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Mathematica Slovaca

DOI: 10.1515/ms-2023-0032
Math. Slovaca **73** (2023), No. 2, 425–438

SHARP APPROXIMATIONS FOR THE GENERALIZED ELLIPTIC INTEGRAL OF THE FIRST KIND

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(*Communicated by Marek Balcerzak*)

ABSTRACT. For $a \in (0, 1/2]$, $r \in (0, 1)$, let $\mathcal{K}_a(r)$ ($\mathcal{K}(r)$) be the generalized (complete) elliptic integral of the first kind. In the article, we prove some monotonicity properties of certain combination of functions involving $\mathcal{K}_a(r)$, and thus establish its two sharp inequalities, which extend and improve some well-known results of $\mathcal{K}(r)$.

1. Introduction

For real numbers a, b and c with $c \neq 0, -1, -2, \ldots$, the Gaussian hypergeometric function[1, 7] is defined by

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad x \in (-1, 1),$$

where $(a)_0 = 1$ for $a \neq 0$, $(a)_n = a(a+1)(a+2)\ldots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer (shifted factorial) function, and $\Gamma(x) = \int\limits_0^{\infty} t^{x-1} \mathrm{e}^{-t} \mathrm{d}t$ $(x > 0)$ is the Euler gamma function (cf. [1, 30]). If $a + b = c$, then $F(a, b; c; x)$ is called zero-balanced.

As a special case of the Gaussian hypergeometric function, for $a \in (0, 1)$ and $r \in (0, 1)$, the generalized elliptic integral of the first kind [4, 18] is defined as

$$\begin{cases} \mathcal{K}_a = \mathcal{K}_a(r) = \pi F(a, 1-a; 1; r^2)/2, \\ \mathcal{K}_a(0) = \pi/2, \ \mathcal{K}_a(1^-) = +\infty. \end{cases} \tag{1.1}$$

When $a = 1/2$, $\mathcal{K}_{1/2}(r) = \mathcal{K}(r)$ (cf. [2, 3]) is the Legendre complete elliptic integral of the first kind. By the symmetry of (1.1), we shall assume that $a \in (0, 1/2]$ throughout this article.

It is well known that the above special functions $F(a, b; c; x)$ and $\mathcal{K}_a(r)(\mathcal{K}(r))$ have been widely applied in mathematics and physics. A lot of problems in geometric function theory, theory of mean values, number theory and theory of elasticity all depend on these functions [5,8,11,13,25,35,36,38]. During the past few decades, many mathematicians have expanded a great deal of effort in studying asymptotic properties and establishing sharp bounds for some combinations of functions involving

2020 Mathematics Subject Classification: 33E05; 26D07.
Keywords: Gaussian hypergeometric function, generalized elliptic integral of the first kind, monotonicity, inequality.
The work was supported by the Natural Science Foundation of China (Grant No. 11701176, 61673169).
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$F(a,b;c;x)$, $\mathcal{K}_a(r)$ and $\mathcal{K}(r)$. As applications, a lot of distortion estimates in quasi-conformal mappings and analytic properties of modular functions in Ramanujan modular equation have been derived (cf. [9, 14, 15, 17, 19–22, 30–33, 37, 39]).

Early at the beginning of the 20-th century, Ramanujan established the asymptotic behavior of the zero-balanced Gaussian hypergeometric function (cf. [7: 1.48] or [12: pp. 33–34])

$$B(a,b)F(a,b;a+b;x)+\log(1-x)=R(a,b)+O[(1-x)\log(1-x)] \tag{1.2}$$

as $x\mapsto 1$, where $B(a,b)=\Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function, $R(a,b)=-\Gamma'(a)/\Gamma(a)-\Gamma'(b)/\Gamma(b)-2\gamma$, and $\gamma=\lim\limits_{n\mapsto\infty}(\sum\limits_{k=1}^{n}\frac{1}{k}-\log n)=0.5772\ldots$ denotes the Euler-Mascheroni constant. Estimates for zero-balanced hypergeometric functions are given in [7] whereas the estimates for the non-zero balanced hypergeometric functions are discussed in [23]. See also [10]. Let

$$B(a)=B(a,1-a)=\Gamma(a)\Gamma(1-a)=\frac{\pi}{\sin(\pi a)} \tag{1.3}$$

and

$$R(a)=R(a,1-a)=-\frac{\Gamma'(a)}{\Gamma(a)}-\frac{\Gamma'(1-a)}{\Gamma(1-a)}-2\gamma. \tag{1.4}$$

Then from (1.2) we obtain

$$B(a)F(a,1-a;1;x)+\log(1-x)=R(a)+O[(1-x)\log(1-x)],\quad x\to 1. \tag{1.5}$$

Using (1.1)–(1.5), it is not difficult to get the asymptotic formulas for generalized (complete) elliptic integral of the first kind: for $r\mapsto 1$,

$$\frac{2\mathcal{K}_a(r)}{\sin(\pi a)}+2\log r'=R(a)+O[(1-r^2)\log(1-r^2)],$$

or

$$\mathcal{K}_a(r)\sim\sin(\pi a)\log\left(\frac{\mathrm{e}^{R(a)/2}}{r'}\right)$$

and

$$\mathcal{K}(r)+\log r'=\log 4+O[(1-r^2)\log(1-r^2)],$$

or

$$\mathcal{K}(r)\sim\log\left(\frac{4}{r'}\right),$$

respectively. Here and in what follows we set $r'=\sqrt{1-r^2}$.

