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EVALUATION OF NORM OF (p,q)-BERNSTEIN OPERATORS

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ABSTRACT. In this paper, we aim to study about the estimation of norm of (p,q)-Bernstein operators $\mathcal{B}^n_{p,q}$ in $C[0,1]$ for the case $q > p > 1$ by applying (p,q)-calculus and divided difference analogue of (p,q)-Bernstein operators. Some basic theorem and related results are also discussed in this paper. Here, the extra parameter p shows more flexibility by choosing the value of p.

1. Introduction and definition

In recent years, the application of quantum calculus and post quantum calculus plays an important role in the area of approximation theory, number theory, theoretical physics and many other branches of Mathematics and Physics. G. M. Phillips [28] initiated the convergence of Bernstein polynomial based on q-calculus. Many results and properties were obtained by S. Ostrovska and some researchers (see, for more details, [24–27]). The Bernstein operators, q-Bernstein operators [5, 30] and (p,q)-Bernstein operators [3, 6, 12], their Kantorovich form [4], bivariate form [1, 18] and others were introduced, and their approximation properties made them an area of intensive research. For in-depth knowledge and details see [2]–[15], [17, 31]. Initially, by using the concept of (p,q)-calculus Mursaleen et al. (see [19]–[23]) first applied the concept of (p,q)-calculus in approximation theory and introduced the (p,q)-analogue of Bernstein operators. For the present work, we recall the following definition and notations:

The q-integers and (p,q)-integers are defined, respectively, as:

$$[n]q := \frac{1-q^n}{1-q} \qquad \text{and} \qquad [n]_{p,q} := \frac{p^n-q^n}{p-q}, \qquad n \in \mathbb{N} \cup \{0\}.$$

The (p,q)-binomial expansion is given as:

$$(x+y)^n_{p,q} := \prod_{\nu=0}^{n-1}(p^\nu x - q^\nu y) \qquad \text{and} \qquad (p,q;x) := \prod_{\nu=0}^{n-1}(p^\nu - q^\nu x).$$

It can be seen by induction that

$$(1+x)(p+qx)(p^2+q^2x)\cdots(p^{n-1}+q^{n-1}x) = \sum_{\nu=0}^{n} p^{\frac{(n-\nu)(n-\nu-1)}{2}} q^{\frac{\nu(\nu-1)}{2}} \binom{n}{\nu}_{p,q} x^\nu,$$

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wherein (p,q)-binomial coefficients are defined by

$$\binom{n}{\nu}_{p,q} := \frac{[n]_{p,q}!}{[\nu]_{p,q}![n-\nu]_{p,q}!}.$$

The Euler's identity based on (p,q)-analogue is given by

$$\prod_{\nu=0}^{n-1}(p^{\nu}-q^{\nu}x) := \sum_{\nu=0}^{n} p^{\frac{(n-\nu)(n-\nu-1)}{2}} q^{\frac{\nu(\nu-1)}{2}} \binom{n}{\nu}_{p,q} x^{\nu}.$$

The (p,q)-analogue of Bernstein operator of a bounded function g introduced by Mursaleen et al. (see [19]) is given as follows:

$$\mathcal{B}^{n}_{p,q}(g;x) := \sum_{\nu=0}^{n} g\left(p^{n-\nu}\frac{[k]_{p,q}}{[n]_{p,q}}\right)\mathcal{Q}_{n,\nu}(p,q;x), \qquad n\in\mathbb{N},$$

where polynomial $\mathcal{Q}_{n,\nu}(p,q;x)$ is given as:

$$\mathcal{Q}_{n,\nu}(p,q;x) = \frac{1}{p^{\frac{n(n-1)}{2}}}\binom{n}{\nu}_{p,q} p^{\frac{\nu(\nu-1)}{2}} x^{\nu} \prod_{\nu=0}^{n-\nu-1}(p^{\nu}-q^{\nu}x), \qquad x\in[0,1]. \tag{1.1}$$

For $p=1$, $\mathcal{B}^{n}_{p,q}(g;x)$ reduces into q-Bernstein operators which are studied by several authors (see [9,15,16,29]). Here we mark out end-point interpolation properties of the (p,q)-Bernstein operator $B^{n}_{p,q}(g,x)$, which are valid for the case $q\neq 1$:

$$\mathcal{B}^{n}_{p,q}(g;0) = g(0), \qquad \mathcal{B}^{n}_{p,kq}(g;1) = g(1).$$

