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TOPOLOGICAL PROPERTIES OF JORDAN INTUITIONISTIC FUZZY NORMED SPACES

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ABSTRACT. The article delves into the task of formulation of meticulous sequence spaces whose elements' convergence is a generalized version of the Cauchy convergence in the setting of intuitionistic fuzzy norm. The task of attaining a finite limit is attained via an infinite matrix operator, namely, Jordan totient operator. We discuss some topological and algebraic properties and establish some inclusion relations between the spaces.

1. Introduction

Researchers have been taking initiatives in proposing novel theories that suggests the convergence of sequences by either operating a function or transforming it into an entirely different sequence using a linear operator. This notion of redefining the sequences' convergence which fail to converge in the usual sense falls under the category of summability theory. It has proved to be quite an indispensable tool to analyse the depths of analysis. One can perceive it as a map with domain and codomain as set of sequences and a finite limit, respectively. The introduction of regular summability method [38] ensures the convergence of the originally convergent sequence to its Cauchy limit. Several authors have deployed the notion of one such naturally occuring regular summability method; regular infinite matrix, see [12, 20, 28, 39]. For a given infinite matrix $\mathcal{P} = (p_{ab})$ and sequence spaces $\mathcal{S}, \mathcal{T}$, the map $\mathcal{P}\colon \mathcal{S} \to \mathcal{T}$ defined by $\mathcal{P}(u) = \sum p_{ab}u_b$ for all $u = (u_b) \in \mathcal{S}$ and $a, b \in \mathbb{N}$, is the linear transformation that acts on a sequence space to produce some finite limit. Readers inquisitive about articles on sequences spaces and summability theory by means of matrix transformation are advised to explore [11, 16, 20, 26, 27].

Thereafter the introduction of fuzzy norm as a generalization of usual norm by Katsaras [19] and Felbin [9]; Bag and Samanta [2] established the base of an even general fuzzy norm. The underlying idea was to define a function that interprets the degree of truth of the length of a vector instead of assuming the length to be a fixed specified scalar. Later it was Saadati and Vaezpour [32] who improvised on the conditions of fuzzy norm. In the wake of coming up with the definition of fuzzy norm defined in terms of degree of truth via membership function, Atanassov [1] pointed out a shortcoming in the fuzzy norm and proposed the set be defined using an additional function named

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non-membership function which defines the degree of falsity or non-belongingess of an element in a given set which he named as intuitionistic fuzzy set.

This introduction of a newfangled idea gave the researchers an insight into defining intuitionistic fuzzy metric space [30] and intuitionistic fuzzy normed space [31] which ultimately led to inception of sequences' convergence, statistical convergence [18], lacunary statistical convergence [29], generalized ideal convergence [7] and summability method [33] in the underlying space.

One such definite matrix operator namely Jordan totient matrix operator denoted by Υ^r was introduced in [15] via Jordan totient J_r function whose domain and codomain are $\mathbb{N}$. The function is the number of r tuples $(h_1, h_2, \ldots, h_r)$ such that $1 \leq h_i \leq n$ and $\gcd(h_1, h_2, \ldots, h_r, n) = 1$. It is defined as $J_r(n) = n^r \prod\limits_{p|n} \left(1 - \frac{1}{p^r}\right)$ where $h = p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_k^{\alpha_k}$ for $\alpha \geq 1$ is the prime decomposition of n. Thereupon the Jordan totient matrix operator denoted by $\Upsilon^r = (\nu_{nk}^r)$ is defined as:

$$\nu_{nk}^r = \begin{cases} \dfrac{J_r(k)}{n^r} & \text{if } k|n, \\ 0 & \text{otherwise}, \end{cases} \tag{1.1}$$

and its inverse $(\Upsilon^r)^{-1}$ is given by

$$(\Upsilon^r)^{-1} = \begin{cases} \dfrac{\mu(\frac{n}{k})}{J_r(n)} k^r & \text{if } k|n, \\ 0 & \text{otherwise}, \end{cases} \tag{1.2}$$

where μ is the Möbius function defined as:

$$\mu(n) = \begin{cases} 0 & \text{if } p^2|n \text{ for some prime } p, \\ 1 & \text{if } n = 1, \\ (-1)^i & \text{if } n = \prod\limits_{k=1}^{i} p_k \text{ where } p_k\text{s are distinct.} \end{cases}$$

Later Kara [17], İlkhan [14] and Khan [21] used the operator to study sequence spaces and expounded certain riveting results. In the ongoing article we define sequence spaces with the help of Jordan totient matrix operator in the setting of intuitionistic fuzzy normed spaces, study the ideal convergence of these sequences, present compelling counter examples and study algebraic and topological properties of these spaces.

2. Preliminaries

For two sequence spaces $\mathcal{S}, \mathcal{T}$ and an infinite matrix $\mathcal{P} = (p_{nk})$, the $\mathcal{P}$ transform of $u = (u_k)$ is given by $\mathcal{P}u = \{\mathcal{P}_n(u)\}_{n=1}^{\infty} \in \mathcal{T}$, where

$$\mathcal{P}_n(u) = \sum_{k=1} p_{nk} u_k, \qquad n \in \mathbb{N}.$$

Lemma 2.1 ([38]). *An infinite matrix $\mathcal{P} = (p_{nk})$ is regular iff:*

(i) *for every $n \in \mathbb{N}$, there exists $\mathcal{M} > 0$, such that $\sum\limits_k |p_{nk}| \leq \mathcal{M}$,*

(ii) $\lim\limits_{n\to\infty} p_{nk} = 0$ *for all $k \in \mathbb{N}$,*

(iii) $\lim\limits_{n\to\infty} \sum\limits_k p_{nk} = 1.$

The article will employ the use of the Jordan totient infinite matrix operator $\Upsilon^r = (\nu^r_{nk})$ which is defined as:

$$\nu^r_{nk} = \begin{cases} \dfrac{J_r(k)}{n^r} & \text{if } k|n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Equivalently,

$$\Upsilon^r = \begin{bmatrix} \frac{J_r(1)}{1} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{J_r(2)}{2^r} & \frac{J_r(1)}{2^r} & 0 & 0 & 0 & 0 & \cdots \\ \frac{J_r(1)}{3^r} & 0 & \frac{J_r(3)}{3^r} & 0 & 0 & 0 & \cdots \\ \frac{J_r(1)}{4^r} & \frac{J_r(2)}{4^r} & 0 & \frac{J_r(4)}{4^r} & 0 & 0 & \cdots \\ \frac{J_r(1)}{5^r} & 0 & 0 & 0 & \frac{J_r(5)}{5^r} & 0 & \cdots \\ \frac{J_r(1)}{6^r} & \frac{J_r(2)}{6^r} & \frac{J_r(3)}{6^r} & 0 & 0 & \frac{J_r(6)}{6^r} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The Υ^r transform of $(u_k) \in \omega$ is defined as $\Upsilon^r_n(u) := \frac{1}{n^r}\sum J_r(k)u_k$.

Definition 1 ([25]). A cluster of subsets $\mathcal{I}$ of a non-empty set $\mathcal{X}$ is said to be an ideal in $\mathcal{X}$ if:

- $\emptyset \in \mathcal{I}$,
- $\mathcal{A}, \mathcal{B} \in \mathcal{I} \Rightarrow \mathcal{A} \cup \mathcal{B} \in \mathcal{I}$,
- $\mathcal{A} \in \mathcal{I}, \mathcal{B} \subseteq A \Rightarrow B \in \mathcal{I}$.

An ideal $\mathcal{I} \subseteq 2^{\mathcal{X}}$ such that $\mathcal{I} \neq 2^{\mathcal{X}}$ is a nontrivial ideal. It transforms into an admissible ideal if $\mathcal{I}$ contains each singleton subset of $\mathcal{X}$ and into a maximal ideal if there does not exist any nontrivial ideal $\mathcal{J} \neq \mathcal{I}$ such that $\mathcal{I} \subset \mathcal{J}$.

Definition 2 ([25]). A cluster of subsets $\mathcal{F}$ of a non-empty set $\mathcal{X}$ is said to be a filter in $\mathcal{X}$ if:

- $\emptyset \notin \mathcal{F}$,
- $\mathcal{A}, \mathcal{B} \in \mathcal{F} \Rightarrow \mathcal{A} \cap \mathcal{B} \in \mathcal{F}$,
- $\mathcal{A} \in \mathcal{F}$ and $\mathcal{B} \supset \mathcal{A}$ imply $\mathcal{B} \in \mathcal{F}$.

$\mathcal{F}(\mathcal{I}) = \{\mathcal{K} \subset \mathcal{X} : \mathcal{K}^c \in \mathcal{I}\}$ is the filter associated with ideal $\mathcal{I}$. All through the paper, we take $\mathcal{I}$ as an admissible ideal in $\mathbb{N}$.

Definition 3 ([25]). A sequence $u = (u_k) \in \omega$ is said to be $\mathcal{I}$-convergent to $c \in \mathbb{C}$ if for every $\varepsilon > 0$,

$$\{k \in \mathbb{N} : |u_k - c| \geq \varepsilon\} \in \mathcal{I}. \tag{2.2}$$

The commonly used mathematical notation to depict the above definition of convergence is $\mathcal{I}\text{-}\lim u_k = c$.

Definition 4 ([37]). Let $\mathcal{I} \subseteq P(\mathbb{N})$ is a non trivial ideal, for any two sequences (u_k) and (v_k), we say $u_k = v_k$ for $a.a.k.r.\mathcal{I}$ if $\{k \in \mathbb{N} : u_k \neq v_k\} \in \mathcal{I}$.