In [6: Theorem 1.48], Anderson, Vamanamurthy and Vuorinen showed that the quotient function $r\mapsto\mathcal{K}(r)/\log(4/r')$ is strictly decreasing from $(0,1)$ onto $(1,\pi/(4\log 2))$, from which it follows that

$$\log\left(\frac{4}{r'}\right)<\mathcal{K}(r)<\frac{\pi}{4\log 2}\log\left(\frac{4}{r'}\right),\quad r\in(0,1). \tag{1.6}$$

Later, Qiu, Vamanamurthy and Vuorinen [26: Theorem 1.5] proved that $r\mapsto\mathcal{K}(r)/\log[\alpha r'+(4/r')]$ is piecewise monotone on $(0,1)$ with $\alpha=\mathrm{e}^{\pi/2}-4$. Consequently, they obtained the following refinement

$$\mathcal{K}(r)>\log\left(\alpha r'+\frac{4}{r'}\right),\quad r\in(0,1). \tag{1.7}$$

Furthermore, the authors [26] pointed out that, for $c\in(-\infty,\infty)$, α is the best possible constant such that $\mathcal{K}(r)>\log(cr'+4/r')$ holds for all $r\in(0,1)$.

On the other hand, in 2000, Qiu [24: Theorem 2.3] extended (1.6) to the zero-balanced Gaussian hypergeometric function and proved that:

$$\frac{1}{B(a,b)}\log\left(\frac{e^{R(a,b)}}{1-x}\right) < F(a,b;a+b;x) < \frac{1}{R(a,b)}\log\left(\frac{e^{R(a,b)}}{1-x}\right) \tag{1.8}$$

for $a, b > 0$ with $R(a,b) > 0$ and $x \in (0,1)$. Letting $b = 1-a$ and $r = \sqrt{x}$, then from (1.1), (1.3), (1.4) and (1.8), we have

$$\sin(\pi a)\log\left(\frac{e^{R(a)/2}}{r'}\right) < \mathcal{K}_a(r) < \frac{\pi}{R(a)}\log\left(\frac{e^{R(a)/2}}{r'}\right), \quad r \in (0,1). \tag{1.9}$$

Motivated by (1.6), (1.7) and (1.9), one of the main purposes of this paper is to find analogue of the inequality (1.7) for the generalized elliptic integral of the first kind $\mathcal{K}_a(r)$.

For $r \in (0,1)$, let $\operatorname{arth}(r)$ denote the inverse of the hyperbolic tangent function. Very recently, with the combination of the arithmetic-geometric mean and some other classical bivariate means, Wang et al. [34: Theorems 3.5 and 4.2] proved that the function $r \mapsto [4r\mathcal{K}(r)/\pi - \operatorname{arth}(r) - 3r/(2+r')]/[\operatorname{arth}(r)/2 + r/2 - 3r/(2+r')]$ is strictly increasing from $(0,1)$ onto $(17/44, 8/\pi - 2)$. Consequently, the following sharp inequality has been derived: for $r \in (0,1)$,

$$\begin{aligned}&\frac{\pi}{2}\left[\frac{17}{44}\left(\frac{3\operatorname{arth}(r)}{4r} + \frac{1}{4}\right) + \left(1 - \frac{17}{44}\right)\left(\frac{\operatorname{arth}(r)}{2r} + \frac{3}{2(2+r')}\right)\right] \\ &< \mathcal{K}(r) < \frac{\pi}{2}\left[\left(\frac{8}{\pi} - 2\right)\left(\frac{3\operatorname{arth}(r)}{4r} + \frac{1}{4}\right) + \left(3 - \frac{8}{\pi}\right)\left(\frac{\operatorname{arth}(r)}{2r} + \frac{3}{2(2+r')}\right)\right].\end{aligned} \tag{1.10}$$

Another purpose of this paper is to extend the above result to the generalized elliptic integral of the first kind $\mathcal{K}_a(r)$.

We now state our main results.

Theorem 1.1. *Let $a \in (0, 1/2]$, $c \in (-\infty, +\infty)$,*

$$\lambda_a = e^{\pi/[2\sin(\pi a)]}[\sin(\pi a) - a(1-a)\pi]/[2\sin(\pi a)]$$

and

$$F_c(r) = r' e^{\mathcal{K}_a(r)/\sin(\pi a)} - e^{R(a)/2} - cr'^2, \quad r \in (0,1). \tag{1.11}$$

Then we have the following conclusions:

(1) *The function $F_c(r)$ is strictly decreasing if and only if $c \le \lambda_a$, in which case the range of $F_c(r)$ on $(0,1)$ is $(0, e^{\pi/[2\sin(\pi a)]} - e^{R(a)/2} - c)$. Consequently, the inequality*

$$\begin{aligned}&\sin(\pi a)\log\left(cr' + \frac{e^{R(a)/2}}{r'}\right) \\ &< \mathcal{K}_a(r) < \sin(\pi a)\log\left(cr' + \frac{e^{\pi/[2\sin(\pi a)]} - c}{r'}\right)\end{aligned} \tag{1.12}$$

holds for all $r \in (0,1)$;

(2) *If $c > \lambda_a$, then there exists an $r_0 \in (0,1)$, such that $F_c(r)$ is strictly increasing on $(0, r_0)$ and strictly decreasing on $(r_0, 1)$. Moreover, the inequality*

$$\mathcal{K}_a(r) > \sin(\pi a)\log\left(cr' + \frac{e^{R(a)/2}}{r'}\right) \tag{1.13}$$

holds for all $r \in (0,1)$ if and only if $c \le c_0^* = e^{\pi/[2\sin(\pi a)]} - e^{R(a)/2}$.