2. Statement of main results

In this section we mainly talk about some auxiliary results and statements of our main theorem, which will be provided in the next section. We start with a simple statement for the case $q>p>1$, which gives the better approximation of Bernstein operators $\mathcal{B}^{n}_{p,q}$ and it is useful to understand the asymptomatic behavior as n $\to\infty$. The following result involves the function $(\frac{1}{z};\frac{p}{q})_{\infty}$ analytic in $\mathbb{C}\setminus\{0\}$ and has an essential singularity at 0.

Theorem 2.1.

(i) *The following results whose convergence is uniform on any compact set $M\subset\mathbb{C}\setminus\{0\}$ for $n\in\mathbb{N}$, $z\neq 0$, $q>p>1$,*

$$\lim_{n\to\infty}\frac{p^{\frac{n(n-1)}{2}}\mathcal{Q}_{n,k}(p,q;z)}{(-1)^{n}q^{\frac{n(n-1)}{2}}z^{n}} = \frac{(-1)^{k}(\frac{1}{z};\frac{p}{q})_{\infty}}{q^{\frac{k(k-1)}{2}}(\frac{p}{q};\frac{p}{q})_{k}}.$$

(ii) *The following results whose convergence is uniform on any compact set $M\subset\mathbb{C}\setminus\{0\}$*

$$\lim_{n\to\infty}\frac{p^{nk}\mathcal{Q}_{n,n-k}(p,q;z)}{q^{nk}z^{n}} = \frac{(-1)^{k}(\frac{1}{z};\frac{p}{q})_{k}}{q^{\frac{k(k+1)}{2}}(\frac{p}{q};\frac{p}{q})_{k}}.$$

The next corollary shows that outside the set $\mathbb{T}_{p,q} = \{0\}\cup\{p^{k}q^{-k}\}_{k=0}^{\infty}$, the polynomial $\mathcal{Q}_{n,k}(p,q;x)$ given in (1.1) tends to infinity.

Corollary 2.1.1. *The undermentioned result holds true:*

$$\lim_{n\to\infty}\mathcal{Q}(p;q;z)=\begin{cases}0 & \text{if } z\in\mathbb{T}_{p,q}\setminus\{0\},\\ 0 & \text{if } z=0,\ k\neq 0,\\ 1 & \text{if } z=0,\ k=0,\\ \infty & \text{otherwise.}\end{cases}$$

Theorem 2.1 suggests that it is interesting to investigate the behavior of polynomial (1.1) in Banach space of analytic function

$$\binom{n}{\nu}_{p,q}:=\frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}=\frac{q^{\frac{n(n+1)}{2}}\left(\frac{p}{q},\frac{p}{q}\right)_n}{q^{\frac{k(k+1)}{2}}q^{\frac{(n-k)(n-k-1)}{2}}\left(\frac{p}{q},\frac{p}{q}\right)_k\left(\frac{p}{q},\frac{p}{q}\right)_{n-k}}\sim\frac{q^{k(n-k)}}{p^{k(n-k)}}. \tag{2.1}$$

Theorem 2.2. *Let* $g_{n,k}(p,q;x)=x^k\prod_{j=0}^{n-k-1}(p^j-q^jx)$. *Then*

$$\|g_{n,n}(p,q;x)\|=\|g_{n,n}(p,q;x)\|_0=1$$

and

$$\|g_{n,k}(p,q;x)\|=\|g_{n,k}(p,q;x)\|_0=\frac{q^{\frac{(n-k)(n-k-1)}{2}}}{n.\,p^{\frac{(n-k)(n-k-1)}{2}}},\qquad k=0,1,\ldots,n-1.$$

Corollary 2.2.1. *For $q>p>1$ and $k=0,1,2\ldots n-1$, the following asymptotic estimate holds true:*

$$\|\mathcal{Q}_{n,k}(p,q;x)\|=\|\mathcal{Q}_{n,k}(p,q;x)\|_0=\frac{q^{\frac{(n-k)(n-k-1)}{2}}}{n.\,p^{\frac{(n-k)(n-k-1)}{2}}},\qquad k=0,1,\ldots,n-1.$$

