colon Z \twoheadrightarrow Z$ satisfying $\rho_Z(z, h(z)) < U$, for all points z of Z, is called a *U-homeomorphism* [10: Definition 1.4.58].

The next theorem is [10: Theorem 1.4.59].

Theorem 2.6. *Let Z be a connected Tychonoff space and let $\mathfrak{U}$ be a uniformity that induces the topology of Z. If Z has the uniform property of Effros with respect to $\mathfrak{U}$, then Z is a homogeneous space.*

Throughout the paper, all the spaces are Tychonoff spaces.

3. The uniform property of Effros vs. local homogeneity

We compare the uniform property of Effros with various types of local homogeneity. We begin with the following technical lemma which is useful for the proof of several results.

Lemma 3.1. *Let Z be a compactum. For each element z of Z, let W_z be an open subset of Z containing z. Suppose that U belongs to $\mathfrak{U}_Z$ (*Remark 2.2*) and satisfies that given $z' \in W_z$, for some z in Z, there exists a U-homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(z) = z'$. If $V \in \mathfrak{U}_Z$ and $\mathfrak{C}(V)$ refines the open cover $\mathfrak{W} = \{W_z \mid z \in Z\}$, then the following is true:*

($\star$) *If z_1 and z_2 belong to Z and $\rho_Z(z_1, z_2) < V$, then there exists a homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(z_1) = z_2$ and $\rho_Z(z, h(z)) < 2U$, for all points z in Z.*

Proof. Since Z is a compactum, by Theorem 2.3, there exists $V \in \mathfrak{U}_Z$ (Remark 2.2) such that $\mathfrak{C}(V)$ refines $\mathfrak{W}$. Let z_1 and z_2 be two elements of Z such that $\rho_Z(z_1, z_2) < V$. Since $\mathfrak{C}(V)$ refines $\mathfrak{W}$, there exists an element z_0 of Z such that $\{z_1, z_2\} \subset W_{z_0}$. By our assumption, there exist two U-homeomorphisms $h_2, h_1\colon Z \twoheadrightarrow Z$ such that $h_1(z_0) = z_1$, $h_2(z_0) = z_2$. Let $h = h_2 \circ h_1^{-1}$. Then $h\colon Z \twoheadrightarrow Z$ is a homeomorphism and $h(z_1) = z_2$. Let z be an element of Z. Since $\rho_Z(z, h(z)) = \rho_Z(z, h_2 \circ h_1^{-1}(z))$, $\rho_Z(z, h_1^{-1}(z)) = \rho_Z(h_1 \circ h_1^{-1}(z), h_1^{-1}(z)) < U$ and $\rho_Z(h_1^{-1}(z), h_2 \circ h_1^{-1}(z)) < U$, we have that $\rho_Z(z, h(z)) < 2U$. □

We start with the definition of the uniform local homogeneity given by Ungar [13]. A Tychonoff space Z is *uniformly locally homogeneous with respect to* $\mathfrak{U}$ (a uniformity that induces the topology of Z), provided that for each element z of Z and every $U \in \mathfrak{U}$, there exists an open subset $O_{U,z}$ with $z \in O_{U,z}$ such that if $z' \in O_{U,z}$, there exists a homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(z) = z'$ and $\Gamma(h) \subset U$.

Theorem 3.2. *Let Z be a Tychonoff space and let $\mathfrak{U}$ be a uniformity that induces the topology of Z. If Z has the uniform property of Effros with respect to $\mathfrak{U}$, then Z is uniformly locally homogeneous with respect to $\mathfrak{U}$.*

Proof. Let z be a point of Z and let $U \in \mathfrak{U}$. Since Z has the uniform property of Effros with respect to $\mathfrak{U}$, there exists an Effros entourage V for U. Note that $\mathrm{Int}_Z(B(z,V))$ is an open subset of Z containing z. Let $z' \in \mathrm{Int}_Z(B(z,V))$. Then $\rho_Z(z,z') < V$. Thus, there exists a U-homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(z) = z'$. Since h is a U-homeomorphism, we have that $\Gamma(h) \subset U$ (Remark 2.4). Therefore, Z is uniformly locally homogeneous with respect to $\mathfrak{U}$. $\square$

The next result shows that, for compacta, the converse implication of Theorem 3.2 is true.