Theorem 1.2. *Let* $a \in (0,1/2]$ *and*

$$G(r) = \frac{2r\mathcal{K}_a(r)/\pi + (6a^2-6a+1)\operatorname{arth}(r) - 6(3a^2-3a+1)r/(2+r')}{\operatorname{arth}(r) - 6r/(2+r') + r}, \quad r \in (0,1).$$

Then $G(r)$ *is strictly increasing from* $(0,1)$ *onto* $((45a^4-90a^3+96a^2-51a+11)/11,$ $2\sin(\pi a)/\pi + 6a^2-6a+1)$. *Consequently, the inequality*

$$\begin{aligned}
&\frac{15\pi}{22}a(a-1)(3a^2-3a-1)\frac{\operatorname{arth}(r)}{r}\\
&\quad + \left(-\frac{45}{11}a^4 + \frac{90}{11}a^3 - \frac{63}{11}a^2 + \frac{18}{11}a\right)\frac{3\pi}{2+r'}\\
&\quad + \frac{\pi}{2}\left(\frac{45}{11}a^4 - \frac{90}{11}a^3 + \frac{96}{11}a^2 - \frac{51}{11}a + 1\right)\\
&< \mathcal{K}_a(r) < \sin(\pi a)\frac{\operatorname{arth}(r)}{r}\\
&\quad + \left(-3a^2 + 3a - \frac{2\sin(\pi a)}{\pi}\right)\frac{3\pi}{2+r'}\\
&\quad + \frac{\pi}{2}\left(\frac{2\sin(\pi a)}{\pi} + 6a^2 - 6a + 1\right)
\end{aligned} \tag{1.14}$$

holds for all $r \in (0,1)$.

Remark 1. If we take $a = 1/2$, then the inequalities (1.13) and (1.14) reduce to (1.7) and (1.10), respectively.

2. Lemmas

In order to prove our main results, we shall need several formulas and lemmas which we present in this section. First of all, for $a \in (0,1)$ and $r \in (0,1)$, the generalized elliptic integral of the second kind [4,18] is defined as

$$\mathcal{E}_a = \mathcal{E}_a(r) = \frac{\pi}{2}F(a-1,1-a;1;r^2), \quad \mathcal{E}_a(0^+) = \frac{\pi}{2}, \quad \mathcal{E}_a(1^-) = \frac{\sin(\pi a)}{2(1-a)}. \tag{2.1}$$

In the particular case $a = 1/2$, then $\mathcal{E}_a$ reduces to the complete elliptic integral of the second kind $\mathcal{E}$ (cf. [2,7]). We also need the following derivative formulas (cf. [4: Theorem 4.1]):

$$\frac{\mathrm{d}\mathcal{K}_a}{dr} = \frac{2(1-a)\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)}{rr'^2}, \qquad \frac{\mathrm{d}\mathcal{E}_a}{dr} = \frac{2(a-1)\left(\mathcal{K}_a - \mathcal{E}_a\right)}{r},$$

$$\frac{\mathrm{d}\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)}{\mathrm{d}r} = 2ar\mathcal{K}_a, \qquad \frac{\mathrm{d}\left(\mathcal{K}_a - \mathcal{E}_a\right)}{\mathrm{d}r} = \frac{2(1-a)r\mathcal{E}_a}{r'^2}.$$

Lemma 2.1 (cf. [7: Theorem 1.25])**.** *Let* $-\infty < a < b < \infty$, $f, g\colon [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b), *and* $g'(x) \neq 0$ *on* (a,b). *If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b), *then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2.2 (cf. [16] or [23: Lemma 2.1]). *Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ both converge for $|x| < R$, and $b_n > 0$ for all $n \in \{0, 1, 2, \dots\}$. If the non-constant sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing (decreasing), then $h(x) = f(x)/g(x)$ is strictly increasing (decreasing) on $(0, R)$.*

Lemma 2.2 is basically due to [16] and in this form with a general setting was stated in [23] along with many applications which were later adopted by a number of researchers.

Lemma 2.3. *Let $a \in (0, 1/2]$. Then the following statements are true:*

1. *The function $\phi_1(r) = \left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)/r^2$ is strictly increasing from $(0, 1)$ onto $(\pi a/2, \sin(\pi a)/[2(1-a)])$;*
2. *The function $\phi_2(r) = r'^c\mathcal{K}_a$ is strictly decreasing from $(0, 1)$ onto $(0, \pi/2)$ if and only if $c \geq 2a(1-a)$. Moreover, $r \mapsto \sqrt{r'}\mathcal{K}_a$ is strictly decreasing for each $a \in (0, 1/2]$;*
3. *The function $\phi_3(r) = \mathcal{K}_a/\sin(\pi a) + \log r'$ is strictly decreasing from $(0, 1)$ onto $(R(a)/2, \pi/(2\sin(\pi a)))$;*
4. *The function $\phi_4(r) = \left[(1 + r'^2)\sin(\pi a) - 2(1-a)\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)\right]/r'^{2(1+a)(2-a)/3}$ is strictly decreasing from $(0, 1)$ onto $(0, 2\sin(\pi a))$.*

Proof. Parts (1), (2) and (3) can be found in [4: Lemmas 5.2(1), 5.4(1) and 5.5(1)].

For part (4), let $\phi_{41}(r) = (1 + r'^2)\sin(\pi a) - 2(1-a)\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)$ and $\phi_{42}(r) = r'^{2(1+a)(2-a)/3}$. Simple computations lead to that

$$\phi_4(r) = \frac{\phi_{41}(r)}{\phi_{42}(r)}, \qquad \phi_{41}(1^-) = \phi_{42}(1^-) = 0 \tag{2.2}$$

and

$$\frac{\phi'_{41}(r)}{\phi'_{42}(r)} = \frac{3}{(1+a)(2-a)}\left[r'^{2(a^2-a+1)/3}\sin(\pi a) + 2a(1-a)r'^{2(a^2-a+1)/3}\mathcal{K}_a\right]. \tag{2.3}$$

It is not difficult to verify that $r'^{2(a^2-a+1)/3}\mathcal{K}_a$ is strictly decreasing on $(0, 1)$ from part (2), so that $\phi'_{41}(r)/\phi'_{42}(r)$ is strictly decreasing on $(0, 1)$ by (2.3). Combining with (2.2) and applying Lemma 2.1, we derive that $\phi_4(r)$ is strictly decreasing on $(0, 1)$.

