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ON THE BIVARIATE GENERALIZED GAMMA-LINDLEY DISTRIBUTION

Mohamed Ali Ghorbel[c] — Rahmouna Mecene

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ABSTRACT. In this research paper, we propose a new class of bivariate distributions called Bivariate Generalized Gamma-Lindley (BGGL) distribution with four parameters. This model is a mixture of independent Gamma random variables and bivariate generalized Lindley distribution. We investigate various properties of the new bivariate distribution such as graphical representation, joint moments and correlation. Furthermore, we derive a measure of entropy of this bivariate distribution. We also derive the distributions of the random variables $X_1 + X_2$, $X_1/(X_1 + X_2)$, X_1/X_2 and X_1X_2 as well as the corresponding moment properties when X_1 and X_2 follow the BGGL distribution. Additionally, we address two approximations of the product of the proposed model and assess their goodness of fit. Next, we elaborate the expectation maximization (E.M) algorithm in order to estimate the BGGL model parameters. Finally, we provide two concrete examples to demonstrate the applicability of the results.

1. Introduction

Statistical distributions are intrinsic in terms of describing and predicting real world phenomena. They are frequently used in life data analysis and reability engineering. However, many of these classical distributions do not provide enough flexibility for analysing different types of lifetime data and are not sufficient for modelization in several other areas such the discipline of signal processing applications or the area of image processing and computer vision (for more details, see [15]). Therefore, a new class of distribution was introduced by Lindley [12] to analyse failure time data, especially in application modeling stress-strength reability (see, for example, [5]). This new class of distribution can be also used in a wide variety of fields including biology, engineering and medicine. The one-parameter Lindley distribution is defined by its probability density function (pdf) expressed as

$$f(x;\theta) = \frac{\theta^2(1+x)\mathrm{e}^{-\theta x}}{1+\theta}, \qquad x > 0,\ \theta > 0. \tag{1}$$

It has been explored and generalized by numerous authors. Basically, Ghitany et al. [5] provided a treatment of the mathematical properties of equation (1). Next, Shanker et al. [22] derived a two-parameter Lindley distribution of which the one parameter Lindley distribution is a particular case, and they demonstrated its applicability in modeling waiting and survival times data. In [26], Zakerzadeh and Dolati introduced a three-parameter generalization of the one-parameter Lindley distribution and examined multiple properties of this new distribution. Furthermore,

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[c] Corresponding author.

Torabi et al. [24] set forward a four parameters extension of the generalized Lindley distribution. In addition, discrete versions of the Lindley distribution were proposed in literature by Gómez-Déniz and Calderín-Ojeda [7], Gómez-Déniz et al. [8], Hussain et al. [10] and many other authors. Zeghdoudi and Nedjar [27] proposed another extension of Lindley distribution called Gamma Lindley distribution which offers a more flexible model for modeling lifetime data and they investigated its properties. Recently, Laribi et al. [13] derived a new generalized Gamma-Lindley distribution through mixing the Gamma and the three-parameters generalized Lindley distributions. Its pdf is (for x, α, $\theta > 0$, $\gamma \geq 0$ and $\beta \geq 1$) indicated by

$$f\left(x;\alpha,\beta,\theta,\gamma\right) = \frac{\theta^{\alpha+1}x^{\alpha-1}\mathrm{e}^{-\theta x}}{\beta\Gamma(\alpha+1)(\gamma+\theta)}\left(\alpha + \left(\beta\gamma+\beta\theta-\theta\right)x\right), \tag{2}$$

where $\Gamma(\alpha) := \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x$ is the Euler Gamma function.

In this paper, we generate a new class of bivariate distributions called Bivariate Generalized Gamma-Lindley (BGGL) distribution. The joint pdf of this new distribution corresponds to

$$f_\Theta(x_1,x_2) = \frac{\theta^{2\alpha+1}(x_1x_2)^{\alpha-1}\mathrm{e}^{-\theta(x_1+x_2)}}{\beta(\gamma+\theta)\Gamma^2(\alpha+1)}\left(\alpha^2 + \theta\left(\beta\gamma+\beta\theta-\theta\right)x_1x_2\right), \tag{3}$$

where x_1, x_2, α $, \theta > 0$, $\gamma \geq 0$, $\beta \geq 1$ and $\Theta := (\alpha,\beta,\gamma,\theta)$.

The paper is organized as follows: In Section 2, we display some properties of our distribution such as joint moments, correlation and mixture construction of the BGGL distribution defined by equation (3). Additionally, for different values of parameters, we provide graphically a variety of forms of the BGGL density. In Section 3, we derive the exact densities and moment functions of the random variables $Y = X_1 + X_2$, $Z = X_1/(x_1+x_2)$, $V = X_1/X_2$ and $W = X_1X_2$, where (X_1, X_2) follows the BGGL distribution. Furthermore, we exhibit approximations for the distribution of $W = X_1X_2$, and discuss evidence for their robustness. The objective consists in providing simple approximations in terms of the beta and inverted beta distributions so as to facilitate the use of the known procedures for inference, prediction, etc. In Section 4, we determine the exact forms of the Rényi and Shannon entropies for the BGGL distribution. In Section 5, we investigate the estimation of the BGGL distribution parameters by the E.M algorithm. In the last section, we provide an application of the results within the field of precipitation data about rain and snow from France and we tackle another application in the field of demographic studies in USA.

2. Properties

In this section, we handle several outstanding properties of the BGGL distribution. This bivariate distribution is based essentially on a mixture of the product of two independent Gamma random variables (denoted by gamma(α,θ)) and bivariate generalized Lindley distribution (denoted by BGL(α,γ,θ)), introduced by Zakerzadeh and Dolati [26]. Both distributions are defined by the joint pdfs

$$f_{gg}(x_1,x_2;\alpha,\theta) = \frac{\theta^{2\alpha}(x_1x_2)^{\alpha-1}}{\Gamma^2(\alpha)}\mathrm{e}^{-\theta(x_1+x_2)}, \qquad x_1,x_2,\alpha,\ \theta > 0,$$

and

$$f_{BL}\left(x_1,x_2;\alpha,\theta,\gamma\right) = \frac{\theta^{2\alpha+1}(x_1x_2)^{\alpha-1}\left(\alpha^2+\gamma\theta x_1x_2\right)}{(\gamma+\theta)\Gamma^2(\alpha+1)}\mathrm{e}^{-\theta(x_1+x_2)},$$

where x_1, x_2, α, $\theta > 0$, $\gamma \geq 0$. Let (Y_1,Y_2) and (Z_1,Z_2) be two random vectors, such that Y_1 and Y_2 are independent random variables distributed according to gamma$(\alpha+1,\theta)$ and (Z_1,Z_2) is a vector distributed according to BGL(α, γ, θ). For $\beta \geq 1$, consider the random pair (X_1, X_2) which stands for the mixture between (Y_1, Y_2) with probability $\frac{(\beta-1)}{\beta}$, and (Z_1, Z_2) with probability $\frac{1}{\beta}$. This mixture law is called BGGL distribution and the joint pdf of (X_1, X_2) is obtained by using the total probability law:

$$\begin{aligned} f_\Theta(x_1, x_2) &= \frac{(\beta-1)}{\beta} f_{gg}\left(x_1, x_2; \alpha+1, \theta\right) + \frac{1}{\beta} f_{BL}\left(x_1, x_2; \alpha, \theta, \gamma\right) \\ &= \frac{\theta^{2\alpha+1}(x_1x_2)^{\alpha-1}\mathrm{e}^{-\theta(x_1+x_2)}}{\beta(\gamma+\theta)\Gamma^2(\alpha+1)} \left(\alpha^2 + \theta\left(\beta\gamma+\beta\theta-\theta\right)x_1x_2\right), \end{aligned} \tag{4}$$

where x_1, x_2, α, $\theta > 0$, $\gamma \geq 0$ and $\Theta := (\alpha, \beta, \gamma, \theta)$, which coincides with equation (3). Note that, the joint pdf (3) may be written in terms of the gamma function as:

$$f_\Theta(x_1, x_2) = \frac{\theta\left(\alpha^2 + x_1x_2\left(\gamma\theta + (\beta-1)(\gamma+\theta)\theta\right)\right)}{\beta(\gamma+\theta)\alpha^2} f_g\left(x_1; \alpha, \theta\right) f_g\left(x_2; \alpha, \theta\right).$$

It is easy to demonstrate that the univariate marginal density functions of equation (3) are of the form (2). When $\beta = 1$, the BGGL distribution reduces to the bivariate generalized Lindley distributions with three parameters (see [26]). For $\beta = 1$ and $\alpha = 1$, the BGGL distribution reduces to the bivariate generalized Lindley distributions with two parameters (see [23]). For $\beta = 1$ and $\gamma = 0$, the random variables X_1 and X_2 become independent and the joint pdf (3) reduces to the product of two gamma density functions with the same parameters.