Corollary 2.2.2. *The following estimate holds:*

$$\|\mathcal{B}^n_{p,q}\|=\frac{q^{\frac{(n-1)n}{2}}}{n.p^{\frac{(n-1)n}{2}}},\qquad n\to\infty.$$

Theorem 2.3. *The following asymptotic estimate holds:*

$$\|\mathcal{B}^n_{p,q}\|\sim\frac{\left(\frac{p^2}{q^2};\frac{p^2}{q^2}\right)_\infty q^{\frac{n(n-1)}{2}}}{n\cdot \mathrm{e}\,p^{\frac{n(n-1)}{2}}},\qquad n\to\infty.$$

3. Main results

In this section, we prove some results related to approximation of analytic function of (p,q)-Bernstein operators and we handle estimation of norm of the operator $\mathcal{Q}_{n,k}(p,q;z)$.

Proof of Theorem 2.1.

(i) For $z\neq 0$, we have

$$\mathcal{Q}_{n,k}(p,q;z)=\binom{n}{\nu}_{p,q}z^n(-1)^{n-k}q^{\frac{(n-k)(n-k-1)}{2}}\left(\frac{p}{z};\frac{p}{z}\right)_{n-k},$$

which by using (2.1) becomes

$$\frac{\mathcal{Q}_{n,k}(z)}{(-1)^n q^{\frac{n(n-1)}{2}} z^n} = \frac{(-1^k)\left(\frac{p}{q};\frac{p}{q}\right)_n \left(\frac{p}{q};\frac{p}{q}\right)_{n-k}}{\left(\frac{p}{q};\frac{p}{q}\right)_k \left(\frac{p}{q};\frac{p}{q}\right)_{n-k} q^{\frac{k(k-1)}{2}}}. \tag{3.1}$$

As we know that

$$\lim_{n\to\infty}\left(\frac{p}{q};\frac{p}{q}\right)_n = \lim_{n\to\infty}\left(\frac{p}{q};\frac{p}{q}\right)_{n-k} = \left(\frac{p}{q};\frac{p}{q}\right)_\infty, \tag{3.2}$$

while $\left(\frac{p}{q};\frac{p}{q}\right)_{n-k} \to \left(\frac{p}{q};\frac{p}{q}\right)_\infty$ as $n \to \infty$ converges uniformly on any compact set $M \subset \mathbb{C} \setminus \{0\}$. The statement follows from (3.1).

(ii) As we know that

$$\binom{n}{n-k}_{p,q} = \binom{n}{k}_{p,q},$$

we can write

$$\mathcal{Q}_{n,n-k}(p,q;z) = \binom{n}{k}_{p,q} z^n (-1)^k q^{\frac{k(k-1)}{2}} \left(\frac{p}{q};\frac{p}{q}\right)_k.$$

Again, by using (2.1), we have

$$\frac{\mathcal{Q}_{n,n-k}(z)}{q^{nk} z^n} = \frac{(-1)^k \left(\frac{p}{q};\frac{p}{q}\right)_k \left(\frac{p}{q};\frac{p}{q}\right)_n}{\left(\frac{p}{q};\frac{p}{q}\right)_k \left(\frac{p}{q};\frac{p}{q}\right)_{n-k} q^{\frac{k(k+1)}{2}}}.$$

By using (3.2), we get the desired result. □

The following lemma tells us about the norm estimation of $h_k(x)$.

Lemma 3.1. *Let $h_k(x) = x^k(1-x)(qx-p)$, $x \in [0,1]$, $k \in \mathbb{N}$. Then we have the following result:*

$$\|h_k\| \sim \|h_k\|_0 \sim \frac{1}{k}.$$

Proof. It can be easily seen that

$$\max\{x^k(1-x)\} = \left(\frac{k}{k+1}\right)\left(1-\frac{k}{k+1}\right) = \frac{1}{k(k+1)^{k+1}} \sim \frac{1}{k}. \tag{3.3}$$

Therefore,

$$\|h_k\| \le \|h_k\|_0 \le \frac{1}{k}.$$

Suppose that $k \ge \frac{2p}{q-p}$, then $\frac{k}{(k+1)} \in \left(\frac{p}{q};1\right)$ and consequently we have

$$\|h_k\| \ge \|h_k\|_0 \ge \left|h_k\left(\frac{k}{k+1}\right)\right| = \frac{1}{k(k+1)^{k+1}} \frac{k(q-p)-1}{(k+1).p} \ge \frac{1}{k}.$$