Part (1) implies that $\phi_4(0^+) = 2\sin(\pi a)$, and by (2.2), (2.3) and L'Hospital's rule we get $\phi_4(1^-) = 0$. Therefore, part (4) is valid. □

Lemma 2.4. *For $a \in (0, 1/2]$, define ψ on $(0, 1)$ by*

$$\psi(r) = \frac{(1-a)\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)^2 + \sin(\pi a)r'^2\left[ar^2\mathcal{K}_a - \left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)\right]}{r^4}. \tag{2.4}$$

Then $\psi(r)$ is strictly increasing from $(0, 1)$ onto $(\pi a^2(1-a)[\pi + \sin(\pi a)]/4, [\sin^2(\pi a)]/[4(1-a)])$.

Proof. Let $\psi_1(r) = (1-a)\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)^2 + \sin(\pi a)r'^2\left[ar^2\mathcal{K}_a - \left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)\right]$. Then differentiating ψ_1 gives

$$\psi'_1(r) = 2r\left[2a(1-a)\mathcal{K}_a\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right) + (-a^2 + a + 1)\sin(\pi a)\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right) - a\sin(\pi a)r^2\mathcal{K}_a\right],$$

so that

$$\begin{aligned}
r^5\psi'(r) &= r\psi_1'(r) - 4\psi_1(r) \\
&= 2r^2\Big[2a(1-a)\mathcal{K}_a\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right) \\
&\qquad + (-a^2+a+1)\sin(\pi a)\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right) - a\sin(\pi a)r^2\mathcal{K}_a\Big] \\
&\quad - 4(1-a)\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)^2 - 4\sin(\pi a)r'^2\left[ar^2\mathcal{K}_a - \left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)\right] \\
&= 4(1-a)\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)\left[ar^2\mathcal{K}_a - \left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)\right] \\
&\quad + 2a(1-a)\sin(\pi a)r^2\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right) \\
&\quad - 2\sin(\pi a)r^2\left[ar^2\mathcal{K}_a - \left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)\right] \\
&\quad - 4\sin(\pi a)r'^2\left[ar^2\mathcal{K}_a - \left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)\right] \\
&= 2a(1-a)\sin(\pi a)r^2\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right) \\
&\quad \times\left\{1 - \frac{1}{a(1-a)\sin(\pi a)}\left[\frac{r^2}{\mathcal{E}_a - r'^2\mathcal{K}_a}\right]\right. \\
&\qquad \times\left[\frac{(1+r'^2)\sin(\pi a) - 2(1-a)\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)}{r'^{\frac{2}{3}(1+a)(2-a)}}\right] \\
&\qquad \left.\times\left[\frac{r'^{\frac{2}{3}(1+a)(2-a)}\left(ar^2\mathcal{K}_a - \left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)\right)}{r^4}\right]\right\}.
\end{aligned} \tag{2.5}$$

Let $\psi_2(r) = r'^{\frac{2}{3}(1+a)(2-a)}\left[ar^2\mathcal{K}_a - \left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)\right]/r^4$. Then from (1.1) and (2.1) one has

$$\psi_2(r) = \pi[a^2(1-a)/4]r'^{\frac{2}{3}(1+a)(2-a)}F(1+a, 2-a; 3; r^2).$$

Since the function $x \mapsto (1-x)^d F(a,b;c;x)$ with $a,b,c > 0$ and $d \in \mathbb{R}$ is decreasing on $(0,1)$ if and only if $d \geq \max\{a+b-c, ab/c\}$(cf. [27: Lemma 2.15] and also [23]), then $\psi_2(r)$ is strictly decreasing on $(0,1)$.

Obviously,

$$\psi_2(0^+) = \frac{a^2(1-a)\pi}{4} \quad \text{and} \quad \psi_2(1^-) = 0. \tag{2.6}$$

Furthermore, it follows from (2.5), (2.6) together with Lemma 2.3 (1) and (4) that $\psi'(r) > 0$ for all $r \in (0,1)$, so that $\psi(r)$ is strictly increasing on $(0,1)$.

Using (2.6) and Lemma 2.3(1), we get

$$\psi(0^+) = \frac{\pi a^2(1-a)}{4}[\pi + \sin(\pi a)] \quad \text{and} \quad \psi(1^-) = \frac{\sin^2(\pi a)}{4(1-a)}. \tag{2.7}$$

Therefore, Lemma 2.4 follows from (2.7) and the monotonicity of $\psi(r)$. □

Lemma 2.5. *For $a \in (0, 1/2]$, define φ on $(0,1)$ by*

$$\varphi(r) = \mathrm{e}^{\frac{\mathcal{K}_a(r)}{\sin(\pi a)}}\frac{\sin(\pi a) - 2(1-a)\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)/r^2}{r'}. \tag{2.8}$$

Then $\varphi(r)$ is strictly increasing from $(0,1)$ onto $\left(\mathrm{e}^{\pi/[2\sin(\pi a)]}[\sin(\pi a) - a(1-a)\pi], +\infty\right)$.

Proof. Differentiation gives

$$\begin{aligned}\varphi'(r) &= \mathrm{e}^{\frac{\mathcal{K}_a(r)}{\sin(\pi a)}} \frac{2(1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)}{rr'^2\sin(\pi a)} \left[\frac{\sin(\pi a) - 2(1-a)\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)/r^2}{r'}\right] \\ &\quad + \mathrm{e}^{\frac{\mathcal{K}_a(r)}{\sin(\pi a)}} \frac{4(1-a)r'^2\left[(\mathcal{E}_a - r'^2\mathcal{K}_a) - ar^2\mathcal{K}_a\right] + r^2\left[r^2\sin(\pi a) - 2(1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)\right]}{r^3r'^3} \\ &= \frac{\mathrm{e}^{\frac{\mathcal{K}_a(r)}{\sin(\pi a)}}}{r^3r'^3\sin(\pi a)}\Big\{r^4[\sin(\pi a)]^2 - 4(1-a)^2(\mathcal{E}_a - r'^2\mathcal{K}_a) \\ &\quad - 4(1-a)\sin(\pi a)r'^2\left[ar^2\mathcal{K}_a - \left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)\right]\Big\} \\ &= \frac{r[\sin(\pi a)]\mathrm{e}^{\frac{\mathcal{K}_a(r)}{\sin(\pi a)}}}{r'^3}\left[1 - \frac{4(1-a)}{[\sin(\pi a)]^2}\psi(r)\right], \end{aligned} \tag{2.9}$$

where $\psi(r)$, defined in (2.4), is strictly increasing on $(0,1)$.