In addition, Figure 1 depicts a few graphs of the pdf defined by equation (3) for different values of parameters. Here, one can record the wide range of forms that result from the BGGL distribution.

Furthermore, simple calculations yield that for each positive integer m and n, the joint (m, n)-th moment of the variables X_1 and X_2 having BGGL distribution, can be stated as

$$\mathbf{E}\left(X_1^m X_2^n\right) = \frac{\Gamma\left(\alpha+m\right)\Gamma\left(\alpha+n\right)}{\beta(\gamma+\theta)\Gamma^2(\alpha+1)\theta^{m+n}} \left(\left(\beta\theta+\beta\gamma-\theta\right)\left(\alpha+n\right)\left(\alpha+m\right)+\theta\alpha^2\right). \tag{5}$$

Now, substituting appropriately for m and n in equation (5), we obtain

$$\mathbf{E}(X_1X_2) = \frac{\left(\left(\beta\theta+\beta\gamma-\theta\right)\left(\alpha+1\right)^2+\theta\alpha^2\right)}{\beta(\gamma+\theta)\theta^2}.$$

Besides, we can easily show that the correlation coefficient of X_1 and X_2 is indicated by

$$\mathbf{Corr}(X_1, X_2) = \frac{\theta\left(\beta\theta+\beta\gamma-\theta\right)}{\left(\beta^2-1\right)\theta^2+\alpha\beta^2(\gamma+\theta)^2+\beta^2\gamma\left(\gamma+2\theta\right)}.$$

It is straightforward to notice that if $\alpha \to 0$, $\beta = 1$ and $\theta \to \infty$ or if $\alpha \to 0$, $\gamma = 0$ and $\beta = 1$, we have $\mathbf{Corr}(X_1, X_2) \to 1/2$; which corresponds also to the maximum value of the correlation for this family. The BGGL distribution stands for a mixture of i.i.d random variables since the BGL distribution is itself a mixture of independent random variables. Relying upon the result of Drouet-Mari and Kotz [3: Section 3.2.8], we infer that X_1 and X_2 are positively correlated by mixtures.

A bivariate distribution is said to be positively likelihood ratio dependent if the density $f_\Theta(x_1, x_2)$ satisfies

$$f_\Theta(x_1, y_1)f_\Theta(x_2, y_2) \geq f_\Theta\left(x_1, y_2\right) f_\Theta\left(x_2, y_1\right)$$

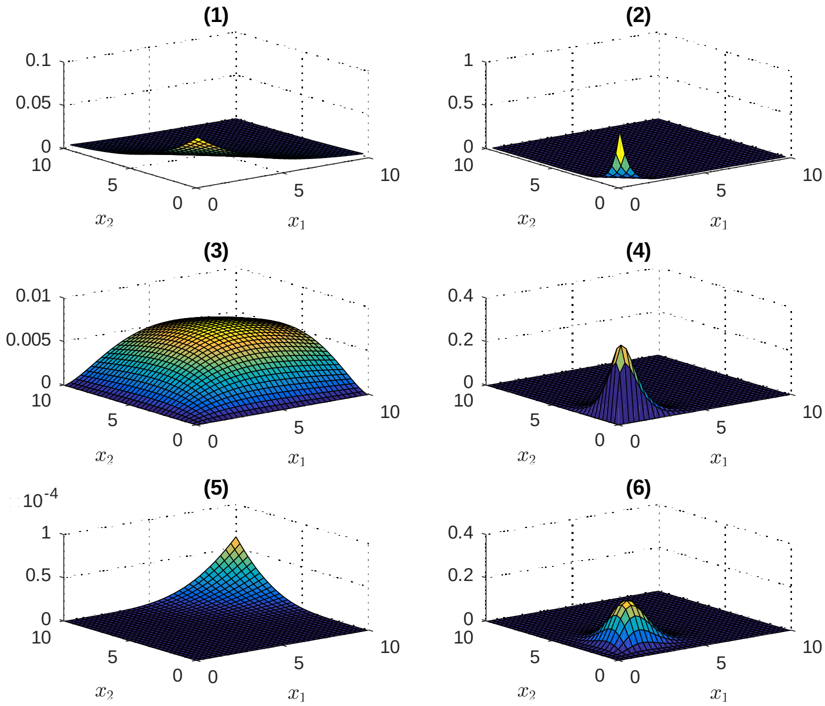

FIGURE 1. Plots of the pdf of BGGL distribution for some selected values of parameters $(\alpha, \beta, \gamma, \theta)$ as: (1) $(0.05, 1.2, 1.5, 0.3)$; (2) $(0.05, 1.2, 1.5, 2)$; (3) $(2, 1.2, 1.5, 0.3)$; (4) $(2, 1.2, 1.5, 2)$; (5) $(7, 1.2, 1.5, 0.3)$; (6) $(7, 1.2, 1.5, 3)$.

for all $x_1 > x_2$ and $y_1 > y_2$ (see Lehmann [14]). For the equation (3), the above inequality reduces to

$$\begin{aligned}&\left(\alpha^2 + \theta\left(\beta\gamma + \beta\theta - \theta\right)x_1 y_1\right)\left(\alpha^2 + \theta\left(\beta\gamma + \beta\theta - \theta\right)x_2 y_2\right)\\ &\geq \left(\alpha^2 + \theta\left(\beta\gamma + \beta\theta - \theta\right)x_1 y_2\right)\left(\alpha^2 + \theta\left(\beta\gamma + \beta\theta - \theta\right)x_2 y_1\right),\end{aligned}$$

or $x_2 y_2 + x_1 y_1 \geq x_1 y_2 + x_2 y_1$, which clearly holds. Hence, the BGGL distribution is positively likelihood ratio dependent.

3. Some transformations

In this section, we derive explicit expressions for the pdfs and moments of $Y = X_1 + X_2$, $Z = X_1/(X_1 + X_2)$, $V = X_1/X_2$ and $W = X_1 X_2$ when the random vector (X_1, X_2) is set according to BGGL distribution. In fact, for given random variables X_1 and X_2, the distributions of the product, sum, and ratio arise in many fields as biology, economics, engineering, genetics, medicine, number theory, physics, etc. (see for example [17]). For instance, if X_1 and X_2 denote the rainfall intensity and the duration of a storm, then $W = X_1 X_2$ will represent the amount of rainfall produced by that storm. Another expressive example is the following: if X_1 and X_2 denote the proportion of days with rain and the proportion of days with snow, then $Y = X_1 + X_2$ will represent the proportion of days with precipitation (rain or snow). Moreover, here is an example of the quotient $V = X_1/X_2$ which consists in representing X_1 as the crude divorce rate and X_2

as the crude marriage rate. Then, V will represent the divorce to marriage ratio. Furthermore, we set forward approximations for the distribution of $W = X_1X_2$, and discuss evidence for their robustness.

3.1. PDFS

Theorems 1–4 and Corollary 2 yield the pdfs of Y, Z, V and W when X_1 and X_2 are distributed according to equation (3).