Theorem 3.3. *Let Z be a compactum. If Z is uniformly locally homogeneous, then Z has the uniform property of Effros.*

Proof. Let U and U' be elements of $\mathfrak{U}_Z$ (Remark 2.2) such that $2U' \subset U$. Since Z is uniformly locally homogeneous, for each z in Z, there exists an open subset $O_{U',z}$ of Z such that if $z' \in O_{U',z}$, then there exists a homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(z) = z'$ and $\Gamma(h) \subset U'$. Observe that $\mathfrak{O} = \{O_{U',z} \mid z \in Z\}$ is an open cover of Z. Since Z is a compactum, by Theorem 2.3, there exists $V \in \mathfrak{U}_Z$ such that $\mathfrak{C}(V)$ refines $\mathfrak{O}$. Let z_1 and z_2 be two elements of Z such that $\rho_Z(z_1,z_2) < V$. By Lemma 3.1 and Remark 2.4, there exists a homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(z_1) = z_2$ and $\rho_Z(z,h(z)) < 2U'$ for all elements z of Z. Since $2U' \subset U$, we obtain that $\rho_Z(z,h(z)) < U$, for every point z in Z. Therefore, V is an Effros entourage for U and Z has the uniform property of Effros. $\square$

Question 3.4. Is Theorem 3.3 true if Z is not a compactum?

A topological space Z is *locally homogeneous$_H$*, according to Ho [6], provided that for each element z in Z and every open subset W of Z containing z, there exists an open subset W' of Z satisfying that $z \in W' \subset W$ and that for each element z' of W', there exists a homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(W') \subset W$ and $h(z) = z'$.

Theorem 3.5. *Let Z be a Tychonoff space and let $\mathfrak{U}$ be a uniformity that induces the topology of Z. If Z has the uniform property of Effros with respect to $\mathfrak{U}$, then Z is locally homogeneous$_H$.*

Proof. Let z be a point of Z and let W be an open subset of Z containing z. Then there exists $U \in \mathfrak{U}$ such that $B(z,U) \subset W$. Let $U' \in \mathfrak{U}$ be such that $2U' \subset U$. Since Z has the uniform property of Effros with respect to $\mathfrak{U}$, there exists an Effros entourage V for U'. Without loss of generality, we assume that $V \subset U'$. Let $z' \in \mathrm{Int}_Z(B(z,V))$. Then $\rho_Z(z,z') < V$. Hence, there exists a U'-homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(z) = z'$. Since h is a U'-homeomorphism, for each $z'' \in \mathrm{Int}_Z(B(z,V))$, we have that $\rho_Z(z'',h(z'')) < U'$. Also, since $\rho_Z(z,z'') < V$ and $V \subset U'$, we obtain that $\rho_Z(z,h(z'')) < 2U'$. Since $2U' \subset U$, we have that $\rho_Z(z,h(z'')) < U$. Hence, $h(\mathrm{Int}_Z(Bz,V)) \subset B(z,U) \subset W$. Therefore, Z is locally homogeneous$_H$. $\square$

In the following theorem, we show that the converse implication of Theorem 3.5 is true for compacta.

Theorem 3.6. *Let Z be a compactum. If Z is locally homogeneous$_H$, then Z has the uniform property of Effros.*

Proof. Let U and U' be elements of $\mathfrak{U}_Z$ (Remark 2.2) such that $2U' \subset U$. For each point z of Z, we consider the open subset $\mathrm{Int}_Z(B(z,U'))$ of Z. Since Z is locally homogeneous$_H$, for each point z of Z, there exists an open subset O_z such that $z \in O_z \subset \mathrm{Int}_Z(B(z,U'))$ and that for each element z' of O_z, there exists a homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(O_z) \subset \mathrm{Int}_Z(B(z, U'))$ and $h(z) = z'$. Note that $\mathfrak{O} = \{O_z \mid z \in Z\}$ is an open cover of Z. By Theorem 2.3, there exists $V \in \mathfrak{U}_Z$ such that $\mathfrak{C}(V)$ refines $\mathfrak{O}$. Let z_1 and z_2 be two elements of Z such that $\rho_Z(z_1, z_2) < V$. By Lemma 3.1, there exists a homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(z_1) = z_2$ and $\rho_Z(z, h(z)) < 2U'$ for all elements z of Z. Since $2U' \subset U$, we obtain that $\rho_Z(z, h(z)) < U$, for every point z in Z. Therefore, V is an Effros entourage for U and Z has the uniform property of Effros. □

***Example* 3.7.** Let X be the set of integers, $\mathbb{Z}$, with the cofinite topology. By [11: p. 133], X is homogeneous$_H$ but not uniformly locally homogenous. Thus, by Theorem 3.2, X does not have the uniform property of Effros. Hence, the converse of Theorem 3.6 is not true if Z is not a compactum.