Hence, we get

$$\|h_k\| \sim \|h_k\|_0 \sim \frac{1}{k}.$$ □

P r o o f o f T h e o r e m 2.2. Since $g_{n,n} = x^n$, the equality

$$\|g_{n,n}(p,q;x)\| = \|g_{n,n}(p,q;x)\|_0 = 1$$

is obvious. The asymptotic relations for $k = n-1$ and $k = n-2$,

$$\|g_{n,n-1}(p,q;x)\| \sim \frac{1}{n} \qquad \text{and} \qquad \|g_{n,n-2}(p,q;x)\| \sim \frac{1}{n},$$

are easily derivable from (3.3) and Lemma 3.1. Also, for points $k = 0,1,\dots,n-3$, we discuss norm of $g_{n,k}(p,q;x)$ as:

$$\|g_{n,k}(p,q;x)\| = \max\big\{\|g_{n,k}\|_{C[0,\ \frac{p^{(n-k-1)}}{q^{(n-k-1)}}]},\ \|g_{n,k}(q,q;x)\|_s,\ s = 0,1,2,\dots,n-k-2\big\}. \tag{3.4}$$

If $x \in [0, \frac{p^{(n-k-1)}}{q^{(n-k-1)}}]$, then $|g_{n,k}(p,q;x)| \le 1$. Wherefrom $\|g_{n,k}(p,q;x)\|_C[0, \frac{p^{(n-k-1)}}{q^{(n-k-1)}}] \le 1$.

Taking $s_0 = n-k-2$, then for $x \in \left(\frac{p^{s_0+1}}{q^{s_0+1}}; \frac{p^{s_0}}{q^{s_0}}\right)$ we get

$$|g_{n,k}(p,q;x)| = x^k \prod_{j=0}^{s_0}(p^j - q^j x)(q^{s_0+1}x - p) \le q^{s_0+1}x - 1 \le q - p \le 1,$$

which shows $\|g_{n,k}(p,q;x)\|_{n-k-2} \le 1$.

For estimating $\|g_{n,k}(p,q;x)\|_0$, consider the norm on the small interval on the right, and choosing $x \in (\frac{p}{q}; 1)$, we have

$$\begin{aligned} |g_{n,k}(p,q;x)| &= x^k(1-x)(qx-p) \prod_{j=2}^{n-k-1}(q^j x - p^j) \\ &= x^{n-2}(1-x)(qx-p)q^{\frac{(n-k)(n-k-1)}{2}-1} \prod_{j=2}^{n-k-1}\Big(1 - \frac{p^j}{q^j}x\Big), \end{aligned}$$

as

$$\begin{aligned} 1 \ge \prod_{j=2}^{n-k-1}\left(1 - \frac{p^j}{q^j x}\right) &= \prod_{j=1}^{n-k-1}\left(1 - q^{-j}\frac{p^j}{qx}\right) \\ &\ge \prod_{j=1}^{n-k-2}\left(1 - \frac{p^j}{q^j}\right) \ge \left(\frac{p}{q}; \frac{p}{q}\right)_\infty. \end{aligned}$$

Now, by applying Lemma 3.1,

$$\|g_{n,k}(p,q;x)\|_0 \sim \frac{q^{\frac{(n-k)(n-k-1)}{2}}}{p^{\frac{(n-k)(n-k-1)}{2}}}\ \|h_{n-2}(p,q;x)\|_0 \sim \frac{q^{\frac{(n-k)(n-k-1)}{2}}}{n.\,p^{\frac{(n-k)(n-k-1)}{2}}}.$$