Theorem 1. *If X_1 and X_2 are jointly distributed according to equation (3), then the pdf of Y is expressed by*

$$f_Y(y) = \frac{\theta^{2\alpha+1} y^{2\alpha-1} \mathrm{e}^{-\theta y}}{\beta(\gamma+\theta)\Gamma(2\alpha)} \left(1 + \frac{(\beta\theta+\beta\gamma-\theta)\theta y^2}{2\alpha(2\alpha+1)}\right), \qquad y > 0, \tag{6}$$

and the pdf of Z is given by

$$f_Z(z) = \frac{z^{\alpha-1}(1-z)^{\alpha-1}\Gamma(2\alpha)}{\beta(\gamma+\theta)\Gamma^2(\alpha+1)} \left(2\alpha(2\alpha+1)(\beta\theta+\beta\gamma-\theta)z(1-z) + \alpha^2\theta\right), \qquad z \in (0,1). \tag{7}$$

Proof. Substituting $x_1 = yz$ and $x_2 = y(1-z)$ with the Jacobian $J(x_1, x_2 \to y, z) = y$, in the joint pdf of X_1 and X_2, we obtain the joint pdf of Y and Z as

$$f_{Y,Z}(y,z) = \frac{\theta^{2\alpha+1} z^{\alpha-1}(1-z)^{\alpha-1} y^{2\alpha-1} \mathrm{e}^{-\theta y}}{\beta(\gamma+\theta)\Gamma^2(\alpha+1)} \left[(\beta\theta+\beta\gamma-\theta)\theta y^2 z(1-z) + \alpha^2\right], \tag{8}$$

where $z \in (0,1)$ and $y > 0$. Now, integrating this equation appropriately, we obtain marginal densities of Y and Z. □

Remark 1. Simple calculations reveal that equation (8) can be rewritten as

$$f_{Y,Z}(y,z) = \frac{\theta}{\beta(\gamma+\theta)} f_g(y; 2\alpha, \theta) f_b(z; \alpha, \alpha) + \frac{(\beta\theta+\beta\gamma-\theta)}{\beta(\gamma+\theta)} f_g(y; 2\alpha+2, \theta) f_b(z; \alpha+1, \alpha+1),$$

where $f_g(y; a, b)$ and $f_b(z; a, b)$ denote the pdfs of the gamma and the beta distributions with the parameters a and b, respectively, for all $0 < z < 1$ and $y > 0$.

Corollary 2. *If X_1 and X_2 are jointly distributed according to equation (3), then the pdf of the variable $V = X_1/X_2$ is expressed by*

$$f_V(v) = \frac{v^{\alpha-1}\Gamma(2\alpha)}{\beta(\gamma+\theta)(1+v)^{2\alpha}\Gamma^2(\alpha+1)} \left(\frac{2\alpha(2\alpha+1)(\beta\theta+\beta\gamma-\theta)v}{(1+v)^2} + \alpha^2\theta\right), \qquad v > 0. \tag{9}$$

Proof. We can state that

$$\frac{X_1}{X_1+X_2} = \frac{X_1/X_2}{X_1/X_2+1}$$

which, in terms of $Z = X_1/(X_1+X_2)$ and $V = X_1/X_2$, can be written as $Z = V/(1+V)$. Therefore, making the transformation $Z = V/(1+V)$ with the Jacobian $J(z \to v) = (1+v)^{-2}$ in the pdf of Z given in (7), we get the pdf of V. □

In the following, we derive the exact pdf of the product $W = X_1X_2$. The calculations involve the modified Bessel function of the second kind defined by:

$$K_\nu(x) = \frac{\pi(I_{-\nu}(x) - I_\nu(x))}{2\sin(\nu\pi)},$$

where $I_\nu(\cdot)$ denotes the modified Bessel function of the first kind of order ν defined by

$$I_\nu(x) = \frac{x^\nu}{2^\nu\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{1}{(\nu+1)_k\, k!} \left(\frac{x^2}{4}\right)^k.$$

Furthermore, $K_0(\cdot)$ is interpreted as the limit $K_0(x) = \lim_{\nu \to 0} K_\nu(x)$. We also need the following important lemma:

Lemma 3 ([18: Equation 2.3.16.1]). *For $p > 0$ and $q > 0$,*

$$\int_0^\infty x^{\alpha-1} \mathrm{e}^{-px-q/x} \mathrm{d}x = 2\left(\frac{q}{p}\right)^{\alpha/2} K_\alpha\left(2\sqrt{pq}\right).$$

Theorem 4. *If X_1 and X_2 are jointly distributed according to equation* (3)*, then the pdf of the variable $W = X_1 X_2$ is provided by*

$$f_W(w) = \frac{2\theta^{2\alpha+1} w^{\alpha-1}\left((\beta\theta + \beta\gamma - \theta)\,\theta w + \alpha^2\right)}{\beta(\gamma+\theta)\Gamma^2(\alpha+1)} K_0\left(2\theta\sqrt{w}\right), \qquad w > 0. \tag{10}$$

Proof. From (3), the joint pdf of $(X_1, W) = (X_1, X_1X_2)$ with the Jacobian $J(x_1, x_2 \to x_1, w) = x_1^{-1}$, becomes

$$f_{X_1,W}(x_1, w) = \frac{\theta^{2\alpha+1} w^{\alpha-1}\left((\beta\theta + \beta\gamma - \theta)\,\theta w + \alpha^2\right)}{\beta(\gamma+\theta)\Gamma^2(\alpha+1)} x_1^{-1} \mathrm{e}^{-\theta\left(x_1 + \frac{w}{x_1}\right)}.$$

Hence, integrating this equation with respect to x_1 we find that

$$f_W(w) = \frac{\theta^{2\alpha+1} w^{\alpha-1}\left((\beta\theta + \beta\gamma - \theta)\,\theta w + \alpha^2\right)}{\beta(\gamma+\theta)\Gamma^2(\alpha+1)} \int_0^\infty x_1^{-1} \mathrm{e}^{-\theta\left(x_1 + \frac{w}{x_1}\right)} \mathrm{d}x_1. \tag{11}$$

Direct application of Lemma 3 implies that

$$\int_0^\infty x_1^{-1} \mathrm{e}^{-\theta\left(x_1 + \frac{w}{x_1}\right)} \mathrm{d}x_1 = 2K_0(2\theta\sqrt{w}). \tag{12}$$

The result of the theorem follows by combining equations (11) and (12). □

By using the mathematical software MATLAB, we can present numerically in Figure 2, the graphs of pdfs of Y and V defined by (6) and (9), respectively, for different values of parameters. The graph of W will be given later.

3.2. Moments

In this part, we derive the moments of the variables $Y = X_1 + X_2$, $Z = X_1/(X_1 + X_2)$, $V = X_1/X_2$ and $W = X_1X_2$ defined in the previous section. We establish the following:

Theorem 5. *If X_1 and X_2 are jointly distributed according to equation* (3)*, then the moment functions of the variables Y, Z, V and W are, respectively,*

$$\mathbf{E}\left(Y^n\right) = \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma\left(n-k+\alpha\right)\Gamma\left(\alpha+k\right)}{\beta\left(\gamma+\theta\right)\Gamma^2(\alpha+1)\theta^n} \left(\left(\beta\theta + \beta\gamma - \theta\right)\left(\alpha+n-k\right)\left(\alpha+k\right) + \theta\alpha^2\right), \tag{13}$$

$$\mathbf{E}\left(Z^n\right) = \frac{\alpha\Gamma(2\alpha)\Gamma\left(\alpha+n\right)}{\beta(\gamma+\theta)\Gamma(\alpha+1)\Gamma(2\alpha+n)} \left(\frac{2\left(\beta\theta+\beta\gamma-\theta\right)\left(2\alpha+1\right)\left(\alpha+n\right)}{\left(2\alpha+n\right)\left(2\alpha+n+1\right)} + \theta\right), \tag{14}$$

$$\mathbf{E}\left(V^n\right) = \frac{\Gamma(\alpha+n)\Gamma\left(\alpha-n\right)}{\beta(\gamma+\theta)\Gamma^2(\alpha+1)} \left(\left(\beta\theta+\beta\gamma-\theta\right)\left(\alpha^2-n^2\right) + \theta\alpha^2\right), \qquad \alpha > n, \tag{15}$$

$$\mathbf{E}\left(W^n\right) = \frac{\theta^{-2n}\Gamma^2\left(\alpha+n\right)}{\beta(\gamma+\theta)\Gamma^2(\alpha+1)} \left(\left(\alpha+n\right)^2\left(\beta\theta+\beta\gamma-\theta\right) + \theta\alpha^2\right). \tag{16}$$