Definition 5 ([25]). A sequence $u = (u_k) \in \omega$ is said to be $\mathcal{I}$-Cauchy if, for each $\varepsilon > 0$, there exists a number $\mathcal{J} = \mathcal{J}(\varepsilon)$ such that the set

$$\{k \in \mathbb{N} : |u_k - u_{\mathcal{J}}| \geq \varepsilon\} \in \mathcal{I}.$$

Definition 6 ([34]). A binary operation $*\colon [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if:

- $*$ is associative and commutative,

- $*$ is continuous,
- $a * 1 = a$ for all $a \in [0, 1]$,
- $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Definition 7 ([34])**.** A binary operation $\diamond\colon [0,1]\times[0,1] \to [0,1]$ is said to be a continuous t-conorm if:

- $\diamond$ is associative and commutative,
- $\diamond$ is continuous,
- $a \diamond 0 = a$ for all $a \in [0, 1]$,
- $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Definition 8 ([31])**.** Let $\mathcal{X}$ is a linear space, $*$ & $\diamond$ are continuous t-norm and t-conorm respectively and ϕ, ψ are fuzzy sets on $\mathcal{X} \times (0, \infty)$. The five-tuple $(\mathcal{X}, \phi, \psi, *, \diamond)$ is an intuitionistic fuzzy normed space if for every $u, v \in \mathcal{X}$ and $s, t > 0$ the following conditions hold:

- $\phi(u,t) + \psi(u,t) \leq 1$,
- $\phi(u,t) > 0$,
- $\phi(u,t) = 1$ if and only if $u = 0$,
- $\phi(\alpha u, t) = \phi\big(u, \frac{t}{\alpha}\big)$ for each $\alpha \neq 0$,
- $\phi(u,t) * \phi(v,s) \leq \phi(u+v, t+s)$,
- $\phi(u, \cdot)\colon (0,\infty) \to [0,1]$ is continuous,
- $\lim\limits_{t\to\infty} \phi(u,t) = 1$ and $\lim\limits_{t\to 0} \phi(u,t) = 0$,
- $\psi(u,t) < 1$,
- $\psi(u,t) = 0$ if and only if $u = 0$,
- $\psi(\alpha u, t) = \psi\big(u, \frac{t}{\alpha}\big)$ for each $\alpha \neq 0$,
- $\psi(u,t) * \psi(v,s) \geq \psi(u+v, t+s)$,
- $\psi(u, \cdot)\colon (0,\infty) \to [0,1]$ is continuous,
- $\lim\limits_{t\to\infty} \psi(u,t) = 0$ and $\lim\limits_{t\to 0} \psi(u,t) = 1$.

Consequently the pair (ϕ, ψ) is called intuitionistic fuzzy norm.

Definition 9 ([22])**.** Let $(\mathcal{X}, \phi, \psi, *, \diamond)$ be an IFNS. A sequence $u = (u_k)$ is said to be $\mathcal{I}$-convergent to $l \in \mathcal{X}$ with respect to intuitionistic fuzzy norms (ϕ, ψ), if for every $\varepsilon > 0$ and $t > 0$, the set

$$\big\{k \in \mathbb{N} : \phi\big(u_k - l, t\big) \leq 1 - \varepsilon \text{ or } \psi\big(u_k - l, t\big) \geq \varepsilon\big\} \in \mathcal{I}.$$

The notation $\mathcal{I}^{(\phi,\psi)}\text{-}\lim u_k = l$ will be used in the article to denote the ideal convergence of the sequence (u_k) to l with respect to the intuitionistic fuzzy norm (ϕ, ψ).

Definition 10 ([29])**.** Let $(\mathcal{X}, \phi, \psi, *, \diamond)$ be an IFNS. A sequence $u = (u_k)$ is said to be $\mathcal{I}$-Cauchy sequence with respect to (ϕ, ψ), if for every $\varepsilon > 0$ and $t > 0$, there exists $\mathcal{N} = \mathcal{N}(\varepsilon) \in \mathbb{N}$ such that the set

$$\big\{k \in \mathbb{N} : \phi(u_k - u_{\mathcal{N}}, t) \leq 1 - \varepsilon \text{ or } \psi(u_k - u_{\mathcal{N}}, t) \geq \varepsilon\big\} \in \mathcal{I}.$$

Definition 11. Let $(\mathcal{X}, \phi, \psi, *, \diamond)$ be an IFNS. Then $(\mathcal{X}, \phi, \psi, *, \diamond)$ is said to be complete if every Cauchy sequence is convergent with respect to the intuitionistic fuzzy norm (ϕ, ψ).

3. Main results

Through the whole of the current section we presume the ideal $\mathcal{I}$ to be a nontrivial admissible ideal of subset of $\mathbb{N}$. We set out the following sequence spaces:

$$c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r) = \big\{u=(u_k)\in\omega : \{n\in\mathbb{N} : \text{for some } l\in\mathbb{C}, \phi\big(\Upsilon^r_n(u)-l,t\big)\le 1-\varepsilon \text{ or } \psi\big(\Upsilon^r_n(u)-l,t\big)\ge\varepsilon\}\in\mathcal{I}\big\}, \tag{3.1}$$

$$c^{\mathcal{I}}_{0(\phi,\psi)}(\Upsilon^r) = \big\{u=(u_k)\in\omega : \{n\in\mathbb{N} :\ \phi\big(\Upsilon^r_n(u),t\big)\le 1-\varepsilon \text{ or } \psi\big(\Upsilon^r_n(u),t\big)\ge\varepsilon\}\in\mathcal{I}\big\}, \tag{3.2}$$

$$\ell^{\mathcal{I}}_{\infty(\phi,\psi)}(\Upsilon^r) = \big\{u=(u_k)\in\omega : \{n\in\mathbb{N} :\ \exists\,\xi\in(0,1), \phi\big(\Upsilon^r_n(u),t\big)\le 1-\xi \text{ or } \psi\big(\Upsilon^r_n(u),t\big)\ge\xi\}\in\mathcal{I}\big\}. \tag{3.3}$$

$$\ell_{\infty(\phi,\psi)}(\Upsilon^r) = \big\{u=(u_k)\in\omega : \{n\in\mathbb{N} :\ \exists\,\xi\in(0,1),\ \phi\big(\Upsilon^r_n(u),t\big)\ge 1-\xi \text{ or } \psi\big(\Upsilon^r_n(u),t\big)\le\xi\}. \tag{3.4}$$

We bring forth the following definitions of open ball and closed ball with centre at u and radius $r>0$ with respect to parameter of fuzziness $\varepsilon\in(0,1)$ as follows:

$$\mathcal{B}^{\mathcal{I}}_u(r,\varepsilon)(\Upsilon^r) = \big\{v=(v_k)\in\omega : \{n\in\mathbb{N} : \phi\big(\Upsilon^r_n(u)-\Upsilon^r_n(v),r\big)\le 1-\varepsilon \text{ or } \psi\big(\Upsilon^r_n(u)-\Upsilon^r_n(v),r\big)\ge\varepsilon\}\in\mathcal{I}\big\} \tag{3.5}$$

$$\mathcal{B}^{\mathcal{I}}_u[r,\varepsilon](\Upsilon^r) = \big\{v=(v_k)\in\omega : \{n\in\mathbb{N} : \phi\big(\Upsilon^r_n(u)-\Upsilon^r_n(v),r\big)< 1-\varepsilon \text{ or } \psi\big(\Upsilon^r_n(u)-\Upsilon^r_n(v),r\big)>\varepsilon\}\in\mathcal{I}\big\}. \tag{3.6}$$

Theorem 3.1. *The spaces $c^{\mathcal{I}}_{0(\phi,\psi)}(\Upsilon^r)$ and $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ are linear spaces.*

Proof. The proof of linearity of the space $c^{\mathcal{I}}_{0(\phi,\psi)}(\Upsilon^r)$ can be conclusively drawn on parallels of $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$. Given arbitrary sequences $a=(a_k), b=(b_k)\in c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ imply existence of $a_0,b_0\in\mathbb{C}$ so that (a_k) and (b_k) $\mathcal{I}$–converge to a_0 and b_0, respectively. For $t>0, 0<\varepsilon<1$ and $\gamma,\lambda\in\mathbb{R}$, consider the subsequent sets:

$$\mathscr{A} = \Big\{n\in\mathbb{N} : \phi\Big(\Upsilon^r_n(a)-a_0,\frac{t}{2|\gamma|}\Big)\le 1-\varepsilon \text{ or } \psi\Big(\Upsilon^r_n(a)-a_0,\frac{t}{2|\gamma|}\Big)\ge\varepsilon\Big\}\in\mathcal{I},$$
$$\mathscr{A}^c = \Big\{n\in\mathbb{N} : \phi\Big(\Upsilon^r_n(a)-a_0,\frac{t}{2|\gamma|}\Big)> 1-\varepsilon \text{ or } \psi\Big(\Upsilon^r_n(a)-a_0,\frac{t}{2|\gamma|}\Big)<\varepsilon\Big\}\in\mathcal{F}(\mathcal{I}),$$
$$\mathscr{B} = \Big\{n\in\mathbb{N} : \phi\Big(\Upsilon^r_n(b)-b_0,\frac{t}{2|\lambda|}\Big)\le 1-\varepsilon \text{ or } \psi\Big(\Upsilon^r_n(b)-b_0,\frac{t}{2|\lambda|}\Big)\ge\varepsilon\Big\}\in\mathcal{I},$$
$$\mathscr{B}^c = \Big\{n\in\mathbb{N} : \phi\Big(\Upsilon^r_n(b)-b_0,\frac{t}{2|\lambda|}\Big)> 1-\varepsilon \text{ or } \psi\Big(\Upsilon^r_n(a)-a_0,\frac{t}{2|\lambda|}\Big)<\varepsilon\Big\}\in\mathcal{F}(\mathcal{I}).$$