By (1.1), (2.1) and Lemma 2.3 (1), (3) together with L'Hospital's rule, we obtain that

$$\varphi(0^+) = \mathrm{e}^{\frac{\pi}{2\sin(\pi a)}}[\sin(\pi a) - a(1-a)\pi] \tag{2.10}$$

and

$$\begin{aligned}\varphi(1^-) &= \lim_{r\mapsto 1^-} \mathrm{e}^{\frac{\mathcal{K}_a(r)}{\sin(\pi a)} + \log r'} \cdot \lim_{r\mapsto 1^-} \frac{\sin(\pi a) - 2(1-a)\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)/r^2}{r'^2} \\ &= 2(1-a)\mathrm{e}^{\frac{R(a)}{2}} \lim_{r\mapsto 1^-} \frac{ar^2\mathcal{K}_a - \left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)}{r^4} = +\infty. \end{aligned} \tag{2.11}$$

Therefore, Lemma 2.5 follows from (2.10), (2.11) and the monotonicity of $\varphi(r)$. □

Lemma 2.6. *For $a \in (0,1/2]$, define the function h on $(0,1)$ by*

$$h(r) = \frac{r'^2\left[a(1-2a)r^2(\mathcal{E}_a - r'^2\mathcal{K}_a) - (a^2-2a+2)r^2(\mathcal{K}_a - \mathcal{E}_a) - a^2(1-2a)r^4\mathcal{K}_a + 2(\mathcal{K}_a - \mathcal{E}_a) - 2(1-a)r^2\mathcal{E}_a\right]}{r^2\left[(1-a)r^2\mathcal{E}_a - a(1-2a)r^2(\mathcal{E}_a - r'^2\mathcal{K}_a) - r'^2(\mathcal{K}_a - \mathcal{E}_a)\right]}.$$

Then $h(r)$ is strictly decreasing from $(0,1)$ onto $(0, a(1-a)(7a^2-7a+4)/[3(5a^2-5a+2)])$.

Proof. Let

$$\begin{aligned} h_1(r) = r'^2\big[&a(1-2a)r^2\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right) - (a^2-2a+2)r^2\left(\mathcal{K}_a - \mathcal{E}_a\right) \\ &- a^2(1-2a)r^4\mathcal{K}_a + 2\left(\mathcal{K}_a - \mathcal{E}_a\right) - 2(1-a)r^2\mathcal{E}_a\big] \end{aligned}$$

and

$$h_2(r) = r^2\left[(1-a)r^2\mathcal{E}_a - a(1-2a)r^2\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right) - r'^2\left(\mathcal{K}_a - \mathcal{E}_a\right)\right].$$

Then $h(r)=h_1(r)/h_2(r)$ and tedious computations lead to that

$$\begin{aligned}
\frac{2}{\pi}h_1(r) = r'^2\bigg[&\sum_{n=0}^{\infty}\frac{a^2(1-2a)(a)_n(1-a)_n}{(n+1)(n!)^2}r^{2n+4}+\sum_{n=0}^{\infty}\frac{-(a^2-2a+2)(a)_n(1-a)_{n+1}}{(n+1)(n!)^2}r^{2n+4}\\
&+\sum_{n=0}^{\infty}\frac{-a^2(1-2a)(a)_n(1-a)_n}{(n!)^2}r^{2n+4}+\sum_{n=0}^{\infty}\frac{2(a)_n(1-a)_{n+1}}{(n+1)(n!)^2}r^{2n+2}\\
&+\sum_{n=0}^{\infty}\frac{-2(1-a)(a-1)_n(1-a)_n}{(n!)^2}r^{2n+2}\bigg]\\
= r'^2\bigg[&\sum_{n=0}^{\infty}\frac{a^2(1-2a)(a)_n(1-a)_n}{(n+1)(n!)^2}r^{2n+4}+\sum_{n=0}^{\infty}\frac{-(a^2-2a+2)(a)_n(1-a)_{n+1}}{(n+1)(n!)^2}r^{2n+4}\\
&+\sum_{n=0}^{\infty}\frac{-a^2(1-2a)(a)_n(1-a)_n}{(n!)^2}r^{2n+4}+\sum_{n=0}^{\infty}\frac{2(a^2-2a+n+2)(a)_n(1-a)_{n+1}}{(n+2)(n+1)(n!)^2}r^{2n+4}\bigg]\\
= r'^2\sum_{n=0}^{\infty}\bigg[&\frac{a^2(1-2a)(a)_n(1-a)_n}{(n+1)(n!)^2}+\frac{-(a^2-2a+2)(a)_n(1-a)_{n+1}}{(n+1)(n!)^2}\\
&+\frac{-a^2(1-2a)(a)_n(1-a)_n}{(n!)^2}+\frac{2(a^2-2a+n+2)(a)_n(1-a)_{n+1}}{(n+2)(n+1)(n!)^2}\bigg]r^{2n+4}\\
= \sum_{n=1}^{\infty}&\frac{a(a)_n(1-a)_n[2(a^2-a+1)n+5a^2-5a+2]}{(n+2)(n+1)n[(n-1)!]^2}r^{2n+4}\\
-&\sum_{n=0}^{\infty}\frac{a(a)_n(1-a)_n[2(a^2-a+1)n+5a^2-5a+2]}{(n+2)(n+1)n[(n-1)!]^2}r^{2n+6}\\
= -&\sum_{n=0}^{\infty}\frac{a(a)_n(1-a)_n}{(n+3)(n+2)(n+1)(n!)^2}\Big[2(a^2-a+1)n^2+2(a^4-2a^3+6a^2-5a+1)n\\
&-a(1-a)(7a^2-7a+4)\Big]r^{2n+6}
\end{aligned}$$

and

$$\frac{2}{\pi}h_2(r)=\sum_{n=0}^{\infty}\frac{a(a)_n(1-a)_n\left[2(a^2-a+1)n+5a^2-5a+2\right]}{(n+2)(n+1)(n!)^2}r^{2n+6}.$$