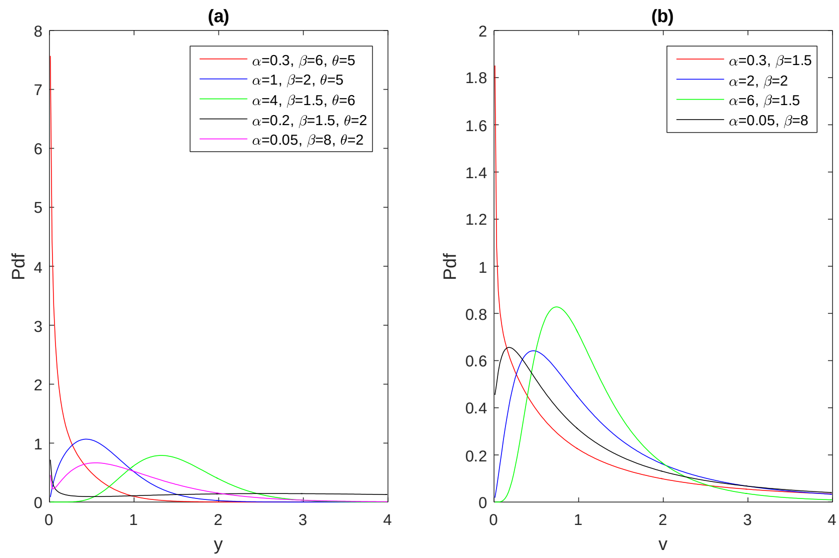

FIGURE 2. PDF of Y for the choices of α, β, θ and fixed parameter $\gamma = 2$ (a) and PDF of V for the choices of α, β and fixed parameters $\theta = 4$, $\gamma = 3$ (b).

P r o o f. Firstly, we have

$$\mathbf{E}\left[(X_1 + X_2)^n\right] = \sum_{k=0}^{n} \binom{n}{k} \mathbf{E}\left(X_1^{n-k} X_2^k\right). \tag{17}$$

By using (5), it is clear that

$$\mathbf{E}\left(X_1^{n-k} X_2^k\right) = \frac{\Gamma(n-k+\alpha)\,\Gamma(\alpha+k)}{\beta(\gamma+\theta)\Gamma^2(\alpha+1)\theta^n} \left((\beta\theta + \beta\gamma - \theta)(\alpha+n-k)(\alpha+k) + \theta\alpha^2\right). \tag{18}$$

Then, substituting (18) into (17), the result in (13) follows. Next, using (7), we obtain

$$\mathbf{E}\left(Z^n\right) = \frac{\Gamma(2\alpha)}{\beta(\gamma+\theta)\,\Gamma^2(\alpha+1)} \int_0^1 z^{n+\alpha-1}(1-z)^{\alpha-1}\left(2\alpha(2\alpha+1)(\beta\theta+\beta\gamma-\theta)\,z\,(1-z) + \theta\alpha^2\right) \mathrm{d}z.$$

After performing simple calculations, we obtain (14). Similarly, equation (15) follows by using the pdf of V given in (9) and the fact that

$$\int_0^\infty x^{a-1}(1+x)^{-a-b}\mathrm{d}x = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} \qquad \text{for } x > 0 \text{ and } a,\ b > 0.$$

Finally, the result in equation (16) follows from (5) by taking $n = m$. □

In order to measure the symmetry and tailedness of V, W, Y and Z, we compute their skewness and kurtosis. Indeed, for the different selected values of α, β and fixed values $\gamma = 3$ and $\theta = 5$, the distributions of V and W are highly skewed (skewness > 1), the kurtosis is greater than 3 and the distributions are leptokurtic. However, the distribution of Y is approximately symmetric (the skewness is between $-1/2$ and $1/2$) and platykurtic since the kurtosis is less than 3. The distribution of Z is symmetric for the three cases and approximately mesokurtic (kurtosis ≈ 3) for greater values of α. The numerical values of the skewness and the kurtosis of V, W, Y and Z with their variance and ordinary moments (μ_r, $r = 3, 4$) are given in Table 1.

Table 1. Numerical values for the ordinary moments, variance, skewness and kurtosis of V, W, Y and Z for some selected parameter values.

Moments of V, W, Y and Z for $\alpha = 5$ and $\beta = 1.5$.				
Transformations	V	W	Y	Z
μ_3	2.2500	0.8909	1.7312	—
μ_4	23.7163	3.3723	8.2386	0.0011
Variance	0.7762	0.6672	2.7464	0.0207
Skewness	3.2900	1.6348	0.3804	—
Kurtosis	39.3608	7.5758	1.0923	2.5871

Moments of V, W, Y and Z for $\alpha = 5$ and $\beta = 3$.				
Transformations	V	W	Y	Z
μ_3	1.8748	0.9590	1.8009	—
μ_4	16.9959	3.7467	9.0536	0.0010
Variance	0.7182	0.7164	2.9349	0.0200
Skewness	3.0801	1.5815	0.3582	—
Kurtosis	32.9477	7.3002	1.0511	2.5965

Moments of V, W, Y and Z for $\alpha = 10$ and $\beta = 4$.				
Transformations	V	W	Y	Z
μ_3	0.2156	10.2281	5.9593	0.1110×10^{-5}
μ_4	0.5741	98.1441	74.2882	0.0003
Variance	0.2624	4.3751	9.8565	0.0110
Skewness	1.6040	1.1177	0.1926	0.9582×10^{-13}
Kurtosis	8.3383	5.1274	0.7647	2.7594

3.3. Approximations of the product

In this section, we derive the approximate distribution of the product $W = X_1X_2$. It is clear that the random variable $P := 1/(1+W)$ has support in the interval $(0, 1)$. For this reason, in what follows we shall account for the approximate distribution by the beta distribution with parameters $a, b > 0$ (say $P \overset{d}{\sim} \beta(a, b)$) and pdf

$$f_P(y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1} (1-y)^{b-1}. \tag{19}$$

The idea of approximating distributions including complicated formulae with the beta distribution is frequently tackled in the statistics literature, based on the interesting work of Dasgupta [1] (see also [4,11,16] and [21]). Note that recently, Ghorbel [6] developed this single beta distribution of the form (19) for the bivariate beta distribution. Subsequently, this transformation was opted for owing to its popularity, simplicity and its association with the derived distribution. The basic reason for doing this doesn't lie in the fact that the calculation of equation (10) cannot be handled; it rather resides in providing a simple approximation in terms of the beta distribution so that one can use the known procedures for inference, prediction, etc. Furthermore, the presented approximation seems to be useful especially to practitioners who not only avoid the use of the modified Bessel function of the second kind but also regard the beta distribution as widely accessible in standard statistical packages.

The choice of the beta parameters a and b is enacted using the method of moments. The first two moments of P can be written as

$$\mathbf{E}(P) = \frac{a}{a+b} \qquad \text{and} \qquad \mathbf{E}(P^2) = \frac{a(a+1)}{(a+b)(a+b+1)}.$$

Solving simultaneously the two expressions above, it is straightforward to find that

$$a = \mathbf{E}(P) \frac{\mathbf{E}(P) - \mathbf{E}\left(P^2\right)}{\mathbf{E}(P^2) - \mathbf{E}^2(P)} \tag{20}$$

and

$$b = (1 - \mathbf{E}(P)) \frac{\mathbf{E}(P) - \mathbf{E}\left(P^2\right)}{\mathbf{E}(P^2) - \mathbf{E}^2(P)}. \tag{21}$$

where the two moments $\mathbf{E}(P)$ and $\mathbf{E}(P^2)$ can be computed numerically by evaluating the two integrals

$$I_1 = \int_0^\infty \frac{f_W(w)}{1+w} \mathrm{d}w \text{ and } I_2 = \int_0^\infty \frac{f_W(w)}{(1+w)^2} \mathrm{d}w$$

for the given parameters α, β, γ and θ, where $f_W(w)$ is presented by (10). To demonstrate the closeness of the proposed approximation, we use equations (20) and (21) with eight selected values of the parameters $(\alpha, \beta, \gamma, \theta)$. Both integrals as well as the corresponding estimated values of the parameters a and b are depicted in Table 2.