The set $\mathscr{C}=\mathscr{A}^c\cap\mathscr{B}^c$ being a non-empty set lies in $\mathcal{F}(\mathcal{I})$, so consider $n\in\mathscr{C}$, then

$$\begin{aligned}\phi\Big(\Upsilon^r_n(\gamma a+\lambda b)-(\gamma a_0+\lambda b_0),t\Big) &\ge \phi\Big(\gamma\Upsilon^r_n(a)-\gamma a_0,\frac{t}{2}\Big)*\phi\Big(\lambda\Upsilon^r_n(b)-\lambda b_0,\frac{t}{2}\Big)\\ &= \phi\Big(\Upsilon^r_n(a)-a_0,\frac{t}{2|\gamma|}\Big)*\phi\Big(\Upsilon^r_n(b)-b_0,\frac{t}{2|\lambda|}\Big)\\ &> (1-\varepsilon)*(1-\varepsilon)=1-\varepsilon\end{aligned}$$

$$\implies\quad \phi\Big(\Upsilon^r_n(\gamma a+\lambda b)-(\gamma a_0+\lambda b_0),t\Big)>1-\varepsilon$$

and

$$\begin{aligned}\psi\Big(\Upsilon_n^r(\gamma a+\lambda b)-(\gamma a_0+\lambda b_0),t\Big) &\le \psi\Big(\gamma\Upsilon_n^r(a)-\gamma a_0,\frac{t}{2}\Big)\diamond\psi\Big(\lambda\Upsilon_n^r(b)-\lambda b_0,\frac{t}{2}\Big)\\ &=\psi\Big(\Upsilon_n^r(a)-a_0,\frac{t}{2|\gamma|}\Big)\diamond\psi\Big(\Upsilon_n^r(b)-b_0,\frac{t}{2|\lambda|}\Big)\\ &<\varepsilon\diamond\varepsilon=\varepsilon\\ \implies\quad \psi\Big(\Upsilon_n^r(\gamma a+\lambda b)-(\gamma a_0+\lambda b_0),t\Big)&<\varepsilon.\end{aligned}$$

Thereupon, we conclude

$$\begin{aligned}\mathscr{C}\subset &\Big\{n\in\mathbb{N}:\phi\Big(\Upsilon_n^r(\gamma a+\lambda b)-(\gamma a_0+\lambda b_0),t\Big)>1-\varepsilon\Big\}\\ \text{or}\quad &\Big\{n\in\mathbb{N}:\psi\Big(\Upsilon_n^r(\gamma a+\lambda b)-(\gamma a_0+\lambda b_0),t\Big)<\varepsilon\Big\}.\end{aligned}$$

By employing the properties of $\mathcal{F}(\mathcal{I})$, we have

$$\begin{aligned}&\big\{n\in\mathbb{N}:\phi\big(\Upsilon_n^r(\gamma a+\lambda b)-(\gamma a_0+\lambda b_0),t\big)>1-\varepsilon\\ &\text{or }\psi\big(\Upsilon_n^r(\gamma a+\lambda b)-(\gamma a_0+\lambda b_0),t\big)<\varepsilon\big\}\in\mathcal{F}(\mathcal{I}),\end{aligned}$$

which implies that the sequence $(\gamma a_k+\lambda b_k)$ $\mathcal{I}$-converges to $\gamma a_0+\lambda b_0$. Therefore, $(\gamma a_k+\lambda b_k)\in c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$. Hence, $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ is a linear space. □

Theorem 3.2. *The inclusion relation $c^{\mathcal{I}}_{0(\phi,\psi)}(\Upsilon^r)\subset c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)\subset c^{\mathcal{I}}_{\infty(\phi,\psi)}(\Upsilon^r)$ holds.*

Proof. The inclusion of $c^{\mathcal{I}}_{0(\phi,\psi)}(\Upsilon^r)$ in $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ is pretty evident. We demonstrate that $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)\subset c^{\mathcal{I}}_{\infty(\phi,\psi)}(\Upsilon^r)$. Consider the sequence $u=(u_k)\in c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$. Then there exists $l\in\mathbb{C}$ such that $\mathcal{I}_{(\phi,\psi)}(\Upsilon^r)$-$\lim(u_k)=l$, and for every $0<\varepsilon<1$ and $t>0$, the set

$$\mathscr{X}=\Big\{n\in\mathbb{N}:\phi\Big(\Upsilon_n^r(u)-l,\frac{t}{2}\Big)>1-\varepsilon\text{ or }\psi\Big(\Upsilon_n^r(u)-l,\frac{t}{2}\Big)<\varepsilon\Big\}\in\mathcal{F}(\mathcal{I}).$$

Let $\phi\big(l,\frac{t}{2}\big)=p$ and $\psi\big(l,\frac{t}{2}\big)=q$ where $p,q\in(0,1)$, $t>0$ and $0<\varepsilon<1$, there exist $c,d\in(0,1)$ such that $(1-\varepsilon)*p>1-c$ and $\varepsilon\diamond q<d$. So for $n\in\mathscr{X}$, we have

$$\phi\big(\Upsilon_n^r(u),t\big)=\phi\big(\Upsilon_n^r(u)-l+l,t\big)\ge\phi\Big(\Upsilon_n^r(u)-l,\frac{t}{2}\Big)*\phi\Big(l,\frac{t}{2}\Big)>(1-\varepsilon)*p>1-c$$

and

$$\psi\big(\Upsilon_n^r(u),t\big)=\psi\big(\Upsilon_n^r(u)-l+l,t\big)\le\psi\Big(\Upsilon_n^r(u)-l,\frac{t}{2}\Big)\diamond\psi\Big(l,\frac{t}{2}\Big)<(1-\varepsilon)\diamond q<d.$$

Taking $h=\max\{c,d\}$, we have

$$\begin{aligned}&\big\{n\in\mathbb{N},\exists\,h\in(0,1):\phi\big(\Upsilon_n^r(u)-l,t\big)>1-h\text{ or }\psi\big(\Upsilon_n^r(u)-l,t\big)<h\big\}\in\mathcal{F}(\mathcal{I})\\ &\implies\quad u=(u_k)\in c^{\mathcal{I}}_{\infty(\phi,\psi)}(\Upsilon^r).\end{aligned}$$ □

The converse of the inclusion relation does not hold. We present the following examples in support of our claim.

***Example* 1.** Let $(\mathbb{R},\|.\|)$ be a normed space equipped with supremum norm, $a*b=\min\{a,b\}$ and $a\diamond b=\max\{a,b\}$ for all $a,b\in(0,1)$. Consider the norms (ϕ,ψ) on $\mathcal{X}^2\times(0,\infty)$ as follows:

$$\phi(u,t)=\frac{t}{t+\|u\|}\qquad\text{and}\qquad\psi(u,t)=\frac{\|u\|}{t+\|u\|}.$$

Then $(\mathbb{R},\phi,\psi,*,\diamond)$ is a standard IFNS. Consider the sequence $(u_k)=\{u_0+\frac{1}{k}\}$ where $u_0\neq 0\in\mathbb{R}$. The sequence (u_k) distinctly lies in $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)\smallsetminus c^{\mathcal{I}}_{0(\phi,\psi)}(\Upsilon^r)$.

***Example* 2.** Let $(\mathbb{R}, \|\cdot\|)$ be the normed space equipped with the intutionistic fuzzy norms (ϕ, ψ) as aforementioned above. Consider the sequence $(u_k) = \sin(\frac{1}{k})$. Then, $(u_k) \in c^{\mathcal{I}}_{\infty(\phi,\psi)}(\Upsilon^r) \smallsetminus c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$.

Theorem 3.3. *Every open ball with centre at* z *and radius* $r > 0$ *with respect to parameter of fuzziness* $0 < \varepsilon < 1$*, i.e.,* $\mathcal{B}^{\mathcal{I}}_z(r,\varepsilon)(\Upsilon^r)$ *is an open set in* $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$.

P r o o f. Consider the open ball with centre at z and radius $r > 0$ with parameter of fuzziness $0 < \varepsilon < 1$,

$$\begin{aligned} \mathcal{B}^{\mathcal{I}}_z\big(r,\varepsilon\big)(\Upsilon^r_n) &= \big\{y = (y_k) \in \omega : \{n \in \mathbb{N} : \phi\big(\Upsilon^r_n(z) - \Upsilon^r_n(y), r\big) \le 1-\varepsilon \\ &\qquad \text{or } \psi\big(\Upsilon^r_n(z) - \Upsilon^r_n(y), r\big) \ge \varepsilon\} \in \mathcal{I}\big\} \\ \Longrightarrow \quad \mathcal{B}^{\mathcal{I}}_z\big(r,\varepsilon\big)(\Upsilon^r_n) &= \big\{y = (y_k) \in \omega : \{n \in \mathbb{N} : \phi\big(\Upsilon^r_n(z) - \Upsilon^r_n(y), r\big) > 1-\varepsilon \\ &\qquad \text{or } \psi\big(\Upsilon^r_n(z) - \Upsilon^r_n(y), r\big) < \varepsilon\} \in \mathcal{F}(\mathcal{I})\big\}. \end{aligned}$$

Consider the element $y = (y_k) \in \mathcal{B}^{\mathcal{I}}_z\big(r,\varepsilon\big)(\Upsilon^r_n)$. Then its corresponding set

$$\big\{n \in \mathbb{N} : \phi\big(\Upsilon^r_n(z) - \Upsilon^r_n(y), r\big) > 1-\varepsilon \text{ or } \psi\big(\Upsilon^r_n(z) - \Upsilon^r_n(y), r\big) < \varepsilon\big\} \in \mathcal{F}(\mathcal{I}).$$