For $s_0 = 1, 2, \dots, n-k-3$, we take $x \in \left(\frac{p^{s_0+1}}{q^{s_0+1}}; \frac{p^{s_0}}{q^{s_0}}\right)$ so that $y = \frac{q^{s_0}}{p^{s_0}} x \in (\frac{p}{q}, 1)$.
After calculating, we get

$$\begin{aligned}
&|g_{n,k}(p,q;x)| \\
&= x^k (p^{s_0} - q^{s_0}x)(q^{s_0+1}x - p^{s_0+1}) \prod_{s=0}^{s_0-1} (p^s - q^s x) \prod_{s=s_0+2}^{n-k-1} (q^s x - p^s) \\
&= q^{(s_0+2)+\dots+(n-k-1)} x^{n-k-s_0-2} x^k (1 - q^{s_0}x)(q^{s_0+1}x - 1) \prod_{s=0}^{s_0-1} (p^s - q^s x) \prod_{s=s_0}^{n-k-1} \left(1 - \frac{p^s}{q^s x}\right) \\
&\quad \times q^{\frac{(n-k)(n-k-1)}{2} - s_0(2n-1-s_0)/2 - 1} h_{n-s_0-2}(y) \prod_{s=0}^{s_0-1} (p^s - q^s x) \prod_{s=s_0+2}^{n-k-1} \left(1 - \frac{p^s}{q^s x}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
1 \geq \prod_{s=0}^{s_0-1} (p^s - q^s x) &= \prod_{t=1}^{s_0} (1 - q^{-t}(q^{s_0}x)) \\
&\geq \prod_{s=1}^{s_0} (1 - q^{-s}) \geq \left(\frac{p}{q}; \frac{p}{q}\right)_\infty
\end{aligned}$$

and

$$\begin{aligned}
1 \geq \prod_{s=s_0+2}^{n-k-1} \left(1 - \frac{p^s}{q^s x}\right) &= \prod_{t=1}^{n-k-s_0-2} \left(1 - q^{-t} \frac{1}{q^{s_0+1}x}\right) \\
&\geq \prod_{t=1}^{n-k-s_0-2} (1 - q^{-t}) \geq (1 - q^{-t}) \geq \left(\frac{p}{q}; \frac{p}{q}\right)_\infty .
\end{aligned}$$

We have

$$\|g_{n,k}(p,q;x)\|_{s_0} \sim q^{(n-k-1)(n-k)/2 - s_0(2n-k-1)/2} / (n - s_0 - 2).$$

Lastly, by using (3.4), we get the required result

$$\|g_{n,k}(p,q;x)\| \sim \|g_{n,k}(p,q;x)\|_0 \sim \frac{q^{\frac{(n-k-1)(n-k)}{2}}}{n.\, p^{\frac{(n-k-1)(n-k)}{2}}}. \qquad \square$$

Proof of Theorem 2.3. Since we have

$$\|\mathcal{B}^n_{p,q}(x)\| \geq \|\mathcal{Q}_{n,0}(p,q;.)\|,$$

using Corollary 2.2.1 we get

$$\|\mathcal{B}^n_{p,q}(x)\| \geq \frac{q^{\frac{n(n-1)}{2}}}{2}.$$

Again,

$$\begin{aligned}
\|\mathcal{B}^n_{p,q}(x)\| \leq \sum_{k=0}^{n} \| \mathcal{Q}_{n,k}(p,q;.)\| &\leq 1 + \sum_{k=0}^{n-1} \frac{q^{\frac{(n-k)(n+k-1)}{2}}}{n} \\
&\leq 1 + \frac{q^{\frac{n(n-1)}{2}}}{n} \sum_{k=0}^{n-1} q^{\frac{-k(k-1)}{2}}.
\end{aligned}$$

As $\sum_{k=0}^{\infty} q^{\frac{-k(k-1)}{2}} < \infty$, we deduce that $\sum_{k=0}^{n-1} q^{\frac{-k(k-1)}{2}} < \infty$ and hence $\|\mathcal{B}^n_{p,q}(x)\| \le \frac{q^{\frac{n(n-1)}{2}}}{n}$.
Thus,

$$\|\mathcal{B}^n_{p,q}(x)\| \sim \frac{q^{\frac{n(n-1)}{2}}}{n}. \qquad \square$$

Finally, we estimate the strong asymptotic order of the norm of $\|\mathcal{B}^n_{p,q}(x)\|$.