Then

$$h(r)=\frac{h_1(r)}{h_2(r)}=-\frac{\sum_{n=0}^{\infty}A_nr^{2n}}{\sum_{n=0}^{\infty}B_nr^{2n}}, \tag{2.12}$$

where

$$\begin{aligned}
A_n =&\frac{a(a)_n(1-a)_n}{(n+3)(n+2)(n+1)(n!)^2}\\
&\times\left[2(a^2-a+1)n^2+2(a^4-2a^3+6a^2-5a+1)n-a(1-a)(7a^2-7a+4)\right]
\end{aligned}$$

and

$$B_n=\frac{a(a)_n(1-a)_n\left[2(a^2-a+1)n+5a^2-5a+2\right]}{(n+2)(n+1)(n!)^2}>0$$

for all $n \in \mathbb{N}$. Let $C_n = A_n/B_n$, one has

$$C_n = \frac{2(a^2-a+1)n^2 + 2(a^4-2a^3+6a^2-5a+1)n - a(1-a)(7a^2-7a+4)}{[2(a^2-a+1)n+5a^2-5a+2](n+3)}$$

and

$$\begin{aligned} C_{n+1} - C_n = {} & \frac{(a+1)(2-a)}{[2(a^2-a+1)n+7a^2-7a+4][2(a^2-a+1)n+5a^2-5a+2](n+3)(n+4)} \\ & \times \left\{2\left[(2a^2-2a+3)n+(16a^2-16a+9)\right](a^2-a+1)n + 61a^4-122a^3+113a^2-52a+12\right\}. \end{aligned}$$

It is not difficult to verify that $61a^4-122a^3+113a^2-52a+12 > 0$ for all $a \in (0,1/2]$. Thus, $C_{n+1} - C_n > 0$, i.e. C_n is strictly increasing for all $n \in \mathbb{N}$, and $h(r)$ is strictly decreasing on $(0,1)$ by Lemma 2.2 and (2.12).

Note that

$$h(0^+) = -\frac{A_0}{B_0} = \frac{a(1-a)(7a^2-7a+4)}{3(5a^2-5a+2)}, \quad h(1^-) = 0. \tag{2.13}$$

Therefore, Lemma 2.6 follows from (2.13) and the monotonicity of $h(r)$. □

Lemma 2.7. *For $a \in (0,1/2]$ and $c \in (-\infty,+\infty)$, define g_c on $(0,1)$ by*

$$g_c(r) = \frac{r'^c\left[(1-a)r^2\mathcal{E}_a - a(1-2a)r^2(\mathcal{E}_a - r'^2\mathcal{K}_a) - r'^2(\mathcal{K}_a-\mathcal{E}_a)\right]}{r^4}. \tag{2.14}$$

Then the following statements are true:

1. *$g_c(r)$ is strictly decreasing on $(0,1)$ if and only if $c \ge 2a(1-a)(7a^2-7a+4)/[3(5a^2-5a+2)]$;*
2. *$g_c(r)$ is strictly increasing on $(0,1)$ if and only if $c \le 0$.*

Proof. Differentiating g_c gives

$$g_c'(r) = \frac{r'^{c-2}\left[(1-a)r^2\mathcal{E}_a - a(1-2a)r^2\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right) - r'^2\left(\mathcal{K}_a-\mathcal{E}_a\right)\right]}{r^3}\left[2h(r)-c\right], \tag{2.15}$$

where $h(r)$ is defined in Lemma 2.6. The proof of Lemma 2.6 shows that the function $r \mapsto (1-a)r^2\mathcal{E}_a - a(1-2a)r^2\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right) - r'^2\left(\mathcal{K}_a-\mathcal{E}_a\right)$ is positive on $(0,1)$. Combining with (2.15), we obtain that $g_c'(r) < 0$ for $r \in (0,1)$ if and only if $c \ge 2a(1-a)(7a^2-7a+4)/[3(5a^2-5a+2)]$, and $g_c'(r) > 0$ for $r \in (0,1)$ if and only if $c \le 0$. Therefore, Lemma 2.7 follows. □

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Differentiating F_c in (1.11) gives

$$\begin{aligned} F_c'(r) &= -\frac{r}{r'}e^{\frac{\mathcal{K}_a(r)}{\sin(\pi a)}} + e^{\frac{\mathcal{K}_a(r)}{\sin(\pi a)}}\frac{2(1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)}{rr'\sin(\pi a)} + 2cr \\ &= \frac{e^{\frac{\mathcal{K}_a(r)}{\sin(\pi a)}}}{rr'\sin(\pi a)}\left[2(1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a) - r^2\sin(\pi a)\right] + 2cr \\ &= r\left[2c - \frac{\varphi(r)}{\sin(\pi a)}\right], \end{aligned} \tag{3.1}$$

where $\varphi(r)$ is defined in (2.8). The assertion about monotonicity properties of F_c in parts (1) and (2) directly follows from (3.1) and Lemma 2.5. Employing (1.1) and Lemma 2.3 (3) gives

$$F_c(0^+) = e^{\pi/(2\sin(\pi a))} - e^{R(a)/2} - c, \quad F_c(1^-) = \lim_{r\mapsto 1^-}\left[e^{\frac{\mathcal{K}_a(r)}{\sin(\pi a)}+\log r'} - e^{\frac{R(a)}{2}} - cr'^2\right] = 0. \tag{3.2}$$

Therefore the inequality (1.12) holds.