For $\phi\big(\Upsilon^r_n(z) - \Upsilon^r_n(y), r\big) > 1-\varepsilon$ and $\psi\big(\Upsilon^r_n(z) - \Upsilon^r_n(y), r\big) < \varepsilon$, there exists $r_0 \in (0,r)$ such that $\phi\big(\Upsilon^r_n(z) - \Upsilon^r_n(y), r_0\big) > 1-\varepsilon$ and $\psi\big(\Upsilon^r_n(z) - \Upsilon^r_n(y), r_0\big) < \varepsilon$. Setting $\varepsilon_0 = \phi\big(\Upsilon^r_n(z) - \Upsilon^r_n(y), r_0\big)$ we get $\varepsilon_0 > 1-\varepsilon$ which further concludes the existence of an element $s \in (0,1)$ such that $\varepsilon_0 > 1-s > 1-\varepsilon$. For a given $\varepsilon_0 > 1-s$, we can find $\varepsilon_1, \varepsilon_2 \in (0,1)$ such that $\varepsilon_0 * \varepsilon_1 > 1-s$ and $(1-\varepsilon_0) \diamond (1-\varepsilon_2) < s$. Assume $\varepsilon_3 = \max\{\varepsilon_1, \varepsilon_2\}$. Consider the open ball $\mathcal{B}^{\mathcal{I}}_y\big(r-r_0, 1-\varepsilon_3\big)(\Upsilon^r_n)$. The containment of $\mathcal{B}^{\mathcal{I}}_y\big(r-r_0, 1-\varepsilon_3\big)(\Upsilon^r_n)$ in $\mathcal{B}^{\mathcal{I}}_z\big(r,\varepsilon\big)(\Upsilon^r_n)$ will give us the desired result.
Let $w = (w_k) \in \mathcal{B}^{\mathcal{I}}_y\big(r-r_0, 1-\varepsilon_3\big)(\Upsilon^r_n)$, then $\{n \in \mathbb{N} : \phi\big(\Upsilon^r_n(y) - \Upsilon^r_n(w), r-r_0\big) > \varepsilon_3$ or $\psi\big(\Upsilon^r_n(y) - \Upsilon^r_n(w), r-r_0\big) < 1-\varepsilon_3\} \in \mathcal{F}(\mathcal{I})$. Therefore,

$$\begin{aligned} \phi\big(\Upsilon^r_n(z) - \Upsilon^r_n(w), r\big) &\ge \phi\big(\Upsilon^r_n(z) - \Upsilon^r_n(y), r_0\big) * \phi\big(\Upsilon^r_n(y) - \Upsilon^r_n(w), r-r_0\big) \\ &\ge \varepsilon_0 * \varepsilon_3 \ge \varepsilon_0 * \varepsilon_1 > (1-s) > (1-\varepsilon) \\ \Longrightarrow \quad &\big\{n \in \mathbb{N} : \phi\big(\Upsilon^r_n(z) - \Upsilon^r_n(w), r\big) > 1-\varepsilon\big\} \in \mathcal{F}(\mathcal{I}), \end{aligned}$$

and correspondingly

$$\begin{aligned} \psi\big(\Upsilon^r_n(z) - \Upsilon^r_n(w), r\big) &\le \psi\big(\Upsilon^r_n(z) - \Upsilon^r_n(y), r_0\big) \diamond \psi\big(\Upsilon^r_n(y) - \Upsilon^r_n(w), r-r_0\big) \\ &\le (1-\varepsilon_0) \diamond (1-\varepsilon_3) \le (1-\varepsilon_0) \diamond (1-\varepsilon_2) < s < \varepsilon \\ \Longrightarrow \quad &\big\{n \in \mathbb{N} : \psi\big(\Upsilon^r_n(z) - \Upsilon^r_n(w), r\big) < \varepsilon\big\} \in \mathcal{F}(\mathcal{I}). \end{aligned}$$

Thus the set

$$\begin{aligned} &\big\{n \in \mathbb{N} : \phi\big(\Upsilon^r_n(z) - \Upsilon^r_n(w), r\big) > 1-\varepsilon \text{ or } \psi\big(\Upsilon^r_n(z) - \Upsilon^r_n(w), r\big) < \varepsilon\big\} \in \mathcal{F}(\mathcal{I}) \\ &\qquad \Longrightarrow \quad \mathcal{B}^{\mathcal{I}}_y\big(r-r_0, 1-\varepsilon_3\big)(\Upsilon^r_n) \subset \mathcal{B}^{\mathcal{I}}_z\big(r,\varepsilon\big)(\Upsilon^r_n). \end{aligned}$$

□

Remark 1. The spaces $c^{\mathcal{I}}_{0(\phi,\psi)}(\Upsilon^r)$ and $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ are IFNS with respect to intuitionistic fuzzy norms (ϕ, ψ).

Remark 2. $\tau^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r) = \big\{\mathscr{A} \subset c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$: for each $z = (z_k) \in \mathscr{A}$, there exist $r > 0$ and $\varepsilon \in (0,1)$ such that $\mathcal{B}^{\mathcal{I}}_z\big(r,\varepsilon\big)(\Upsilon^r_n) \subset \mathscr{A}\big\}$. Then $\tau^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ defines a topology on the sequence space $\tau^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$. The collection defined by $\mathcal{B} = \big\{\mathcal{B}^{\mathcal{I}}_z(r,\varepsilon) : z \in c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r), r > 0 \text{ and } \varepsilon \in (0,1)\big\}$ is a base for the topology $\tau^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ on the space $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$.

Theorem 3.4. *The spaces* $c^{\mathcal{I}}_{0(\phi,\psi)}(\Upsilon^r)$ *and* $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ *are Hausdorff spaces.*

Proof. Let $v = (v_k)$ and $w = (w_k) \in c^{\mathcal{I}}_{0(\phi,\psi)}(\Upsilon^r)$ such that $v \neq w$. Then for each $n \in \mathbb{N}$ and $r > 0$, implies $0 < \phi\big(\Upsilon^r_n(v) - \Upsilon^r_n(w), r\big) < 1$ and $0 < \psi\big(\Upsilon^r_n(v) - \Upsilon^r_n(w), r\big) < 1$.

Putting $\varepsilon_1 = \phi\big(\Upsilon^r_n(v) - \Upsilon^r_n(w), r\big)$, $\varepsilon_2 = \psi\big(\Upsilon^r_n(v) - \Upsilon^r_n(w), r\big)$ and $\varepsilon = \max\{\varepsilon_1, 1-\varepsilon_2\}$. Then for each $\varepsilon_0 > \varepsilon$, there exist $\varepsilon_3, \varepsilon_4 \in (0,1)$ such that $\varepsilon_3 * \varepsilon_3 \geq \varepsilon_0$ and $(1-\varepsilon_4)\diamond(1-\varepsilon_4) \leq (1-\varepsilon_0)$. Assigning $\varepsilon_5 = \max\{\varepsilon_3, \varepsilon_4\}$, consider the open balls $\mathcal{B}^{\mathcal{I}}_v\big(1-\varepsilon_5, \frac{r}{2}\big)(\Upsilon^r_n)$ and $\mathcal{B}^{\mathcal{I}}_w\big(1-\varepsilon_5, \frac{r}{2}\big)(\Upsilon^r_n)$ centered at v and w respectively. We demonstrate that $\mathcal{B}^{\mathcal{I}}_v\big(1-\varepsilon_5, \frac{r}{2}\big)(\Upsilon^r_n) \cap \mathcal{B}^{\mathcal{I}}_w\big(1-\varepsilon_5, \frac{r}{2}\big)(\Upsilon^r_n) = \phi$. If viable let $z = (z_k) \in \mathcal{B}^{\mathcal{I}}_v\big(1-\varepsilon_5, \frac{r}{2}\big)(\Upsilon^r_n) \cap \mathcal{B}^{\mathcal{I}}_w\big(1-\varepsilon_5, \frac{r}{2}\big)(\Upsilon^r_n)$. Then for the set $\{n \in \mathbb{N}\} \in \mathcal{F}(\mathcal{I})$, we have

$$\begin{aligned} \varepsilon_1 &= \phi\big(\Upsilon^r_n(v) - \Upsilon^r_n(w), r\big) \\ &\geq \phi\Big(\Upsilon^r_n(v) - \Upsilon^r_n(z), \frac{r}{2}\Big) * \phi\Big(\Upsilon^r_n(z) - \Upsilon^r_n(w), \frac{r}{2}\Big) \\ &> \varepsilon_5 * \varepsilon_5 \geq \varepsilon_3 * \varepsilon_3 \geq \varepsilon_0 > \varepsilon_1 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \varepsilon_2 &= \psi\big(\Upsilon^r_n(v) - \Upsilon^r_n(w), r\big) \\ &\leq \psi\Big(\Upsilon^r_n(v) - \Upsilon^r_n(z), \frac{r}{2}\Big) \diamond \psi\Big(\Upsilon^r_n(z) - \Upsilon^r_n(w), \frac{r}{2}\Big) \\ &< (1-\varepsilon_5) \diamond (1-\varepsilon_5) \leq (1-\varepsilon_4) \diamond (1-\varepsilon_4) \leq (1-\varepsilon_0) < \varepsilon_2. \end{aligned} \tag{3.8}$$

Equation (3.7) leads to contradiction.Therefore, $\mathcal{B}^{\mathcal{I}}_v\big(1-\varepsilon_5, \frac{r}{2}\big)(\Upsilon^r_n) \cap \mathcal{B}^{\mathcal{I}}_w\big(1-\varepsilon_5, \frac{r}{2}\big)(\Upsilon^r_n) = \phi$. Hence, the space $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ is a Hausdorff space. □

THEOREM 3.5. *If a sequence $z = (z_k) \in \omega$ is Jordan intuitionistic fuzzy $\mathcal{I}$ convergent then the $\mathcal{I}_{(\phi,\psi)}(\Upsilon^r_n)$-limit is unique.*

Proof. Presuming the sequence $z = (z_k)$ to be Jordan intuitionistic fuzzy $\mathcal{I}$ convergent with non-identical ideal limits l_1 and l_2. Given an $\varepsilon \in (0,1)$, there exists $\varepsilon_1 \in (0,1)$ such that $(1-\varepsilon_1) * (1-\varepsilon_1) > 1-\varepsilon$ and $\varepsilon_1 \diamond \varepsilon_1 < \varepsilon$. Thus the sets

$$\begin{aligned} \mathscr{S} &= \Big\{n \in \mathbb{N} : \phi\Big(\Upsilon^r_n(z) - l_1, \frac{t}{2}\Big) \leq 1-\varepsilon_1 \text{ or } \psi\Big(\Upsilon^r_n(z) - l_1, \frac{t}{2}\Big) \geq \varepsilon_1\Big\} \in \mathcal{I} \\ \Longrightarrow \quad \mathscr{S}^c &= \Big\{n \in \mathbb{N} : \phi\Big(\Upsilon^r_n(z) - l_1, \frac{t}{2}\Big) > 1-\varepsilon_1 \text{ or } \psi\Big(\Upsilon^r_n(z) - l_1, \frac{t}{2}\Big) < \varepsilon_1\Big\} \in \mathcal{F}(\mathcal{I}), \\ \mathscr{T} &= \Big\{n \in \mathbb{N} : \phi\Big(\Upsilon^r_n(z) - l_2, \frac{t}{2}\Big) \leq 1-\varepsilon_1 \text{ or } \psi\Big(\Upsilon^r_n(z) - l_2, \frac{t}{2}\Big) \geq \varepsilon_1\Big\} \in \mathcal{I} \\ \Longrightarrow \quad \mathscr{T}^c &= \Big\{n \in \mathbb{N} : \phi\Big(\Upsilon^r_n(z) - l_2, \frac{t}{2}\Big) > 1-\varepsilon_1 \text{ or } \psi\Big(\Upsilon^r_n(z) - l_2, \frac{t}{2}\Big) < \varepsilon_1\Big\} \in \mathcal{F}(\mathcal{I}). \end{aligned}$$