Now for $k \le n-2$, $x \in (\frac{p}{q};1)$ and by (3.1), we have

$$|\mathcal{Q}_{n,k}(p,q;x)| = q^{\frac{(n-k)(n+k-1)}{2}} x^{n-1}(1-x) \frac{\left(\frac{p}{q};\frac{p}{q}\right)_n \left(\frac{p}{q};\frac{p}{qx}\right)_{n-k-1}}{\left(\frac{p}{q};\frac{p}{q}\right)_k \left(\frac{p}{q};\frac{p}{q}\right)_{n-k}}. \tag{3.5}$$

Therefore, for large value of n, it is seen that $\left(1-\frac{1}{n}\right) \in (\frac{p}{q},1)$ and for $k \le n-2$ we obtain:

$$\begin{aligned} \left|\mathcal{Q}_{n,k}\left(p,\, q;\, (1-\frac{1}{n})\right)\right| &= q^{\frac{(n-k)(n+k-1)}{2}} \left(1-\frac{1}{n}\right)^{n-1} \frac{1}{n} \binom{n}{k}_{1/q} \left(\frac{p\,n}{q(n-1)};\frac{p}{q}\right)_{n-k-1} \\ &\ge q^{\frac{(n-k)(n+k-1)}{2}} \frac{1}{\mathrm{e}\cdot n} \binom{n}{k}_{1/q} \left(\frac{p\cdot n}{q(n-1)};\frac{p}{q}\right)_{\infty}. \end{aligned} \tag{3.6}$$

Putting $n=k+1$, we have

$$\begin{aligned} \mathcal{Q}_{n,n-1}\left(p,q;\left(1-\frac{1}{n}\right)\right) &= [n]_{p,q}\left(1-\frac{1}{n}\right)^{n-1}\frac{1}{n} \\ &\ge \left(\frac{q}{p}\right)^{n-1}[n]_{p,q}\frac{1}{n\cdot \mathrm{e}} \ge \left(\frac{q}{p}\right)^{n-1}[n]_{1/q}1/n\cdot \mathrm{e}\left(\frac{p.n}{q(n-1)};\frac{p}{q}\right)_{\infty}, \end{aligned}$$

which agree with (3.6).

For $k=n$ and $n \ge 2$

$$\mathcal{Q}_{n,n}\left(1-\frac{1}{n}\right) = \left(1-\frac{1}{n}\right)^n > \frac{1}{\mathrm{e}}$$

and (3.6) is also true. It follows that

$$\|\mathcal{B}^n_{p,q}\| = \max_{x\in[0,1]}\left(\sum_{k=0}^{n}|\mathcal{Q}_{n,k}|\right) \ge \sum_{k=0}^{n}\left|\mathcal{Q}_{n,k}\left(1-\frac{1}{n}\right)\right|.$$

By using (3.6), we can write

$$q^{\frac{-n(n-1)}{2}}\, n\cdot \mathrm{e}\,\|\mathcal{B}^n_{p,q}\| \ge \left(\frac{p.n}{q(n-1)};\frac{p}{q}\right)_{\infty}\sum_{k=0}^{n} q^{\frac{-k(k-1)}{2}}\binom{n}{k}_{1/q}.$$

Now by using the Rothe's Identity, we get

$$q^{\frac{-n(n-1)}{2}} n\cdot \mathrm{e}\|\mathcal{B}^n_{p,q}\| \ge \left(\frac{n.p}{q(n-1)};\frac{p}{q}\right)_{\infty}\left(-1;\frac{q}{p}\right)_n.$$

Since $g(x) = \left(\frac{p}{q}x;\frac{p}{q}\right)$ is continuous at $x=1$, the limit of right-hand side as $n\to\infty$ exists and equals

$$\left(\frac{p}{q};\frac{p}{q}\right)\left(-1;\frac{p}{q}\right)_{\infty} = 2\left(\frac{p^2}{q^2};\frac{p^2}{q^2}\right).$$

As a result we obtain

$$\liminf_{n\to\infty} q^{\frac{-n(n-1)}{2}} n\cdot \mathrm{e}\,\|\mathcal{B}^n_{p,q}\| \geq\ 2\left(\frac{p^2}{q^2};\frac{p^2}{q^2}\right)_\infty.$$