TABLE 2. Estimates of (I_1, I_2, a, b) for selected $(\alpha, \beta, \gamma, \theta)$.

α	β	γ	θ	I_1	I_2	a	b
0.5	1.5	0.5	2.2	0.8662	0.7807	2.4314	0.3757
1.5	1.5	1.5	6	0.9087	0.8339	8.2760	0.8318
4	1.5	4	1.5	0.1269	0.0234	1.8010	12.3888
5	1.5	3	3	0.2700	0.0886	3.1271	8.4546
4	1.5	0.5	3.2	0.4106	0.1960	3.2063	4.6032
2	1.5	2	2	0.4666	0.2650	1.9894	2.2745
3.5	1.5	0.5	7	0.7849	0.6309	8.1810	2.2418
1	1.5	2	2	0.6835	0.5200	2.1181	0.9808

The corresponding graphics are outlined in Figure 3, demonstrating comparison between exact and approximate densities of $P = 1/(1+W)$. The exact pdf corresponds to the solid curve and approximate pdf stands for the broken curve. It is evident that the approximate density is quite close to the exact density. Note that the exact pdf of P is stated by (with $0 < p < 1$)

$$f_P(P) = \frac{2\theta^{2\alpha+1}(1-p)^{\alpha-1}\left(\theta(\beta\theta + \beta\gamma - \theta)(1-p)p^{-1} + \alpha^2\right)}{\beta(\gamma+\theta)\Gamma^2(\alpha+1)p^{\alpha+1}} K_0\left(2\theta\sqrt{\frac{1-p}{p}}\right),$$

where $K_0(\cdot)$ is interpreted as the limit (as $\nu \to 0$) of the modified Bessel function of the second kind $K_\nu(x)$. Similar findings were recorded when this exercise was repeated for many other combinations of $(\alpha, \beta, \gamma, \theta)$. Finally, It is worth noting that the numerical results are obtained by the use of besselk function in mathematical software MATLAB.

Afterwards, based on the performed relationship in the first approximation, we present another direct approximation of W by the inverted beta distribution of the two parameters $p > 0$ and $q > 0$ say $\beta'(p, q)$, defined by the pdf

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{p-1}(1+x)^{-(p+q)}, \qquad x > 0. \tag{22}$$

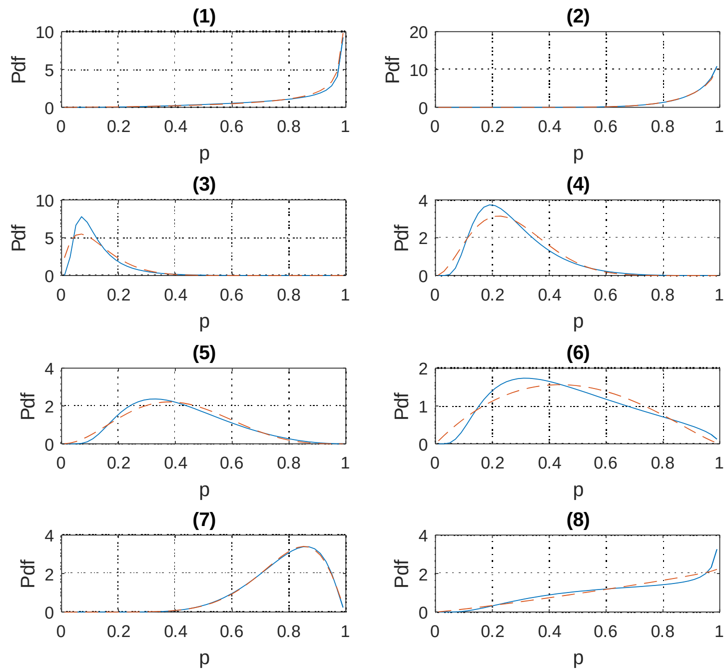

FIGURE 3. The exact pdf of P (solid curve) and its approximated function (broken curve) for $\beta = 1.5$ and (1): $(\alpha, \gamma, \theta) = (0.5, 0.5, 2.2)$; (2): $(\alpha, \gamma, \theta) = (1.5, 1.5, 6)$; (3): $(\alpha, \gamma, \theta) = (4, 4, 1.5)$; (4): $(\alpha, \gamma, \theta) = (5, 3, 3)$; (5): $(\alpha, \gamma, \theta) = (4, 0.5, 3.2)$; (6): $(\alpha, \gamma, \theta) = (2, 2, 2)$; (7): $(\alpha, \gamma, \theta) = (3.5, 0.5, 7)$; (8): $(\alpha, \gamma, \theta) = (1, 2, 2)$.

From this, we use the method of moments to determine the parameters p and q. The first two moments of W become

$$\mathbf{E}\left(W\right) = \frac{q}{p-1} \text{ and } \mathbf{E}\left(W^2\right) = \frac{q\left(q+1\right)}{\left(p-1\right)\left(p-2\right)}.$$

After some algebraic manipulation, it is straightforward to verify that

$$p = 1 + \frac{\mathbf{E}\left(W^2\right) - \mathbf{E}\left(W\right)}{\mathbf{E}\left(W^2\right) - \mathbf{E}^2(W)} \tag{23}$$

and

$$q = \mathbf{E}\left(W\right) \frac{\mathbf{E}\left(W^2\right) - \mathbf{E}\left(W\right)}{\mathbf{E}\left(W^2\right) - \mathbf{E}^2(W)}, \tag{24}$$

where both moments $\mathbf{E}\left(W\right)$ and $\mathbf{E}\left(W^2\right)$ follow from (16) by taking $n = 1$ and $n = 2$, respectively.

In order to corroborate the robustness as well as goodness of fit of the second approximation, we use (23) and (24) with given selected values of the parameters $(\alpha, \beta, \gamma, \theta)$. The corresponding estimated values of the parameters p and q are summarized in Table 3. Afterwards, the notion of robustness is assessed by comparing the exact and the approximated pdfs of W as exhibited in (10) and (22). These comparisons are illustrated in Figure 4.

Table 3. Estimates of (a, b) for selected $(\alpha, \beta, \gamma, \theta)$

α	β	γ	θ	p	q
1	1.5	0.5	1.7	1.9208	0.7820
0.5	1.5	0.5	2	1.3898	0.1153
2	1.5	0.5	2.5	1.9962	0.9918
0.5	1.5	1	1	2.1309	1.7906
2.5	1.5	2.5	2	2.7739	4.6442
4	1.5	1.5	4	2.4129	1.8223
2	1.5	0.5	2	2.3193	2.0888
1.5	1.5	0.7	2	2.0428	1.1144

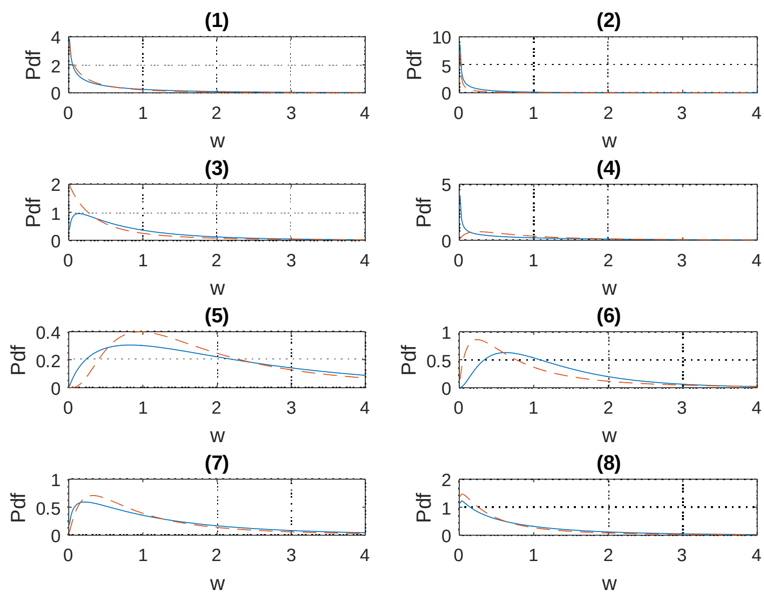

Figure 4. The exact pdf of W (solid curve) and its approximated function (broken curve) for $\beta = 1.5$ and (1): $(\alpha, \gamma, \theta) = (1, 0.5, 1.7)$; (2): $(\alpha, \gamma, \theta) = (0.5, 0.5, 2)$; (3): $(\alpha, \gamma, \theta) = (2, 0.5, 2.5)$; (4): $(\alpha, \gamma, \theta) = (0.5, 1, 1)$; (5): $(\alpha, \gamma, \theta) = (2.5, 2.5, 2)$; (6): $(\alpha, \gamma, \theta) = (4, 1.5, 4)$; (7): $(\alpha, \gamma, \theta) = (2, 0.5, 2)$; (8): $(\alpha, \gamma, \theta) = (1.5, 0.7, 2)$.