Then $\mathscr{S}^c \cap \mathscr{T}^c \neq \phi$. Taking $n \in \mathscr{S}^c \cap \mathscr{T}^c$, we have

$$\begin{aligned} \phi\Big(l_1 - l_2, t\Big) &\geq \phi\Big(\Upsilon^r_n(z) - l_1, \frac{t}{2}\Big) * \phi\Big(\Upsilon^r_n(z) - l_2, \frac{t}{2}\Big) \\ &> (1-\varepsilon_1) * (1-\varepsilon_1) > (1-\varepsilon) \end{aligned}$$

and

$$\begin{aligned} \psi\Big(l_1 - l_2, t\Big) &\leq \psi\Big(\Upsilon^r_n(z) - l_1, \frac{t}{2}\Big) \diamond \psi\Big(\Upsilon^r_n(z) - l_2, \frac{t}{2}\Big) \\ &< \varepsilon_1 \diamond \varepsilon_1 < \varepsilon. \end{aligned}$$

$\varepsilon \in (0,1)$ being arbitrary; $l_1 = l_2$. That is $\mathcal{I}_{(\phi,\psi)}(\Upsilon^r_n)$-limit is unique. □

THEOREM 3.6. *A sequence $z = (z_k) \in \omega$ is Jordan intuitionistic fuzzy $\mathcal{I}$ convergent with respect to intuitionistic fuzzy norms (ϕ,ψ) iff it is Jordan intuitionistic fuzzy $\mathcal{I}$-Cauchy with reference to the identical norms.*

P r o o f. Let $z=(z_k)\in\omega$ is Jordan intuitionistic fuzzy $\mathcal{I}$ convergent with respect to intuitionistic fuzzy norms (ϕ,ψ) such that $\mathcal{I}_{(\phi,\psi)}(\Upsilon^r_n)$-lim$(z_k)=l$ and there exists $\varepsilon_1\in(0,1)$ such that $(1-\varepsilon_1)*(1-\varepsilon_1)>1-\varepsilon$ and $\varepsilon_1\diamond\varepsilon_1<\varepsilon$ for a given $\varepsilon\in(0,1)$. Thus for all $t>0$,

$$\mathscr{P}=\Big\{n\in\mathbb{N}:\phi\Big(\Upsilon^r_n(z)-l_1,t\Big)\le 1-\varepsilon_1 \text{ or } \psi\Big(\Upsilon^r_n(z)-l_1,t\Big)\ge\varepsilon_1\Big\}\in\mathcal{I}$$

$$\implies\quad \mathscr{P}^c=\Big\{n\in\mathbb{N}:\phi\Big(\Upsilon^r_n(z)-l_1,t\Big)>1-\varepsilon_1 \text{ or } \psi\Big(\Upsilon^r_n(z)-l_1,t\Big)<\varepsilon_1\Big\}\in\mathcal{F}(\mathcal{I}).$$

For $n\in\mathscr{P}^c$, $\phi\big(\Upsilon^r_n(z)-l_1,t\big)>1-\varepsilon_1$ or $\psi\big(\Upsilon^r_n(z)-l_1,t\big)<\varepsilon_1$. For a particular $k\in\mathscr{P}^c$, we can say

$$\mathscr{Q}=\big\{n\in\mathbb{N}:\phi\big(\Upsilon^r_n(z)-\Upsilon^r_k(z),t\big)\le 1-\varepsilon \text{ or } \psi\big(\Upsilon^r_n(z)-\Upsilon^r_k(z),t\big)\ge\varepsilon\big\}.$$

Let

$$n\in\mathscr{Q}\implies \phi\big(\Upsilon^r_n(z)-\Upsilon^r_k(z),t\big)\le 1-\varepsilon \ \text{ or }\ \psi\big(\Upsilon^r_n(z)-\Upsilon^r_k(z),t\big)\ge\varepsilon.$$

On the contrary, let $\phi\big(\Upsilon^r_n(z)-\Upsilon^r_k(z),t\big)>1-\varepsilon$. Then

$$\begin{aligned}1-\varepsilon&\ge\phi\big(\Upsilon^r_n(z)-\Upsilon^r_k(z),t\big)\\&\ge\phi\Big(\Upsilon^r_n(z)-l,\frac{t}{2}\Big)*\phi\Big(\Upsilon^r_k(z)-l,\frac{t}{2}\Big)\\&>(1-\varepsilon_1)*(1-\varepsilon_1)>(1-\varepsilon),\end{aligned}$$

which is a contradiction. Likewise, consider $\psi\big(\Upsilon^r_n(z)-\Upsilon^r_k(z),t\big)\ge\varepsilon$ such that $\psi\big(\Upsilon^r_n(z)-l,\frac{t}{2}\big)\ge\varepsilon_1$. On the contrary, let $\psi\big(\Upsilon^r_n(z)-l,\frac{t}{2}\big)<\varepsilon_1$. Hence

$$\begin{aligned}\varepsilon&\le\psi\big(\Upsilon^r_n(z)-\Upsilon^r_k(z),t\big)\\&\le\psi\Big(\Upsilon^r_n(z)-l,\frac{t}{2}\Big)\diamond\psi\Big(\Upsilon^r_k(z)-l,\frac{t}{2}\Big)\\&<\varepsilon_1\diamond\varepsilon_1<\varepsilon,\end{aligned}$$

which is a contradiction too. Therefore, for $n\in\mathscr{Q}$, we have $\phi\big(\Upsilon^r_n(z)-l,t\big)\le 1-\varepsilon_1$ and $\phi\big(\Upsilon^r_n(z)-l,t\big)\ge\varepsilon_1$, which imply $n\in\mathscr{P}$. Therefore, $\mathscr{Q}\subset\mathscr{P}$ and $\mathscr{Q}\in\mathcal{I}$. Consequently the sequence $z=(z_k)$ is Jordan intuitionistic fuzzy $\mathcal{I}$-Cauchy with respect to norms (ϕ,ψ).

Conversely, presuppose $z=(z_k)$ is Jordan intuitionistic fuzzy $\mathcal{I}$- Cauchy with respect to the norms (ϕ,ψ) and is not Jordan intuitionistic fuzzy $\mathcal{I}$-convergent. As a result, there exists $k\in\mathbb{N}$ such that

$$\mathscr{A}=\big\{n\in\mathbb{N}:\phi\big(\Upsilon^r_n(z)-\Upsilon^r_k(z),t\big)\le 1-\varepsilon \ \text{ or }\ \psi\big(\Upsilon^r_n(z)-\Upsilon^r_k(z),t\big)\ge\varepsilon\big\}\in\mathcal{I}$$

and

$$\mathscr{B}=\big\{n\in\mathbb{N}:\phi\big(\Upsilon^r_n(z)-l,t\big)>1-\varepsilon \text{ or } \psi\big(\Upsilon^r_n(z)-l,t\big)<\varepsilon\big\}\in\mathcal{I}$$

$$\begin{aligned}\implies\quad 1-\varepsilon&\ge\phi\big(\Upsilon^r_n(z)-\Upsilon^r_k(z),t\big)\\&\ge\phi\Big(\Upsilon^r_n(z)-l,\frac{t}{2}\Big)*\phi\Big(\Upsilon^r_n(z)-l,\frac{t}{2}\Big)\\&>(1-\varepsilon_1)*(1-\varepsilon_1)>1-\varepsilon.\end{aligned}$$

Simultaneously,

$$\begin{aligned}\varepsilon&\le\psi\big(\Upsilon^r_n(z)-\Upsilon^r_k(z),t\big)\\&\le\psi\Big(\Upsilon^r_n(z)-l,\frac{t}{2}\Big)\diamond\psi\Big(\Upsilon^r_n(z)-l,\frac{t}{2}\Big)<\varepsilon_1\diamond\varepsilon_1<\varepsilon,\end{aligned}$$

which lead to contradiction. Therefore, $\mathscr{B}\in\mathcal{F}(\mathcal{I})$ and hence, $z=(z_k)$ is Jordan intuitionistic fuzzy $\mathcal{I}$-convergent. □

Theorem 3.7. *Consider IFNS $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ and $\tau^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ be the topology on $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$. Let $(z^j) = (z^j_k)^{\infty}_{j=1}$ be a sequence in $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$. The sequence $z^j \to z$ as $j \to \infty$ if and only if $\phi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), t\big) \to 1$ and $\psi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), t\big) \to 0$ as $n \to \infty$.*

Proof. Let $z^j \to z$ as $j \to \infty$. Fix a designated $r > 0$ and $0 < \varepsilon < 1$, there exists $n \in \mathbb{N}$ such that $(z^j) \in \mathcal{B}^{\mathcal{I}}_z\big(r, \varepsilon\big)(\Upsilon^r_n)$ for all $j \geq k$. Then,

$$\mathscr{S} = \big\{n \in \mathbb{N} : \phi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), r\big) \leq 1 - \varepsilon \text{ or } \psi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), r\big) \geq \varepsilon\big\} \in \mathcal{I},$$

or equivalently,

$$\mathscr{S}^c = \big\{n \in \mathbb{N} : \phi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), r\big) > 1 - \varepsilon \text{ or } \psi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), r\big) < \varepsilon\big\} \in \mathcal{F}(\mathcal{I}).$$