From Theorems 3.2, 3.3, 3.5, and 3.6, we obtain:

Theorem 3.8. *Let Z be a compactum. Then the following are equivalent:*

(1) *Z has the uniform property of Effros.*

(2) *Z is uniformly locally homogeneous.*

(3) *Z is locally homogeneous$_H$.*

A topological space Z is *locally homogeneous$_F$*, *according to Fora* [4], provided that for each point z of Z, there exists an open subset W of Z containing z such that for every element z' of W, there exists a homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(z) = z'$.

Theorem 3.9. *Let Z be a Tychonoff space and let $\mathfrak{U}$ be a uniformity that induces the topology of Z. If Z has the uniform property of Effros with respect to $\mathfrak{U}$, then Z is locally homogeneous$_F$.*

Proof. Suppose Z has the uniform property of Effros with respect to $\mathfrak{U}$. By Theorem 3.5, we have that Z is locally homogeneous$_H$. By [11: Theorem 4], we obtain that Z is locally homogeneous$_F$. □

Remark 3.10. The converse of Theorem 3.9 is false. By [11: p. 132], we have that local homogeneity$_H$ does not follow from local homogeneity$_F$. Thus, by Theorem 3.5, the uniform property of Effros does not follow from the local homogeneity$_F$.

The next theorem gives us properties of Effros continua.

Theorem 3.11. *Let Z be an Effros continuum. Then:*

(1) *Z has the uniform property of Effros.*

(2) *Z uniformly locally homogeneous.*

(3) *Z is locally homogeneous$_H$.*

(4) *Z is locally homogeneous$_F$.*

Proof. Suppose Z is an Effros continuum. By [10: Theorem 1.4.60], Z has the uniform property of Effros. Now, the theorem follows from Theorems 3.2, 3.5 and 3.9. □

A topological space Z is *strongly locally homogeneous*, *according to Ford* [5], provided that for any element z of Z and any open subset W of Z containing z, there exists an open subset $O_{W,z}$ of Z with $z \in O_{W,z} \subset W$ such that if z' is an element of $O_{W,z}$, then there exists a homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(z) = z'$ and $h|_{Z\setminus O_{W,z}} = 1_{Z\setminus O_{W,z}}$.

Theorem 3.12. *Let Z be a compactum. If Z is strongly locally homogeneous, then Z has the uniform property of Effros.*

P r o o f. Let U and U' be elements of $\mathfrak{U}_Z$ (Remark 2.2) such that $4U' \subset U$. For each point z of Z, we consider the open subset $\mathrm{Int}_Z(B(z, U'))$ of Z. Since Z is strongly locally homogeneous, for each z in Z, there exists an open subset $O_{U',z}$ of Z such that $z \in O_{U',z} \subset \mathrm{Int}_Z(B(z, U'))$ and if $z' \in O_{U,z}$, then there exists a homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(z) = z'$ and $h|_{Z\setminus O_{U',z}} = 1_{Z\setminus O_{U',z}}$. Observe that $\mathfrak{O} = \{O_{U',z} \mid z \in Z\}$ is an open cover of Z. Since Z is a compactum, by Theorem 2.3, there exists $V \in \mathfrak{U}_Z$ such that $\mathfrak{C}(V)$ refines $\mathfrak{O}$. Let z_1 and z_2 be two elements of Z such that $\rho_Z(z_1, z_2) < V$. Since $\mathfrak{C}(V)$ refines $\mathfrak{O}$, there exists an element z_0 of Z such that $\{z_1, z_2\} \subset O_{U',z_0}$. By our assumption, there exist two homeomorphisms $h_2, h_1\colon Z \twoheadrightarrow Z$ such that $h_1(z_0) = z_1$, $h_2(z_0) = z_2$,

$$h_1|_{Z\setminus O_{U',z_0}} = 1_{Z\setminus O_{U',z_0}} \quad \text{and} \quad h_2|_{Z\setminus O_{U',z_0}} = 1_{Z\setminus O_{U',z_0}}.$$