4. Entropies

The entropy is a crucial concept in such areas such as classical thermodynamics, where it was first recognized, statistical mechanics and physics, information theory and many other fields. It stands for an excellent tool to quantify the amount of uncertainty involved in the value of a random variable or the outcome of a random process. Several measures of entropy have been examined and compared in literature. The simplest known entropy is the Shannon entropy (see [20]). Let $(\mathcal{X}, \mathcal{B}, \mathcal{P})$ be a probability space. Consider a pdf f associated with $\mathcal{P}$, dominated by σ-finite measure μ on $\mathcal{X}$. Thus, the Shannon entropy is defined by

$$H_{SH}(f) = -\int_{\mathcal{X}} f(x) \log f(x) \mathrm{d}\mu.$$

One of the main extensions of the Shannon entropy was established by Rényi (see [19]). This generalized entropy measure is expressed in terms of

$$H_R(\mu, f) = \frac{\log G(\mu)}{1-\mu} \qquad (\text{for } \mu > 0 \text{ and } \mu \neq 1), \tag{25}$$

where

$$G(\mu) = \int_{\mathcal{X}} f^{\mu} \mathrm{d}\mu.$$

Note that the Shannon entropy is the particular case of the Rényi entropy for $\mu \uparrow 1$. The calculations of these entropies involve the hypergeometric function identified as

$${}_3F_0(a, b, c; -; x) := \sum_{k=0}^{\infty} (a)_k (b)_k (c)_k \frac{x^k}{k!}, \qquad |x| < 1,$$

where $(i)_k := i(i+1)\dots(i+k-1)$ denotes the ascending factorial.

Before deriving these entropies, we need the following lemma:

Lemma 6. *Let* $g(\alpha, \beta, \gamma, \theta) = \lim_{\mu \to 1} h(\mu)$, *where*

$$h(\mu) = \frac{\mathrm{d}}{\mathrm{d}\mu}\, {}_3F_0\Big(-\mu, \mu(\alpha-1)+1, \mu(\alpha-1)+1; -; \frac{\beta\theta + \beta\gamma - \theta}{\theta\alpha^2\mu^2}\Big).$$

Therefore,

$$g(\alpha, \beta, \gamma, \theta) = \frac{(2-\alpha)(\beta\theta + \beta\gamma - \theta)}{\theta\alpha}. \tag{26}$$

Proof. Expanding ${}_3F_0$ in a series form, $h(\mu)$ is equal to

$$\frac{\mathrm{d}}{\mathrm{d}\mu} \sum_{k=0}^{\infty} \frac{(-1)^k \Delta_k(\mu)}{k!} \Big(\frac{\beta\theta + \beta\gamma - \theta}{\theta\alpha^2}\Big)^k = \sum_{k=0}^{\infty} \frac{(-1)^k (\beta\theta + \beta\gamma - \theta)^k}{k! (\theta\alpha^2)^k} \Big[\frac{\mathrm{d}}{\mathrm{d}\mu} \Delta_k(\mu)\Big], \tag{27}$$

where

$$\begin{aligned} \Delta_k(\mu) &= \frac{(-1)^k \Gamma(-\mu + k) \Gamma^2(\mu(\alpha-1)+1+k)}{\Gamma(-\mu) \Gamma^2(\mu(\alpha-1)+1) \mu^{2k}} \\ &= \frac{\Gamma(1+\mu) \Gamma^2(\mu(\alpha-1)+1+k)}{\Gamma(1+\mu-k) \Gamma^2(\mu(\alpha-1)+1)\mu^{2k}}. \end{aligned}$$

Now, differentiating the logarithm of $\Delta_k(\mu)$ with respect to μ, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\mu} \Delta_k(\mu) = \Delta_k(\mu) \Big[& \Psi(1+\mu) + 2(\alpha-1)(\Psi(\mu(\alpha-1)+1+k)) \\ & - \Psi(\mu(\alpha-1)+1) - \Psi(1+\mu-k) - \frac{2k}{\mu}\Big], \end{aligned} \tag{28}$$

where $\Psi(\cdot) = \Gamma'(\cdot)/\Gamma(\cdot)$ is the digamma function. Finally, substituting equation (28) into equation (27) and taking $\mu \to 1$, we get

$$\begin{aligned} g(\alpha, \beta, \gamma, \theta) = \sum_{k=0}^{1} & \frac{(\beta\theta + \beta\gamma - \theta)^k}{k! (\theta\alpha^2)^k} \frac{(-1)^k \Gamma^2(\alpha+k)}{\Gamma(2-k)\Gamma^2(\alpha)} \\ & \times [\Psi(2) + 2(\alpha-1)(\Psi(\alpha+k) - \Psi(\alpha)) - \Psi(2-k) - 2k]. \end{aligned}$$

From this equation, the result follows. □

Theorem 7. *Consider the random pair* (X_1, X_2) *defined by its pdf* (3). *The Rényi and the Shannon entropies are indicated by*

$$H_R(\mu, f) = \frac{1}{1-\mu}\Big[(3\mu-2)\log(\theta) + 2\log\Gamma(\mu(\alpha-1)+1) - \mu\log(\beta) - 2(\mu(\alpha-1)+1)\log(\mu) - \mu\log(\gamma+\theta) - 2\mu\log\Gamma(\alpha) + \log\ {}_3F_0(-\mu, \mu(\alpha-1)+1, \mu(\alpha-1)+1; -; \frac{\beta\theta+\beta\gamma-\theta}{\theta\alpha^2\mu^2})\Big]$$

and

$$H_{SH}(f) = -3\log\theta - 2(\alpha-1)\Psi(\alpha) + \log\beta + 2\alpha + \log(\gamma+\theta) + 2\log\Gamma(\alpha) - \frac{\theta}{\theta(2-\beta)-\beta\gamma} g(\alpha,\beta,\gamma,\theta),$$

respectively, where $g(\alpha,\beta,\gamma,\theta)$ *is provided by equation* (26).

P r o o f. For $\mu>0$ and $\mu\neq 1$, using the joint pdf of X_1 and X_2 presented by (3), we have

$$G(\mu) = \int_0^\infty\int_0^\infty f_\Theta^\mu(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 = \frac{\alpha^{2\mu}\theta^{\mu(2\alpha+1)}}{\beta^\mu(\gamma+\theta)^\mu\Gamma^{2\mu}(\alpha+1)}\int_0^\infty x_1^{\mu(\alpha-1)}\mathrm{e}^{-\theta\mu x_1} \times \int_0^\infty x_2^{\mu(\alpha-1)}\mathrm{e}^{-\theta\mu x_2}\left(1+\frac{(\beta\theta+\beta\gamma-\theta)\theta x_1}{\alpha^2}x_2\right)^\mu \mathrm{d}x_2\mathrm{d}x_1.$$

Hence, using [9: Equation (3.383.5)], we obtain

$$G(\mu) = \frac{\alpha^{2\mu}\theta^{\mu(2\alpha+1)}}{\beta^\mu(\gamma+\theta)^\mu\Gamma^{2\mu}(\alpha+1)}\sum_{k=0}^\infty \frac{(\mu(\alpha-1)+1)_k(-\mu)_k}{k!\,(\alpha^2\theta\mu)^k}((\beta\theta+\beta\gamma-\theta)\theta)^k \times \Gamma(\mu(\alpha-1)+1)(\theta\mu)^{-\mu(\alpha-1)-1}\int_0^\infty x_1^{\mu(\alpha-1)+k}\mathrm{e}^{-\theta\mu x_1}\mathrm{d}x_1.$$