For $\big\{n \in \mathbb{N}\big\} \subseteq \mathscr{S}^c$, $\phi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), r\big) > 1 - \varepsilon \implies 1 - \phi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), r\big) < \varepsilon$ and $\psi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), r\big) < \varepsilon$. Therefore, for $n \to \infty$, $1 - \phi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), r\big) \to 0$ and $\psi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), r\big) \to 0$. This implies that $\phi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), r\big) \to 1$ and $\psi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), r\big) \to 0$ as $n \to \infty$.
Conversely, suppose that for each $t > 0$, $\phi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), t\big) \to 1$ and $\psi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), t\big) \to 0$ as $n \to \infty$. Then for each $\varepsilon \in (0,1)$, there exists $k \in \mathbb{N}$ such that $1 - \phi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), t\big) < \varepsilon$ and $\psi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), t\big) < \varepsilon$ for all $n \geq k \implies \phi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), t\big) > 1 - \varepsilon$ and $\psi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), t\big) < \varepsilon$ for all $n \geq k$. Consider the ideal $\mathcal{I}$ generated by the set $\{n \in \mathbb{N} : n < k\}$, implies that the collection of sets generated by the set $\{n \in \mathbb{N} : n \geq k\}$ belongs to $\mathcal{F}(\mathcal{I})$. Thus $\{n \in \mathbb{N} : \phi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), t\big) > 1 - \varepsilon$ or $\phi\big(\Upsilon^r_n(z^j) - \Upsilon^r_n(z), t\big) < \varepsilon\} \in \mathcal{F}(\mathcal{I}) \implies (z^j) \in \mathcal{B}^{\mathcal{I}}_z\big(r, \varepsilon\big)(\Upsilon^r_n)$ for all $n \geq k$. Hence, $z^j \to z$ as $j \to \infty$. □

Theorem 3.8. *Let $z = (z_k) \in c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$. Then for some $l \in \mathbb{C}$, $z_k \xrightarrow{\mathcal{I}_{(\phi,\psi)}(\Upsilon^r)} l$ if and only if for every $\varepsilon \in (0,1)$ and $t > 0$, there exist positive integers $\mathcal{N} = \mathcal{N}(z, \varepsilon, t)$ such that*

$$\Big\{\mathcal{N} \in \mathbb{N} : \phi\Big(\Upsilon^r_{\mathcal{N}}(z) - l, \frac{t}{2}\Big) > 1 - \varepsilon \text{ or } \psi\Big(\Upsilon^r_{\mathcal{N}}(z) - l, \frac{t}{2}\Big) < \varepsilon\Big\} \in \mathcal{F}(\mathcal{I}).$$

Proof. Suppose $z_k \xrightarrow{\mathcal{I}_{(\phi,\psi)}(\Upsilon^r)} l$ for some $l \in \mathbb{C}$. For given $\varepsilon \in (0,1)$, there exists $r \in (0,1)$ such that $(1-\varepsilon) * (1-\varepsilon) > 1 - r$ and $\varepsilon \diamond \varepsilon < r$. Since $z_k \xrightarrow{\mathcal{I}_{(\phi,\psi)}(\Upsilon^r)} l$, for all $t > 0$,

$$\mathscr{X} = \Big\{n \in \mathbb{N} : \phi\Big(\Upsilon^r_n(z) - l, \frac{t}{2}\Big) \leq 1 - \varepsilon \text{ or } \psi\Big(\Upsilon^r_n(z) - l, \frac{t}{2}\Big) \geq \varepsilon\Big\} \in \mathcal{I};$$

which implies that

$$\mathscr{X}^c = \Big\{n \in \mathbb{N} : \phi\Big(\Upsilon^r_n(z) - l, \frac{t}{2}\Big) > 1 - \varepsilon \text{ or } \psi\Big(\Upsilon^r_n(z) - l, \frac{t}{2}\Big) < \varepsilon\Big\} \in \mathcal{F}(\mathcal{I}).$$

Conversely, let us choose $\mathcal{N} \in \mathscr{X}^c$. Then

$$\phi\Big(\Upsilon^r_{\mathcal{N}}(z) - l, \frac{t}{2}\Big) > 1 - \varepsilon \text{ or } \psi\Big(\Upsilon^r_{\mathcal{N}}(z) - l, \frac{t}{2}\Big) < \varepsilon.$$

We show the existence of a positive integer $\mathcal{N} = \mathcal{N}(z, \varepsilon, t)$ such that

$$\mathscr{P} = \big\{n \in \mathbb{N} : \phi\big(\Upsilon^r_n(z) - \Upsilon^r_{\mathcal{N}}(z), t\big) \leq 1 - r \text{ or } \psi\big(\Upsilon^r_n(z) - \Upsilon^r_{\mathcal{N}}(z), t\big) \geq r\big\} \in \mathcal{I}.$$

We shall show that $\mathscr{P} \subseteq \mathscr{X}$. On the contrary, let $\mathscr{P} \nsubseteq \mathscr{X}$, i.e., there exists $n \in \mathscr{P}$ such that $n \notin \mathscr{X}$. Then

$$\phi\big(\Upsilon^r_n(z) - \Upsilon^r_{\mathcal{N}}(z), t\big) \leq 1 - r \quad \text{and} \quad \phi\Big(\Upsilon^r_n(z) - l, \frac{t}{2}\Big) > 1 - \varepsilon.$$

Particularly,

$$\phi\Big(\Upsilon^r_{\mathcal{N}}(z) - l, \frac{t}{2}\Big) > 1 - \varepsilon.$$

Therefore, we have

$$\begin{aligned}1-r &\geq \phi\big(\Upsilon_n^r(z)-\Upsilon_{\mathcal{N}}^r(z),t\big)\\ &\geq \phi\Big(\Upsilon_n^r(z)-l,\frac{t}{2}\Big)*\phi\Big(\Upsilon_{\mathcal{N}}^r(z)-l,\frac{t}{2}\Big)\geq\ (1-\varepsilon)*(1-\varepsilon)>1-r\end{aligned}$$

which is a contradiction. Similarly,

$$\psi\big(\Upsilon_n^r(z)-\Upsilon_{\mathcal{N}}^r(z),t\big)\geq r \quad \text{and} \quad \psi\big(\Upsilon_n^r(z)-l,t\big)<\varepsilon.$$

Particularly,

$$\psi\Big(\Upsilon_{\mathcal{N}}^r(z)-l,\frac{t}{2}\Big)<\varepsilon.$$

Therefore,

$$\begin{aligned}r &\leq \psi\big(\Upsilon_n^r(z)-\Upsilon_{\mathcal{N}}^r(z),t\big)\\ &\leq \psi\Big(\Upsilon_n^r(z)-l,\frac{t}{2}\Big)\diamond\psi\Big(\Upsilon_{\mathcal{N}}^r(z)-l,\frac{t}{2}\Big)\leq\varepsilon\diamond\varepsilon<r,\end{aligned}$$

which is again a contradiction. Hence, $\mathscr{P}\subseteq\mathscr{X}$ and since $\mathscr{X}\in\mathcal{I}$, implies $\mathscr{P}\in\mathcal{I}$. □

Definition 12. A non-empty set $\mathscr{S}\subset c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ is compact if every open cover of $\mathscr{S}$ defined by the open set of $\tau^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ has a finite subcover.

Theorem 3.9. *Every finite subset $\mathscr{S}$ of $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ is compact.*

P r o o f. Let $\mathscr{S}=\{z_1,z_2,z_3,\dots,z_n\}$ be the finite subset of $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$. For $r>0$ and $0<\varepsilon<1$, let us assume $\{\mathcal{B}^{\mathcal{I}}_z(r,\varepsilon)(\Upsilon^r_n):z\in S\}$ is an open cover of $\mathscr{S}$. Then $\mathscr{S}\subseteq\bigcup_{z\in\mathscr{S}}\mathcal{B}^{\mathcal{I}}_z(r,\varepsilon)(\Upsilon^r_n)$. Now for all $z_i\in\mathscr{S}$, $i=1,2,3,\dots n$, we have $z_i\in\bigcup_{z_i\in s}\mathcal{B}^{\mathcal{I}}_{z_i}(r,\varepsilon)(\Upsilon^r_n)$. That implies $z_i\in\mathcal{B}^{\mathcal{I}}_{z_j}(r,\varepsilon)(\Upsilon^r_n)$ for some $j\in\{1,2,3,\dots,n\}$. Then $\{\mathcal{B}^{\mathcal{I}}_{z_i}(r,\varepsilon)(\Upsilon^r_n):i=1,2,3,\dots,n\}$ is a finite subcover of $\mathscr{S}$. □

Theorem 3.10. *A set $\mathscr{S}\subseteq c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ is compact iff every sequence in $\mathscr{S}$ has a convergent subsequence.*

P r o o f. Suppose $\mathscr{S}$ is a compact subset of $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$. Let $(z^j_k)=(z^j)_{j=1}^{\infty}$ be a sequence in $\mathscr{S}$. For given $0<\varepsilon<1$ and $r>0$, let $\{\mathcal{B}^{\mathcal{I}}_z(\frac{r}{3},\varepsilon)(\Upsilon^r_n):z=(z_k)\in\mathscr{S}\}$ be an open cover of $\mathscr{S}$. This implies, $(z^j)\in\bigcup_{z\in\mathscr{S}}\{\mathcal{B}^{\mathcal{I}}_z(\frac{r}{3},\varepsilon)(\Upsilon^r_n)\}$. Then there exists some $z=(z_k)\in S$ such that $(z^j)\in\mathcal{B}^{\mathcal{I}}_z(\frac{r}{3},\varepsilon)(\Upsilon^r_n)$. Therefore, the set

$$\mathscr{X}=\Big\{n\in\mathbb{N}:\phi\Big(\Upsilon^r_n(z^j)-\Upsilon^r_n(z),\frac{fr}{3}\Big)>1-\varepsilon\ \text{ or }\ \psi\Big(\Upsilon^r_n(z^j)-\Upsilon^r_n(z),\frac{r}{3}\Big)<\varepsilon\Big\}\in\mathcal{F}(\mathcal{I}).$$