From above, we now estimate $\|\mathcal{B}^n_{p,q}\|$ in conjunction with the result (3.5) and Theorem 2.3 implies that for $k+2\leq n$ and n large enough, we have

$$\|\mathcal{Q}_{n,k}(p,q;x)\| \leq q^{\frac{(n-k)(n+k-1)}{2}} \frac{\left(\frac{p}{q};\frac{p}{q}\right)_n \left(\frac{p}{q};\frac{p}{qx}\right)_{n-k-1}}{\left(\frac{p}{q};\frac{p}{q}\right)_k \left(\frac{p}{q};\frac{p}{q}\right)_{n-k}} \max_{x\in[p/q;1]}\left\{x^{n-1}(1-x)\right\},$$

$$\|\mathcal{Q}_{n,k}(p,q;x)\| = q^{\frac{(n-k)(n+k-1)}{2}} \frac{\left(\frac{p}{q};\frac{p}{q}\right)_n}{\left(\frac{p}{q};\frac{p}{q}\right)_k \left(\frac{p}{q};\frac{p}{q}\right)_{n-k}} \left(1-\frac{1}{n}\right)^{n-1}\frac{1}{n}. \tag{3.7}$$

Also, we have $\|\mathcal{Q}_{n,n}(p,q;x)\|=1$ and $\|\mathcal{Q}_{n,n-1}(p,q;x)\|\leq \frac{q^{n-1}}{2}$, that implies

$$\lim_{n\to\infty}\|\mathcal{Q}_{n,n}(p,q;x)\|\, n\cdot\mathrm{e}\, q^{\frac{-n(n-1)}{2}} = \lim_{n\to\infty} q^{\frac{-n(n-1)}{2}} n\cdot \mathrm{e}\,\|\mathcal{Q}_{n,n-1}(p,q;x)\| = 0.$$

Consequently, inadequate to estimate $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n-2}\|\mathcal{Q}_{n,n}(p,q;x)\|\, n\cdot \mathrm{e}^{\,q\,\frac{-n(n-1)}{2}}$.

Using (3.7), we have

$$n\cdot\mathrm{e}\, q^{\frac{-n(n-1)}{2}}\sum_{k=0}^{n-2}\|\mathcal{Q}_{n,n}(p,q;x)\| \leq \mathrm{e}.\left(1-\frac{1}{n}\right)^{n-1}\left(\frac{p}{q};\frac{p}{q}\right)\sum_{k=0}^{n-2}\frac{q^{-k(k-1)/2}}{\left(\frac{p}{q};\frac{p}{q}\right)\left(1-(\frac{p}{q})^{n-k}\right)}.$$

For $k,n\in\mathbb{Z}_+$, we fix

$$c^{k,n}_{p,q} = \begin{cases} \dfrac{p^{\frac{k(k-1)}{2}} q^{\frac{-k(k-1)}{2}}}{\left(\frac{p}{q};\frac{p}{q}\right)_k\left(1-(\frac{p}{q})^{n-k}\right)}, & k<n-2\\ 0, & \text{otherwise.}\end{cases}$$

Evidently,

$$\sum_{k=0}^{n-2}\frac{q^{\frac{-k(k-1)}{2}}}{\left(\frac{p}{q};\frac{p}{q}(1-(\frac{1}{q})^{n-k}\right)} = \sum_{k=0}^{\infty} c^{p,q}_{k,n}$$

and

$$\left|c^{p,q^2}_{k,n}\right| \leq \frac{q^{\frac{-k(k-1)}{2}}}{\left(\frac{p}{q^2};\frac{p}{q^2}(1-(\frac{1}{q^2})^2)\right)} = d_k.$$

Because we have $\sum\limits_{k=0}^{\infty} d_k < \infty$, by applying the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} c^{p,q^2}_{k,n} = \sum_{k=0}^{\infty}\left(\lim_{n\to\infty} c^{p,q^2}_{k,n}\right) = \sum_{k=0}^{\infty}\frac{-k(k-1)/2}{(\frac{1}{q^2};\frac{1}{q^2})_k} = \left(-1;\frac{p}{q^2}\right)_\infty.$$

By using Euler's identity, we get

$$\limsup_{n\to\infty} q^{-n(n-1)/2}\, n\cdot\mathrm{e}\|\mathcal{B}^n_{p,q^2}\| \leq \left(\frac{p}{q^2};\frac{p}{q^2}\right)_\infty\left(-1;\frac{p}{q^2}\right)_\infty = 2\left(\frac{p^2}{q^2};\frac{p^2}{q^2}\right)_\infty. \tag{3.8}$$

Now the required result easily follows from (3.7) and (3.8).

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