Therefore,

$$G(\mu) = \frac{\theta^{3\mu-2}\Gamma^2(\mu(\alpha-1)+1)}{\beta^\mu(\gamma+\theta)^\mu\mu^{2\mu(\alpha-1)+2}\Gamma^{2\mu}(\alpha)} \times \sum_{k=0}^\infty \frac{(-\mu)_k(\mu(\alpha-1)+1)_k(\mu(\alpha-1)+1)_k}{k!}\left(\frac{\beta\theta+\beta\gamma-\theta}{\theta\alpha^2\mu^2}\right)^k.$$

Thus,

$$G(\mu) = \frac{\theta^{3\mu-2}\Gamma^2(\mu(\alpha-1)+1)}{\beta^\mu(\gamma+\theta)^\mu\mu^{2\mu(\alpha-1)+2}\Gamma^{2\mu}(\alpha)} \times {}_3F_0\left(-\mu, \mu(\alpha-1)+1, \mu(\alpha-1)+1; -; \frac{\beta\theta+\beta\gamma-\theta}{\theta\alpha^2\mu^2}\right).$$

Considering the logarithm of $G(\mu)$ and using (25), we get the Rényi entropy. The Shannon entropy is obtained from Rényi entropy by taking $\mu\uparrow 1$ and using L'Hopital's rule. □

5. Estimation by Expectation-Maximization (E.M) Algorithm

The Expectation-Maximization (E.M) algorithm (see [2]) is a prominent way of finding maximum-likelihood estimates (MLEs) for model parameters when data are incomplete. It is an itetarive method to approximate the maximum likelihood function. Each iteration of the E.M algorithm involves two steps. In the first one (E-Step), we compute the conditional expectation denoted by $\mathbb{Q}\left(\Theta|\Theta^{(j)}\right)$, where $\Theta^{(j)} = \left(\alpha^{(j)}, \beta^{(j)}, \gamma^{(j)}, \theta^{(j)}\right)$ is the estimated parameters vector of $\Theta = (\alpha, \beta, \gamma, \theta)$ at the j^{th} iteration. In the second step of the E.M alghorithm (M-Step), we maximize $\mathbb{Q}\left(\Theta|\Theta^{(j)}\right)$ with respect to the vector of parameters Θ.

In this section, we discuss the problem of computing MLEs of the unknown parameters of the BGGL distribution for a given random sample of size n, of the form $\Lambda = \{(x_{11}, x_{12}), \dots, (x_{n1}, x_{n2})\}$. Using the mixture density representation of the BGGL distribution given by (4) , the incomplete likelihood function can be written as

$$L(\Theta) = \prod_{i=1}^{n} \Big(\frac{\beta-1}{\beta} f_{gg}(x_{i1}, x_{i2}, \Theta_1) + \frac{1}{\beta} f_{BL}(x_{i1}, x_{i2}, \Theta_2) \Big),$$

where $\Theta_1 = (\alpha, \theta)$ and $\Theta_2 = (\alpha, \theta, \gamma)$.

To complete the data of this function, we associate a discrete random vector Δ, where Δ follows a bivariate Bernoulli distribution with parameters $p = \frac{\beta-1}{\beta}$ and $1 - p = \frac{1}{\beta}$. This implies that

$$\mathbf{P}(\Delta = \delta_i) = p^{\delta_i} (1-p)^{1-\delta_i},$$

where $\delta_i \in \{0,1\}^2$. The complete samples are as follows:

$$\{(x_{11}, x_{12}, \delta_1), \dots, (x_{n1}, x_{n2}, \delta_n)\}.$$

Resting on the complete observations, the complete likelihood function is

$$L_c(\Theta) = \prod_{i=1}^{n} \Big(\frac{\beta-1}{\beta} f_{gg}(x_{i1}, x_{i2}, \Theta_1) \Big)^{\delta_i} \Big(\frac{1}{\beta} f_{BL}(x_{i1}, x_{i2}, \Theta_2) \Big)^{1-\delta_i}.$$

Thus, the log-likelihood function becomes

$$l_c(\Theta) := \log L_c(\Theta) = \sum_{i=1}^{n} \delta_i \Big(\log\Big(\frac{\beta-1}{\beta}\Big) + \log\left(f_{gg}(x_{i1}, x_{i2}, \Theta_1)\right) \Big)$$
$$+ \sum_{i=1}^{n} (1-\delta_i) \Big(\log\left(f_{BL}(x_{i1}, x_{i2}, \Theta_2)\right) + \log\Big(\frac{1}{\beta}\Big) \Big).$$

Now, we compute the conditional expectation $\mathbb{Q}\left(\Theta|\Theta^{(j)}\right)$ of the complete log likelihood function, resting upon the observed data and the current value $\Theta^{(j)} = \left(\alpha^{(j)}, \beta^{(j)}, \gamma^{(j)}, \theta^{(j)}\right)$ at the j^{th} iteration. We obtain

$$\begin{aligned}
\mathbb{Q}\big(\Theta|\Theta^{(j)}\big) &= \mathbf{E}\big(l_c(\Theta|\Lambda, \Theta^{(j)}\big) \\
&= \sum_{i=1}^{n} \mathbf{E}\big(\delta_i|\Lambda, \Theta^{(j)}\big) \Big(\log\Big(\frac{\beta-1}{\beta}\Big) + \log\left(f_{gg}(x_{i1}, x_{i2}, \Theta_1)\right) \Big) \\
&\quad + \sum_{i=1}^{n} \mathbf{E}\big((1-\delta_i) |\Lambda, \Theta^{(j)}\big) \Big(\log\Big(\frac{1}{\beta}\Big) + \log\left(f_{BL}(x_{i1}, x_{i2}, \Theta_2)\right) \Big),
\end{aligned}$$

where

$$\mathbf{E}\left(\delta_i|\Lambda,\Theta^{(j)}\right) := \tau_i^{(j)} = \frac{\frac{\left(\beta^{(j)}-1\right)}{\beta^{(j)}} f_{gg}(x_{i1},x_{i2},\Theta_1^{(j)})}{\frac{\left(\beta^{(j)}-1\right)}{\beta^{(j)}} f_{gg}(x_{i1},x_{i2},\Theta_1^{(j)}) + \frac{1}{\beta^{(j)}} f_{BL}(x_{i1},x_{i2},\Theta_2^{(j)})},$$

and

$$\mathbf{E}\big(\left(1-\delta_i\right)|\Lambda,\Theta^{(j)}\big) := \varsigma_i^{(j)} = 1-\tau_i^{(j)}.$$

Note that $\tau_i^{(j)}$ is the posterior probability computed in the j^{th} iteration.

In the maximization step of the E.M algorithm, we attempt to find the maximum $\Theta^{(j+1)}$ of the function $\mathbb{Q}\left(\Theta|\Theta^{(j)}\right)$ with respect to $\Theta = (\alpha,\beta,\gamma,\theta)$.

$$\Theta^{(j+1)} = \arg\max_{\Theta} \mathbb{Q}\left(\Theta|\Theta^{(j)}\right),$$

where

$$\begin{aligned}\mathbb{Q}\big(\Theta|\Theta^{(j)}\big) = \sum_{i=1}^{n} \tau_i^{(j)} \Bigg(& \log\left(\frac{\beta-1}{\beta}\right) + (2\alpha+2)\log(\theta) + \alpha\left(\log(x_{i1}) + \log(x_{i2})\right) \\ & - \theta\left(x_{i1}+x_{i2}\right) - 2\log\left(\Gamma\left(\alpha+1\right)\right)\Big) + \varsigma_i^{(j)}\left(-\log(\beta) + (2\alpha+1)\log(\theta)\right. \\ & + (\alpha-1)\left(\log(x_{i1}) + \log(x_{i2})\right) + \log\left(\alpha^2+\gamma\theta x_{i1}x_{i2}\right) - \theta\left(x_{i1}+x_{i2}\right) \\ & - \quad \log(\gamma+\theta) - 2\log\left(\Gamma\left(\alpha+1\right)\right)\Bigg).\end{aligned} \tag{29}$$