Finally, for the inequality (1.13), according to the range of F_{λ_a} on $(0,1)$ in part (1) we conclude that $c_0^* = e^{\pi/[2\sin(\pi a)]} - e^{R(a)/2} > \lambda_a$ for $a \in (0,1/2]$, so that $F_{c_0^*}$ is piecewise monotone on $(0,1)$. This, together with $F_{c_0^*}(0^+) = F_{c_0^*}(1^-) = 0$, shows that $F_{c_0^*}(r) > 0$ for all $r \in (0,1)$. Since the inequality

$$F_c(r) = r' e^{\mathcal{K}_a(r)/\sin(\pi a)} - e^{R(a)/2} - cr'^2 > 0, \quad r \in (0,1)$$

is equivalent to (1.13), we see that

$$\mathcal{K}_a(r) > \sin(\pi a)\log\left(c_0^* r' + \frac{e^{R(a)/2}}{r'}\right)$$

holds for all $r \in (0,1)$. Furthermore, (3.2) implies that (1.13) holds for all $r \in (0,1)$ only if $c \le c_0^*$. Until now, the optimality of c_0^* follows, and the proof is completed. □

Proof of Theorem 1.2. Let

$$\begin{aligned}
G_1(r) &= \frac{2r\mathcal{K}_a(r)}{\pi} + (6a^2-6a+1)\operatorname{arth}(r) - \frac{6(3a^2-3a+1)r}{2+r'},\\
G_2(r) &= \operatorname{arth}(r) - \frac{6r}{2+r'} + r,\\
G_3(r) &= \frac{2\left[r'^2\mathcal{K}_a + 2(1-a)\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right)\right]}{\pi} - \frac{6(3a^2-3a+1)r'(1+2r')}{(2+r')^2} + 6a^2-6a+1,\\
G_4(r) &= 1 - \frac{6r'(1+2r')}{(2+r')^2} + r'^2,\\
G_5(r) &= \frac{2a(1-2a)r^2\mathcal{K}_a - 2(1-a)(\mathcal{K}_a - \mathcal{E}_a)}{\pi r^2} + \frac{3(3a^2-3a+1)(2+7r')}{r'(2+r')^3},\\
G_6(r) &= -1 + \frac{3(2+7r')}{r'(2+r')^3}.
\end{aligned}$$

Then simple computations lead to that

$$\begin{aligned}
&G(r) = G_1(r)/G_2(r),\\
&G_1'(r)/G_2'(r) = G_3(r)/G_4(r),\\
&G_3'(r)/G_4'(r) = G_5(r)/G_6(r),\\
&G_1(0^+) = G_2(0^+) = G_3(0^+) = G_4(0^+) = G_5(0^+) = G_6(0^+) = 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{G_5'(r)}{G_6'(r)} &= 3a^2-3a+1 - \frac{4(1-a)}{3\pi}\frac{r'^{1/2}(2+r')^4}{21r'^2+8r'+4}\\
&\quad\times \frac{r'^{1/2}\left[(1-a)r^2\mathcal{E}_a - a(1-2a)r^2\left(\mathcal{E}_a - r'^2\mathcal{K}_a\right) - r'^2\left(\mathcal{K}_a - \mathcal{E}_a\right)\right]}{r^4}.
\end{aligned} \tag{3.3}$$

It is easy to check that the function $r \mapsto r(2+r^2)^4/(21r^4+8r^2+4)$ is strictly increasing from $(0,1)$ onto $(0,27/11)$, and $1/2 > 2a(1-a)(7a^2-7a+4)/[3(5a^2-5a+2)]$ for $a \in (0,1/2]$. Thus, $G_5'(r)/G_6'(r)$ is strictly increasing on $(0,1)$ by (3.3) and Lemma 2.7. Applying Lemma 2.1 three times, we conclude that $G(r)$ is strictly increasing on $(0,1)$. Moreover,

$$G(0^+) = \lim_{r\to 0^+} \frac{G_5'(r)}{G_6'(r)} = \frac{45a^4-90a^3+96a^2-51a+11}{11} \tag{3.4}$$

and

$$\begin{aligned} G(1^-) &= \lim_{r\to 1^-} \frac{F(a,1-a;1;r^2)+(6a^2-6a+1)F(\frac{1}{2},1;\frac{3}{2};r^2)-3(3a^2-3a+1)}{F(\frac{1}{2},1;\frac{3}{2};r^2)-2} \\ &= \frac{2\sin(\pi a)}{\pi}+6a^2-6a+1. \end{aligned} \tag{3.5}$$

Therefore, Theorem 1.2 follows from (3.4) and (3.5) together with the monotonicity of $G(r)$. □