Let $h = h_2 \circ h_1^{-1}$. Then $h\colon Z \twoheadrightarrow Z$ is a homeomorphism and $h(z_1) = z_2$. Let z be an element of O_{U',z_0}. Since $\{z, h_1^{-1}(z)\} = \{h_1 \circ h_1^{-1}(z), h_1^{-1}(z)\} \subset O_{U',z_0} \subset \mathrm{Int}_Z(B(z_0, U'))$ and $\{h_1^{-1}(z), h_2 \circ h_1^{-1}(z)\} \subset O_{U',z_0} \subset \mathrm{Int}_Z(B(z_0, U'))$, we have that $\rho_Z(z, h_1^{-1}(z)) = \rho_Z(h_1 \circ h_1^{-1}(z), h_1^{-1}(z)) < 2U'$, $\rho_Z(h_1^{-1}(z), h_2 \circ h_1^{-1}(z)) < 2U'$ and $\rho_Z(z, h(z)) < 4U'$. Since $4U' \subset U$, we obtain that $\rho_Z(z, h(z)) < U$. If $z \in Z \setminus O_{U',z_0}$, then $z = h(z)$ and, in particular, $\rho_Z(z, h(z)) < U$. Therefore, V is an Effros entourage for U and Z has the uniform property of Effros. □

As a consequence of Theorems 2.6 and 3.12, we obtain:

Corollary 3.13. *If Z is a strongly locally homogeneous continuum, then Z is a homogenous continuum.*

A topological space Z is *closed-homogeneous*, *according to Fora* [4], provided that for each pair of points z and z' of Z and a closed subset K of Z, with $K \subset Z\setminus\{z, z'\}$, there exists a homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(z) = z'$ and $h|_K = 1_K$.

Theorem 3.14. *Let Z be a compactum. If Z is closed-homogeneous, then Z has the uniform property of Effros.*

P r o o f. Suppose Z is closed-homogeneous compactum, by [11: Theorem 10], Z is strongly locally homogeneous. Hence, by Theorem 3.12, Z has the uniform property of Effros. □

Remark 3.15. Note that, by [11: Theorem 9], each closed-homogeneous space is homogeneous.

A continuum Z is *almost connected im kleinen at a point z of Z*, provided that for each open subset W of Z containing z, there exists a subcontinuum K of Z such that $\mathrm{Int}_Z(K) \neq \emptyset$ and $K \subset W$. The continuum Z is *connected im kleinen at a point z of Z* if for each open subset W of Z containing z, there exists a subcontinuum K of Z such that $z \in \mathrm{Int}_Z(K) \subset K \subset W$.

Theorem 3.16. *If Z is a continuum that is either uniformly locally homogeneous, locally homogeneous$_H$, strongly locally homogeneous or closed-homogeneous, then the following are equivalent:*

(1) *Z is locally connected.*
(2) *Z is locally connected at some point.*
(3) *Z is connected im kleinen at some point.*
(4) *Z is almost connected im kleinen at every point.*
(5) *Z is almost connected im kleinen at some point.*

P r o o f. If Z is uniformly locally homogeneous or locally homogeneous$_H$, by Theorem 3.8, Z has the uniform property of Effros. If Z is strongly locally homogeneous, by Theorem 3.12, Z has the uniform property of Effros. If Z is closed-homogenoeus, by Theorem 3.14, Z has the uniform property of Effros. Now, the theorem follows from [10: Theorem 1.4.61]. □

We end this section with a summary of relationships between the types of spaces we have considered. Let us agree with the following notation.

UPE means: Uniform property of Effros.
ULH means: Uniformly locally homogeneous.
LH$_H$ means: Locally homogeneous$_H$.
LH$_F$ means: Locally homogeneous$_F$.
SLH means: Strongly locally homogeneous.
C-H means: Closed-homogeneous.

Yes* means that the implication is true for compacta.

We include the number of the result where the implication has been proved in the paper. The rest of the implications are taken from the table on [11: p. 136].