According to Jensen's inequality, as $\mathbb{Q}\left(\Theta|\Theta^{(j)}\right)$ is increasing, the log-likelihood function is also increasing (see [2]). Differentiating equation (29) with respect to each parameter, we get the following:

$$\begin{aligned}\frac{\partial\mathbb{Q}}{\partial\alpha} &= 2n\log(\theta) - 2n\Psi\left(\alpha+1\right) + \sum_{i=1}^{n}\left(\log(x_{i1}) + \log(x_{i2})\right) + 2\alpha\sum_{i=1}^{n}\frac{\varsigma_i^{(j)}}{\alpha^2+\gamma\theta x_{i1}x_{i2}}, \\ \frac{\partial\mathbb{Q}}{\partial\beta} &= \frac{1}{\beta(\beta-1)}\sum_{i=1}^{n}\tau_i^{(j)} - \frac{1}{\beta}\sum_{i=1}^{n}\varsigma_i^{(j)}, \\ \frac{\partial\mathbb{Q}}{\partial\theta} &= \frac{2n(\alpha+1)}{\theta} - \sum_{i=1}^{n}\left(x_{i1}+x_{i2}\right) + \sum_{i=1}^{n}\varsigma_i^{(j)}\Big(\frac{\gamma x_{i1}x_{i2}}{\alpha^2+\gamma\theta x_{i1}x_{i2}} - \frac{1}{\gamma+\theta} - \frac{1}{\theta}\Big),\end{aligned}$$

and

$$\frac{\partial\mathbb{Q}}{\partial\gamma} = \sum_{i=1}^{n}\varsigma_i^{(j)}\Big(\frac{\theta x_{i1}x_{i2}}{\alpha^2+\gamma\theta x_{i1}x_{i2}} - \frac{1}{\gamma+\theta}\Big),$$

where $\Psi(.)$ is the digamma function. Next, we put the previous differential equations equal to zero and we use the mathematical software MATLAB to settle this system of equations numerically. From this perspective, we repeat this step until $\left\|\Theta^{(j+1)} - \Theta^{(j)}\right\| > \varepsilon$, where ε is a fixed threshold of convergence. Dempster et al. [2] and Wu [25] investigated the convergence properties of the E.M algorithm. Consequently, the E.M algorithm converges within a finite iterations number and gives the parameters maximum likelihood estimates. These solutions yield the E.M estimators denoted $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{\theta}$. Furthermore, we elaborate the following algorithm to compute the MLEs of the unknown parameters.

Algorithm.

- *Step 1*: Choose some initial guess values of Θ, say $\Theta^{(0)} = \left(\alpha^{(0)}, \beta^{(0)}, \gamma^{(0)}, \theta^{(0)}\right)$.
- *Step 2*: Obtain

$$\Theta^{(1)} = \arg\max_{\Theta} \mathbb{Q}\Big(\Theta|\Theta^{(0)}\Big).$$

- *Step 3*: Continue the process until convergence takes place.

6. Real data applications

We initially consider the data of rain and snow collect from five French cities and present an application of the model given by (3).

Precipitation is any type of water that forms in the Earth's atmosphere and then drops onto the surface of the Earth. Clouds eventually get too full of water vapor, and the precipitation turns into a liquid (rain) or a solid (snow). The rain stands for precipitation that occurs in different sizes, from big, heavy drops to light ones, but snow is one of the solid types of precipitation. It is made up of water that has been frozen.

We consider data available at the website `https://en.tutiempo.net/climate` collected from the following five French cities (in different parts of the country) regarding the rain and snow: Dijon, Lille, Rennes, Paris and Toulouse. The dataset comprises the number of days in each year where rain and snow appeared during the period from 1990 to 2018. We consider the following variables:

- $X_1:$ proportion of days with rain;
- $X_2:$ proportion of days with snow;
- $Y = X_1 + X_2:$ proportion of days with precipitation (rain or snow).

Table 4 displays the estimates of $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{\theta}$, which were obtained using the E.M algorithm considered in the previous section. Furthermore, we exhibit in the same table the estimated values of the moments $\mathbf{E}\,(Y)$ for five cities.

TABLE 4. Estimated values of $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\theta}$ and of $\mathbf{E}\,(Y)$.

City	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\theta}$	$\mathbf{E}\,(Y)$
Dijon	0.7191	1.1963	5.6880	4.9055	0.5431
Lille	0.5399	1.1710	5.6946	4.1811	0.5637
Rennes	0.3396	1.0625	8.1807	3.8836	0.5338
Paris	0.4957	1.0001	7.7070	4.1536	0.5516
Toulouse	0.3318	1.1444	8.6421	4.6344	0.4431

It was reported that the proportion of days with precipitation phenomenon (rain or snow) was similar for the five cities. Toulouse city presented the lowest proportion.

The second application rests upon demographic studies. We will use the quotient as a demographic ratio of two real data, which are the crude divorce rate and the crude marriage rate. In fact, a measure of divorces in a certain population is the divorce to marriage ratio, corresponding to the ratio of the divorce rate to the marriage rate. A measure of 0.5 means, for example, that there has been one divorce for every two new marriages recorded in a given year and in a given region. We now consider the data on crude divorce and marriage rates for women and men in 2009 for 51 states in the USA. The data were published by U.S. Census Bureau. Additionally, the web

TABLE 5. Estimated values of $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\theta}$ and of $\mathbf{E}(V)$.

Sex	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\theta}$	$\mathbf{E}(V)$
Women	7.5147	1.1990	0.3970	0.5534	1.1430
Men	5.9935	1.1984	0.3177	0.4353	1.1830

site of the data is `https://www2.census.gov/library/publications/2011/compendia/statab/131ed/tables/vitstat.pdf`. Rates are recorded per $1,000$ women (respectively men) residing in some area. Let X_1 be the crude divorce rate, namely divorce rate per $1,000$ women (respectively men), and let X_2 be the crude marriage rate, i.e., marriage rate per $1,000$ women (respectively men). Hence, $V = X_1/X_2$ is the divorce to marriage ratio. Table 5 displays the estimates of $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{\theta}$, which were obtained using the E.M algorithm again. It was inferred from Table 5 that the divorce to marriage ratio for women is less than that for men. Furthermore, the average of divorce for a new marriage for women amounts to 1.1430 and that for a new marriage for men equals 1.1830.

Finally, we consider the experimental results using the real dataset of the rain and snow for Dijon city in the reliabilty for different statistical models such as product of two independent Gamma random variables (PG), BGL and BGGL distributions. In order to compare these models, we use the following four criteria: Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Corrected Akaike Information Criterion (CAIC), Hannan-Quinn information criterion (HQIC) that are defined as follows:

$$\text{AIC} = -2\log\hat{L} + 2k, \qquad BIC = -2\log\hat{L} + k\log n,$$
$$\text{CAIC} = \text{AIC} + \frac{2k\,(k+1)}{n-k+1}, \qquad \text{HQIC} = -2\log\hat{L} + 2k\log(\log n),$$

where k is the number of free parameters in the model, n is the sample size and $\log\hat{L}$ is the log-likelihood. For fitting a data set, the best model is a model with the smallest value of AIC, BIC, CAIC and HQIC.

Table 6 lists the Maximum Likelihood Estimates (MLEs) of the parameters from the fitted models and the values of the following statistics: AIC, BIC, CAIC and HQIC. Based on the values of these statistics, we conclude that the BGGL distribution can provide good fits for realdata. Note that similar results and interpretations can be treated for the other datasets.

TABLE 6. MLEs, AICs, BICs, CAICs and HQICs for the real data set of the rain and snow for the city of Dijon.

Distr.	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\theta}$	AIC	BIC	CAIC	HQIC
BGGL	0.733	1.255	5.248	4.999	−59.526	−54.197	−57.926	−57.897
PG	0.964	—	—	3.550	−30.063	−27.399	−29.619	−29.249
BGL	0.719	—	7.767	4.905	−10.355	−6.358	−9.432	−9.133

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Faculté des Sciences de Sfax
Université de Sfax
B P 802, 3018, Sfax
TUNISIA
E-mail: ghorbel.mmed.ali@gmail.com
mecene.rahmouna@gmail.com