There exists a finite subcover $\{\mathcal{B}^{\mathcal{I}}_{z_i}(\frac{r}{3},\varepsilon)(\Upsilon^r_n):z_i\in\mathscr{S}\text{ and }i=1,2,3,\dots m\}$ since $\mathscr{S}$ is compact such that $\mathscr{S}\subseteq\bigcup_{i=1}^{m}\mathcal{B}^{\mathcal{I}}_{z_i}(\frac{r}{3},\varepsilon)(\Upsilon^r_n)$. Let (z^{j_p}) be a subsequence of (z^j). Then $(z^{j_p})\in\bigcup_{i=1}^{m}\mathcal{B}^{\mathcal{I}}_{z_i}(\frac{r}{3},\varepsilon)(\Upsilon^r_n)$, implies $(z^{j_p})\in\mathcal{B}^{\mathcal{I}}_{z_i}(\frac{r}{3},\varepsilon)(\Upsilon^r_n)$, for some $z_i\in\mathscr{S}$. Therefore, the set

$$\mathscr{Y}=\Big\{n\in\mathbb{N}:\phi\Big(\Upsilon^r_n(z^{j_p})-\Upsilon^r_n(z_i),\frac{r}{3}\Big)>1-\varepsilon\text{ or }\psi\Big(\Upsilon^r_n(z^{j_p})-\Upsilon^r_n(z_i),\frac{r}{3}\Big)<\varepsilon\Big\}\in\mathcal{F}(\mathcal{I}).$$

For $n \in \mathscr{X} \cap \mathscr{Y}$,

$$\begin{aligned}\phi\Big(\Upsilon_n^r(z^{j_p}) - \Upsilon_n^r(z), r\Big) &\geq \phi\Big(\Upsilon_n^r(z^{j_p}) - \Upsilon_n^r(z_i), \frac{r}{3}\Big) \\ &\quad * \phi\Big(\Upsilon_n^r(z^{j}) - \Upsilon_n^r(z_i), \frac{r}{3}\Big) * \phi\Big(\Upsilon_n^r(z^{j}) - \Upsilon_n^r(z), \frac{r}{3}\Big) \\ &> (1-\varepsilon) * (1-\varepsilon) * (1-\varepsilon) = (1-\varepsilon).\end{aligned}$$

Simultaneously,

$$\begin{aligned}\psi\Big(\Upsilon_n^r(z^{j_p}) - \Upsilon_n^r(z), r\Big) &\leq \psi\Big(\Upsilon_n^r(z^{j_p}) - \Upsilon_n^r(z_i), \frac{r}{3}\Big) \\ &\quad \diamond \psi\Big(\Upsilon_n^r(z^{j}) - \Upsilon_n^r(z_i), \frac{r}{3}\Big) \diamond \psi\Big(\Upsilon_n^r(z^{j}) - \Upsilon_n^r(z), \frac{r}{3}\Big) \\ &< \varepsilon \diamond \varepsilon \diamond \varepsilon = \varepsilon.\end{aligned}$$

Take $\varepsilon = \frac{1}{n}$. Then

$$\lim_{n\to\infty} \phi\Big(\Upsilon_n^r(z^{j_p}) - \Upsilon_n^r(z), r\Big) = \lim_{n\to\infty} 1 - \frac{1}{n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \psi\Big(\Upsilon_n^r(z^{j_p}) - \Upsilon_n^r(z), r\Big) = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Thus by theorem (3.7), $z^{j_p} \to z$, as $p \to \infty$. Conversely, suppose (z^{j_p}) be the subsequence of a sequence (z^j) in $\mathscr{S}$ such that $(z^{j_p}) \to z$ in $\mathscr{S}$. Let on contrary $\mathscr{S}$ is not a compact subset of $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$. Let $\{\mathcal{B}^{\mathcal{I}}_z(\frac{r}{3}, \varepsilon)(\Upsilon_n^r) : x \in S\}$ be an open cover of $\mathscr{S}$ implies $\mathscr{S} \subseteq \bigcup\limits_{z\in\mathscr{S}} \{\mathcal{B}^{\mathcal{I}}_z(\frac{r}{3}, \varepsilon)(\Upsilon_n^r)$. Therefore, the set

$$\{n \in \mathbb{N} : \phi\big(\Upsilon_n^r(z^j) - \Upsilon_n^r(z), r\big) > 1 - \varepsilon \ \text{ or } \ \psi\big(\Upsilon_n^r(z^j) - \Upsilon_n^r(z), r\big) < \varepsilon\} \in \mathcal{F}(\mathcal{I}).$$

Since $\mathscr{S}$ is not compact, there exists a finite subcover $\{B^{\mathcal{I}}_{z_i}(r, \varepsilon)(\Upsilon_n^r) : z_i \in S, i = 1, 2, 3 \ldots, m\}$ such that $S \nsubseteq \bigcup\limits_{z_i\in\mathscr{S}} \mathcal{B}^{\mathcal{I}}_{z_i}(r, \varepsilon)(\Upsilon_n^r)$, which implies that the set

$$\{n \in \mathbb{N} : \phi\big(\Upsilon_n^r(z^{j_p}) - \Upsilon_n^r(z_i), r\big) > 1 - \varepsilon \text{ or } \psi\big(\Upsilon_n^r(z^{j_p}) - \Upsilon_n^r(z_i), r\big) < \varepsilon\} \notin \mathcal{F}(\mathcal{I})$$

$\implies$ for any $0 < \varepsilon < 1$ and $r > 0$, $(z^{j_p}) \notin \mathcal{B}^{\mathcal{I}}_z(r, \varepsilon)$. Hence, $(z^{j_p}) \nrightarrow z$, which is a contradiction. Hence, S is compact. □

Theorem 3.11. *Consider the IFNS $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$. Let $r > 0$ and $\varepsilon, \varepsilon' \in (0, 1)$ such that $(1-\varepsilon) * (1-\varepsilon) \geq (1-\varepsilon')$ and $\varepsilon \diamond \varepsilon \leq \varepsilon'$. Then for any $z = (z_k) \in c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$, $\overline{\mathcal{B}^{\mathcal{I}}_z(\frac{r}{2}, \varepsilon)(\Upsilon_n^r)} \subseteq \mathcal{B}^{\mathcal{I}}_z(r, \varepsilon')(\Upsilon_n^r)$.*

Proof. Let $q = (q_k) \in \overline{\mathcal{B}^{\mathcal{I}}_z(\frac{r}{2}, \varepsilon)(\Upsilon_n^r)}$ and $\mathcal{B}^{\mathcal{I}}_q(\frac{r}{2}, \varepsilon)(\Upsilon_n^r)$ be an open ball with centre at q and radius ε. Hence, $\mathcal{B}^{\mathcal{I}}_z(\frac{r}{2}, \varepsilon)(\Upsilon_n^r) \cap \mathcal{B}^{\mathcal{I}}_q(\frac{r}{2}, \varepsilon)(\Upsilon_n^r) \neq \phi$. Suppose $v = (v_k) \in \mathcal{B}^{\mathcal{I}}_q(\frac{r}{2}, \varepsilon)(\Upsilon_n^r) \cap \mathcal{B}^{\mathcal{I}}_z(\frac{r}{2}, \varepsilon)(\Upsilon_n^r)$. Then, the sets

$$\begin{aligned}\mathscr{X} &= \Big\{n \in \mathbb{N} : \phi\Big(\Upsilon_n^r(q) - \Upsilon_n^r(v), \frac{r}{2}\Big) > 1 - \varepsilon_1 \text{ or } \psi\Big(\Upsilon_n^r(q) - \Upsilon_n^r(v), \frac{r}{2}\Big) < \varepsilon_1\Big\} \in \mathcal{F}(\mathcal{I}), \\ \mathscr{Y} &= \Big\{n \in \mathbb{N} : \phi\Big(\Upsilon_n^r(z) - \Upsilon_n^r(v), \frac{r}{2}\Big) > 1 - \varepsilon \text{ or } \psi\Big(\Upsilon_n^r(z) - \Upsilon_n^r(v), \frac{r}{2}\Big) < \varepsilon_1\Big\} \in \mathcal{F}(\mathcal{I}).\end{aligned}$$

Consider $n \in \mathscr{X} \cap \mathscr{Y}$. Then

$$\begin{aligned}\phi\Big(\Upsilon_n^r(z) - \Upsilon_n^r(q), r\Big) &\geq \phi\Big(\Upsilon_n^r(z) - \Upsilon_n^r(v), \frac{r}{2}\Big) * \phi\Big(\Upsilon_n^r(q) - \Upsilon_n^r(v), \frac{r}{2}\Big) \\ &> (1-\varepsilon) * (1-\varepsilon) \geq (1-\varepsilon')\end{aligned}$$

and

$$\begin{aligned}\psi\Big(\Upsilon_n^r(z) - \Upsilon_n^r(q), r\Big) &\leq \psi\Big(\Upsilon_n^r(z) - \Upsilon_n^r(v), \frac{r}{2}\Big) \diamond \psi\Big(\Upsilon_n^r(q) - \Upsilon_n^r(v), \frac{r}{2}\Big) \\ &< \varepsilon \diamond \varepsilon \leq \varepsilon'.\end{aligned}$$

Therefore, the set

$$\{n \in \mathbb{N} : \phi\Big(\Upsilon_n^r(z) - \Upsilon_n^r(q), r\Big) > 1 - \varepsilon' \;\text{ or }\; \psi\Big(\Upsilon_n^r(z) - \Upsilon_n^r(q), r\Big) < \varepsilon'\} \in \mathcal{F}(\mathcal{I})$$
$$\implies \quad q = (q_k) \in \mathcal{B}_z^{\mathcal{I}}(r, \varepsilon')(\Upsilon_n^r).$$

Hence, $\overline{\mathcal{B}_z^{\mathcal{I}}(\frac{r}{2}, \varepsilon)(\Upsilon_n^r)} \subseteq \mathcal{B}_z^{\mathcal{I}}(\frac{r}{2}, \varepsilon')(\Upsilon_n^r)$. □