Summary of Relationships

$\Longrightarrow$	**UPE**	**ULH**	**LH**$_H$	**LH**$_F$	**SLH**	**C-H**
UPE	$\star$	Yes (3.2)	Yes (3.4)	Yes (3.7)	?	?
ULH	Yes* (3.3)	$\star$	Yes	Yes	No	No
LH$_H$	Yes* (3.5)	No	$\star$	Yes	No	No
LH$_F$	No (3.8)	No	No	$\star$	No	No
SLH	Yes* (3.10)	Yes	Yes	Yes	$\star$	No
C-H	Yes* (3.12)	No	Yes	Yes	Yes	$\star$

4. Uniform property of Effros

We start by saying that all the undefined terms in this section can be found in [10]. As we mention earlier, a metric continuum X is homogeneous if and only if X has the property of Effros [9: Theorems 4.2.31 and 4.2.38]. Hence, we might take this property for granted when dealing with homogeneous continua. It turns out that there exist homogeneous continua without the property of Kelley; hence, without the uniform property of Effros [2] (this follows from the fact that a continuum with the uniform property of Effros has the property of Kelley [10: Theorem 1.6.22]). In fact, there exist homogeneous continua that are not Effros continua either [1]. Thus, we decided to extend Jones' Aposyndetic Decomposition Theorem and Prajs' Mutual Aposyndetic Decomposition Theorem by showing that the corresponding quotient space has the uniform property of Effros (Theorem 4.2 and Theorem 4.4, respectively). We use the results of the previous section to substitute the hypothesis of uniform property of Effros in Jones' and Prajs' decomposition theorems by uniform local homogeneity, locally homogeneity$_H$, strong local homogeneity and closed-homogeneity (Theorems 4.5 and 4.6, respectively).

We need the following:

Notation 4.1. Given a continuum Z, we define Jones' set function $\mathcal{T}$ as follows: if A is a subset of Z, then

$$\mathcal{T}(A) = Z \setminus \{z \in Z \mid \text{there exists a subcontinuum } K \text{ of } Z \text{ such that } z \in \mathrm{Int}_Z(K) \subset K \subset Z \setminus A\}.$$

A thorough study of Jones' set function $\mathcal{T}$ is given in [10].

The following theorem is Jones' Aposyndetic Decomposition Theorem [10: Theorem 3.3.8], with the addition that the quotient space also has the uniform property of Effros.

Theorem 4.2. *Let Z be a decomposable continuum with the uniform property of Effros. If $\mathcal{G} = \{\mathcal{T}(\{z\}) \mid z \in Z\}$, then the following hold:*

(1) *$\mathcal{G}$ is a continuous, monotone and terminal decomposition of Z.*

(2) *The elements of $\mathcal{G}$ are cell-like, acyclic, homogeneous and mutually homeomorphic continua.*

(3) *The quotient map $q\colon Z \twoheadrightarrow Z/\mathcal{G}$ is uniformly completely regular and atomic.*

(4) *The quotient space $Z/\mathcal{G}$ is an aposyndetic homogeneous continuum with the uniform property of Effros and it does not contain nondegenerate proper terminal subcontinua.*

Proof. We show that $Z/\mathcal{G}$ has the uniform property of Effros. Let $\mathcal{U} \in \mathfrak{U}_{Z/\mathcal{G}}$. Since the quotient map, q, is continuous, we have that $(q\times q)^{-1}(\mathcal{U}) \in \mathfrak{U}_Z$. Since Z has the uniform property of Effros, there exists an Effros entourage V for $(q \times q)^{-1}(\mathcal{U})$. Since q is an open map [10: Corollary 1.1.24], we have that $(q\times q)(V) \in \mathfrak{U}_{Z/\mathcal{G}}$. We prove that $(q\times q)(V)$ is an Effros entourage for $\mathcal{U}$. Let χ_1 and χ_2 be two elements of $Z/\mathcal{G}$ such that $\rho_{Z/\mathcal{G}}(\chi_1, \chi_2) < (q \times q)(V)$. Then there exist two elements z_1 and z_2 in Z such that $q(z_1) = \chi_1$, $q(z_2) = \chi_2$ and $\rho_Z(z_1, z_2) < V$. Since V is an Effros entourage for $(q \times q)^{-1}(\mathcal{U})$, there exists a $(q \times q)^{-1}(\mathcal{U})$-homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(z_1) = z_2$. As in the proof of [10: Theorem 3.3.6], it can be seen that the map $\zeta\colon Z/\mathcal{G} \twoheadrightarrow Z/\mathcal{G}$ given by $\zeta(\chi) = q\circ h(q^{-1}(\chi))$ is a homeomorphism and $\zeta(\chi_1) = \chi_2$. We prove that ζ is a $\mathcal{U}$-homeomorphism. Let χ be a point of $Z/\mathcal{G}$ and let $z \in q^{-1}(\chi)$. Since h is a $(q \times q)^{-1}(\mathcal{U})$-homeomorphism, we have that $\rho_Z(z, h(z)) < (q \times q)^{-1}(\mathcal{U})$; i.e., $(z, h(z)) \in (q \times q)^{-1}(\mathcal{U})$. Hence, $(q \times q)(z, h(z)) \in \mathcal{U}$. Thus, since $q(z) = \chi$, we obtain that $\rho_{Z/\mathcal{G}}(\chi, \zeta(\chi)) < \mathcal{U}$. Therefore, ζ is $\mathcal{U}$-homeomorphism. Hence, $(q \times q)(V)$ is an Effros entourage for $\mathcal{U}$ and $Z/\mathcal{G}$ has the uniform property of Effros. The theorem now follows from [10: Theorem 3.3.8]. $\square$