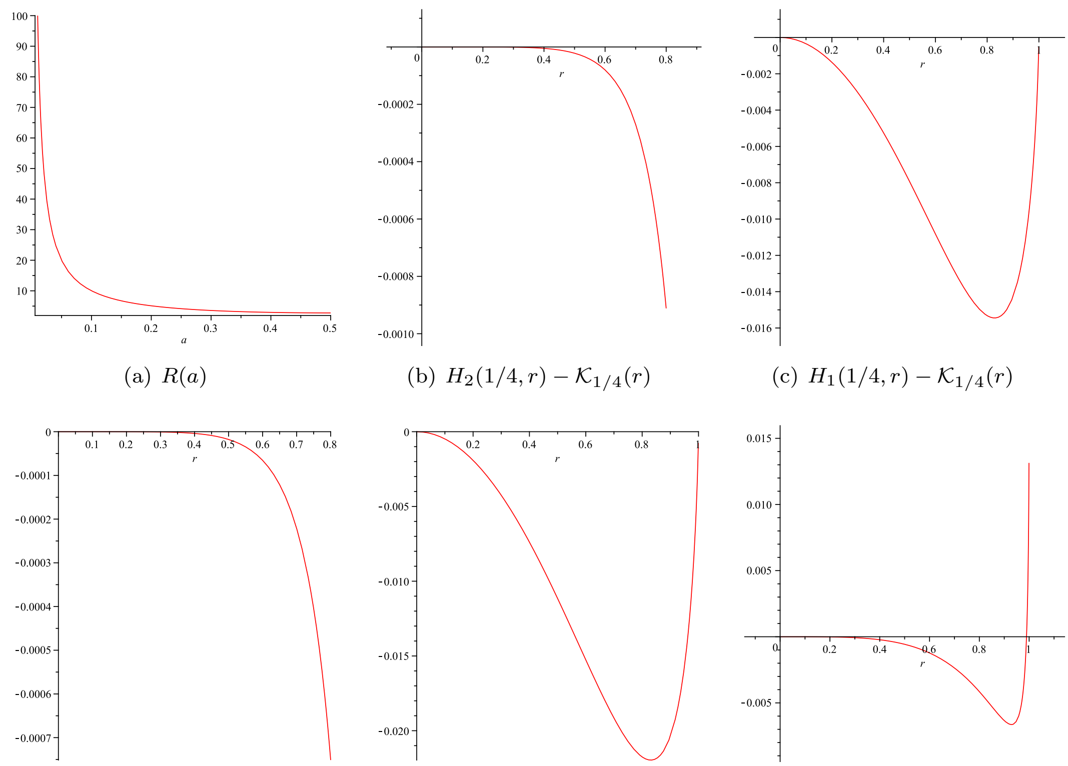
Remark 2. (1) One of the reviewers reminded us that the function $R(a)$ and its properties are very important, and suggested to add its graph in the paper. In fact, it has been shown in [27: Lemma 2.14 (2)] that $a \mapsto [R(a)-\log 16]/(1/2-a)^2$ is strictly decreasing from $(0,1/2]$ onto $(14\zeta(3),\infty)$, so that inequality $R(a) \geq 14\zeta(3)(1/2-a)^2+\log 16 \geq \log 16$ is valid for all $a \in (0,1/2]$, and $a \mapsto R(a)$ is also strictly decreasing from $(0,1/2]$ onto $(\log 16,+\infty)$(cf. Figure 1(a)), where $\zeta(x)$ is the Riemann zeta function. For series expansion and more asymptotic inequalities of $R(a)$, readers can refer to [28, 29].

(2) Numerical experiment results demonstrate that for some given $a \in (0,1/2]$, our lower bound for $\mathcal{K}_a(r)$ in (1.14) is particularly good for $r \in (0,0.8)$, and the upper bounds in Theorems 1.1 and 1.2 are not comparable on the whole interval $(0,1)$. Moreover, the difference of the two sides in (1.13) with $c = c_0^* = \mathrm{e}^{\pi/[2\sin(\pi a)]}-\mathrm{e}^{R(a)/2}$ tends to 0 as r tends to 0 or 1. For the above, see Figure 1(b)–(f) and Table 1. Here, for $(a,r) \in (0,1/2]\times(0,1)$,

$$\begin{aligned} H_1(a,r) &= \sin(\pi a)\log\left(\left(\mathrm{e}^{\pi/[2\sin(\pi a)]}-\mathrm{e}^{R(a)/2}\right)r'+\frac{\mathrm{e}^{R(a)/2}}{r'}\right), \\ H_2(a,r) &= \frac{15\pi a(a-1)(3a^2-3a-1)}{22}\frac{\operatorname{arth}(r)}{r} \\ &\quad+\left(-\frac{45}{11}a^4+\frac{90}{11}a^3-\frac{63}{11}a^2+\frac{18}{11}a\right)\frac{3\pi}{2+r'} \\ &\quad+\frac{\pi}{2}\left(\frac{45}{11}a^4-\frac{90}{11}a^3+\frac{96}{11}a^2-\frac{51}{11}a+1\right), \\ H_3(a,r) &= \sin(\pi a)\frac{\operatorname{arth}(r)}{r}+\left(3a-3a^2-\frac{2\sin(\pi a)}{\pi}\right)\frac{3\pi}{2+r'} \\ &\quad+\frac{\pi}{2}\left(\frac{2\sin(\pi a)}{\pi}+6a^2-6a+1\right) \end{aligned}$$

and

$$\begin{aligned} H_4(a,r) = \sin(\pi a)\log\bigg(&\frac{\mathrm{e}^{\pi/[2\sin(\pi a)]}[\sin(\pi a)-a(1-a)\pi]}{2\sin(\pi a)}r' \\ &+\frac{\mathrm{e}^{\pi/[2\sin(\pi a)]}[\sin(\pi a)+a(1-a)\pi]}{2\sin(\pi a)r'}\bigg). \end{aligned}$$



(a) $R(a)$ (b) $H_2(1/4,r)-\mathcal{K}_{1/4}(r)$ (c) $H_1(1/4,r)-\mathcal{K}_{1/4}(r)$

(d) $H_2(1/3,r)-\mathcal{K}_{1/3}(r)$ (e) $H_1(1/3,r)-\mathcal{K}_{1/3}(r)$ (f) $H_3(1/4,r)-H_4(1/4,r)$

FIGURE 1.

TABLE 1. Relative errors of the approximation of $\mathcal{K}_a(r)$ by $H_2(a,r)$ with $a=1/4$ and $a=1/3$

r	$\frac{\mathcal{K}_{1/4}(r)-H_2(1/4,r)}{\mathcal{K}_{1/4}(r)}\times 100\%$	$\frac{\mathcal{K}_{1/3}(r)-H_2(1/3,r)}{\mathcal{K}_{1/3}(r)}\times 100\%$
0.1	6.02×10^{-10}	1.27×10^{-9}
0.2	3.98×10^{-8}	3.28×10^{-8}
0.3	4.87×10^{-7}	3.94×10^{-7}
0.4	3.01×10^{-6}	2.43×10^{-6}
0.5	1.31×10^{-5}	1.06×10^{-5}
0.6	4.71×10^{-5}	3.79×10^{-5}
0.7	1.52×10^{-4}	1.22×10^{-4}
0.8	4.83×10^{-4}	3.86×10^{-4}
0.9	1.74×10^{-3}	1.38×10^{-3}

Acknowledgement. The authors wish to thank the anonymous referees for helpful comments and suggestions.

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Received 5. 10. 2021
Accepted 20. 3. 2022

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