Theorem 3.12. *Let $z = (z_k) \in \omega$. If there exists a sequence $z' = (z_k') \in c_{(\phi,\psi)}^{\mathcal{I}}(\Upsilon^r)$ such that $\Upsilon_n^r(z) = \Upsilon_n^r(z')$ for almost all n relative to $\mathcal{I}$, then $z \in c_{(\phi,\psi)}^{\mathcal{I}}(\Upsilon^r)$.*

P r o o f. Suppose $\Upsilon_n^r(z) = \Upsilon_n^r(z')$ for almost all n relative to $\mathcal{I}$. Then $\{n \in \mathbb{N} : \Upsilon_n^r(z) \neq \Upsilon_n^r(z')\} \in \mathcal{I}$. This implies $\{n \in \mathbb{N} : \Upsilon_n^r(z) = \Upsilon_n^r(z')\} \in \mathcal{F}(\mathcal{I})$. Therefore, for $n \in \mathcal{F}(\mathcal{I})$ for all $t > 0$,

$$\phi\Big(\Upsilon_n^r(z) - \Upsilon_n^r(z'), \frac{t}{2}\Big) = 1 \quad \text{and} \quad \psi\Big(\Upsilon_n^r(z) - \Upsilon_n^r(z'), \frac{t}{2}\Big) = 0.$$

Since $(z_k') \in c_{(\phi,\psi)}^{\mathcal{I}}(\Upsilon^r)$, let $\mathcal{I}_{(\phi,\psi)}(\Upsilon^r)\text{-}\lim(z_k') = l$. Then for every $\varepsilon \in (0,1)$ and $t > 0$,

$$\mathscr{X} = \Big\{n \in \mathbb{N} : \phi\Big(\Upsilon_n^r(z') - l, \frac{t}{2}\Big) > 1 - \varepsilon \;\text{ or }\; \psi\Big(\Upsilon_n^r(z') - l, \frac{t}{2}\Big) < \varepsilon\Big\} \in \mathcal{F}(\mathcal{I}).$$

Consider the set

$$\mathscr{Y} = \Big\{n \in \mathbb{N} : \phi\Big(\Upsilon_n^r(z) - l, t\Big) > 1 - \varepsilon \;\text{ or }\; \psi\Big(\Upsilon_n^r(z) - l, t\Big) < \varepsilon\Big\}.$$

We show that $\mathscr{X} \subset \mathscr{Y}$. So for $n \in \mathscr{X}$, we have

$$\begin{aligned}\phi\Big(\Upsilon_n^r(z) - l, t\Big) &\geq \phi\Big(\Upsilon_n^r(z) - \Upsilon_n^r(z'), \frac{t}{2}\Big) * \phi\Big(\Upsilon_n^r(z') - l, \frac{t}{2}\Big)\\ &> 1 * (1 - \varepsilon) = 1 - \varepsilon\end{aligned}$$

and

$$\begin{aligned}\psi\Big(\Upsilon_n^r(z) - l, t\Big) &\leq \psi\Big(\Upsilon_n^r(z) - \Upsilon_n^r(z'), \frac{t}{2}\Big) \diamond \psi\Big(\Upsilon_n^r(z') - l, \frac{t}{2}\Big)\\ &< 0 \diamond \varepsilon = \varepsilon.\end{aligned}$$

This implies that $n \in \mathscr{Y}$ and hence, $\mathscr{X} \subset \mathscr{Y}$. Since $\mathscr{X} \in \mathcal{F}(\mathcal{I})$, therefore, $\mathscr{Y} \in \mathcal{F}(\mathcal{I})$. Hence, $z = (z_k) \in c_{(\phi,\psi)}^{\mathcal{I}}(\Upsilon^r)$. □

Theorem 3.13. *A closed ball $\mathcal{B}_z^{\mathcal{I}}[r, \varepsilon](\Upsilon^r)$ is a closed set in $c_{(\phi,\psi)}^{\mathcal{I}}(\Upsilon^r)$.*

P r o o f. Let $y = (y_k) \in \omega$ be such that $y \in \overline{\mathcal{B}_z^{\mathcal{I}}[r, \varepsilon](\Upsilon^r)}$. Thus there exists a sequence $(y^j) = (y_k^j) \in \mathcal{B}_z^{\mathcal{I}}[r, \varepsilon](\Upsilon^r)$ such that $y^j \to y$ as $j \to \infty$. Thus

$$\mathscr{X} = \{n \in \mathbb{N} : \phi\big(\Upsilon_n^r(y^j) - \Upsilon_n^r(z), r\big) \geq 1 - \varepsilon \;\text{ or }\; \psi\big(\Upsilon_n^r(y^j) - \Upsilon_n^r(z), r\big) \leq \varepsilon\} \in \mathcal{F}(\mathcal{I}).$$

Since $y^j \to z$ as $j \to \infty$, by Theorem 3.7, $\phi\big(\Upsilon_n^r(y^j) - \Upsilon_n^r(y), r\big) \to 1$ and $\psi\big(\Upsilon_n^r(y^j) - \Upsilon_n^r(y), r\big) \to 0$, for all $t > 0$ as $n \to \infty$. Hence, for $n \in \mathscr{X}$,

$$\begin{aligned}\phi\big(\Upsilon_n^r(z) - \Upsilon_n^r(y), t + r\big) &\geq \lim_{n\to\infty} \phi\big(\Upsilon_n^r(y^j) - \Upsilon_n^r(y), t\big) * \phi\big(\Upsilon_n^r(y^j) - \Upsilon_n^r(z), r\big)\\ &\geq 1 * (1 - \varepsilon) = 1 - \varepsilon\end{aligned}$$

and

$$\begin{aligned}\psi\big(\Upsilon_n^r(z) - \Upsilon_n^r(y), t + r\big) &\leq \lim_{n\to\infty} \psi\big(\Upsilon_n^r(y^j) - \Upsilon_n^r(y), t\big) \diamond \psi\big(\Upsilon_n^r(y^j) - \Upsilon_n^r(z), r\big)\\ &\leq 0 \diamond \varepsilon = \varepsilon.\end{aligned}$$

A particular $k \in \mathbb{N}$, take $t = \frac{1}{k}$. Then

$$\phi\Big(\Upsilon_n^r(z) - \Upsilon_n^r(y), r\Big) = \lim_{k\to\infty} \phi\Big(\Upsilon_n^r(z) - \Upsilon_n^r(y), r + \frac{1}{k}\Big) \geq 1 - \varepsilon$$

and

$$\psi\Big(\Upsilon_n^r(z) - \Upsilon_n^r(y), r\Big) = \lim_{k\to\infty} \phi\Big(\Upsilon_n^r(z) - \Upsilon_n^r(y), r + \frac{1}{k}\Big) \leq \varepsilon,$$

$\implies$ the set $\big\{n \in \mathbb{N} : \phi\big(\Upsilon_n^r(z) - \Upsilon_n^r(y), r\big) \geq 1 - \varepsilon$ or $\phi\big(\Upsilon_n^r(z) - \Upsilon_n^r(y), r\big) \leq \varepsilon\big\} \in \mathcal{F}(\mathcal{I})$
$\implies$
$y \in \mathcal{B}_z^{\mathcal{I}}[r, \varepsilon](\Upsilon^r)$. Therefore, $\mathcal{B}_z^{\mathcal{I}}[r, \varepsilon](\Upsilon^r)$ is a closed set. □

Theorem 3.14. *Let $\mathscr{S}$ be the compact subset of $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ such that $z = (z_k) \notin \mathscr{S}$. Then there exist two open sets $\mathscr{U}, \mathscr{V}$ in $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ such that $\mathscr{S} \subseteq \mathscr{V}, z \in \mathscr{U}$ and $\mathscr{U} \cap \mathscr{V} = \phi$.*

Proof. Let $\mathscr{S}$ be a compact subset of $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ and $z \notin \mathscr{S}$. Then for any $s \in \mathscr{S}$ we have $z \neq s$. Since $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$ is a Hausdorff space, then for some $r > 0$ and $0 < \varepsilon < 1$ there exist two open balls $\mathscr{U} = \mathcal{B}_z^{\mathcal{I}}(r, \varepsilon)(\Upsilon^r)$ and $\mathscr{V} = \mathcal{B}_s^{\mathcal{I}}(r, \varepsilon)(\Upsilon^r)$ such that $\mathscr{U} \cap \mathscr{V} = \phi$. Consider the open cover $\mathscr{V}_s = \big\{\mathcal{B}_s^{\mathcal{I}}(r, \varepsilon)(\Upsilon^r) : s \in \mathscr{S}\big\}$ of $\mathscr{S}$ and $\mathscr{S}$ is compact, therefore, there exists a finite subcover $\mathscr{S}_{s_i} = \big\{\mathcal{B}_{s_i}^{\mathcal{I}}(r, \varepsilon)(\Upsilon^r) : s_i \in \mathscr{S}$ and $i = 1, 2, 3, \ldots, j\big\}$ such that $\mathscr{S} \subseteq \bigcup_{i=1}^{j} \mathscr{V}_{s_i}$. Taking $\mathscr{V} = \bigcap_{i=1}^{j} \mathscr{V}_{s_i}$ we have $z \notin \mathscr{S}$. Hence, $\mathscr{U}, \mathscr{V}$ are open sets such that $\mathscr{S} \subseteq \mathscr{V}$ and $\mathscr{U} \cap \mathscr{V} = \phi$. □

4. Conclusion

The article deploys the notion of a regular matrix implemented on an initially non-convergent sequence to attain some finite limit. It further explores the convergence of the sequences generated after operating the regular Jordan totient operator in the setting of intuitionistic fuzzy norm via a collection of finite subsets of $\mathbb{N}$. We formulate novel sequence spaces $c^{\mathcal{I}}_{0(\phi,\psi)}(\Upsilon^r)$, $c^{\mathcal{I}}_{(\phi,\psi)}(\Upsilon^r)$, $\ell^{\mathcal{I}}_{\infty(\phi,\psi)}(\Upsilon^r)$ and $\ell_{\infty(\phi,\psi)}(\Upsilon^r)$ and study relations among them. Researchers' interest can further aim at developing function spaces with the aid of a generalized infinite operator and study their properties.

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