We also need the following:

Notation 4.3. Let Z be a continuum and let z be an element of Z. Then

$$\begin{aligned} Q_z = \{z' \in Z \mid {} & \text{if } K_z \text{ and } K_{z'} \text{ are two subcontinua of } Z \text{ such that} \\ & z \in \operatorname{Int}_Z(K_z) \text{ and } z' \in \operatorname{Int}_Z(K_{z'}) \text{ then we have that } K_z \cap K_{z'} \neq \emptyset\}. \end{aligned}$$

The following theorem is Prajs' Mutual Aposyndetic Decomposition Theorem [10: Theorem 3.4.9], with the addition that the quotient space also has the uniform property of Effros. The proof of the part that the quotient space has the uniform property of Effros is similar to the one given for Theorem 4.2.

Theorem 4.4. *Let Z be a decomposable continuum with the uniform property of Effros. If $\mathcal{Q} = \{Q_z \mid z \in Z\}$, then the following hold:*

(1) *$\mathcal{Q}$ is a continuous decomposition of Z.*

(2) *The elements of $\mathcal{Q}$ are homogeneous mutually homeomorphic closed subsets of Z.*

(3) *The quotient map $q\colon Z \twoheadrightarrow Z/\mathcal{Q}$ is uniformly completely regular.*

(4) *The quotient space $Z/\mathcal{Q}$ is a mutually aposyndetic homogeneous continuum with the uniform property of Effros.*

Regarding Jones' Aposyndetic Decomposition Theorem, we have:

Theorem 4.5. *Let Z be a continuum and let $\mathcal{G} = \{\mathcal{T}(\{z\}) \mid z \in Z\}$. If Z is either uniformly locally homogeneous, locally homogeneous$_H$, strongly locally homogeneous or closed-homogeneous, then the following hold:*

(1) *$\mathcal{G}$ is a continuous, monotone and terminal decomposition of Z.*

(2) *The elements of $\mathcal{G}$ are cell-like, acyclic, homogeneous and mutually homeomorphic continua.*

(3) *The quotient map $q\colon Z \twoheadrightarrow Z/\mathcal{G}$ is uniformly completely regular and atomic.*

(4) *The quotient space $Z/\mathcal{G}$ is an aposyndetic homogeneous continuum with the uniform property of Effros and it does not contain nondegenerate proper terminal subcontinua.*

P r o o f. The proof of the fact that in each case Z has the uniform property of Effros is done in Theorem 3.16. Now the theorem follows from Theorem 4.2. □

Regarding Prajs' Mutual Aposyndetic Decomposition Theorem, we have:

Theorem 4.6. *Let Z be a continuum and let $\mathcal{Q} = \{Q_z \mid z \in Z\}$. If Z is either uniformly locally homogeneous, locally homogeneous$_H$, strongly locally homogeneous or closed-homogeneous, then the following hold:*

(1) *$\mathcal{Q}$ is a continuous decomposition of Z.*

(2) *The elements of $\mathcal{Q}$ are homogeneous mutually homeomorphic closed subsets of Z.*

(3) *The quotient map $q\colon Z \twoheadrightarrow Z/\mathcal{Q}$ is uniformly completely regular.*

(4) *The quotient space $Z/\mathcal{Q}$ is a mutually aposyndetic homogeneous continuum with the uniform property of Effros.*

The proof of Theorem 4.6 is similar to the one given for Theorem 4.5 except that we use Theorem 4.4 instead of Theorem 4.2.

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Instituto de Matemáticas
Universidad Nacional Autónoma de México
Circuito Exterior, Ciudad Universitaria
CDMX, C. P. 04510
MÉXICO
E-mail: macias@unam.mx