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ON OBLIQUE DOMAINS OF JANOWSKI FUNCTIONS

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ABSTRACT. We investigate certain properties of tilted (oblique) domains, associated with the Janowski function $(1 + Az)/(1 + Bz)$, where $A, B \in \mathbb{C}$ with $A \neq B$ and $|B| \leq 1$. We find several bounds for these oblique domains and also establish various subordination, radius, argument estimates involving Janowski function with complex parameters. Moreover, some results also generalize earlier well-known results pertaining to Janowski function.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the class of analytic functions defined on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Assume $\mathcal{H}[a, n] := \{f \in \mathcal{H}(\mathbb{D}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}$, where $n = 1, 2, \dots$ with $a \in \mathbb{C}$ and $\mathcal{H}_1 := \mathcal{H}[1, 1]$. Let $\mathcal{A}_n := \{f \in \mathcal{H}(\mathbb{D}) : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots\}$ and $\mathcal{A} := \mathcal{A}_1$. A subclass of $\mathcal{A}$ consisting of all univalent functions is denoted by $\mathcal{S}$. We say, f is subordinate to g, written as $f \prec g$, if $f(z) = g(\omega(z))$, where f, g are analytic functions and $\omega(z)$ is a Schwarz function. Moreover, if g is univalent, then $f \prec g$ if and only if $f(\mathbb{D}) \subseteq g(\mathbb{D})$ and $f(0) = g(0)$. For $-\pi/2 < \lambda < \pi/2$, [23] introduced the tilted Carathéodory class by angle λ as:

$$\mathcal{P}_\lambda := \left\{ p \in \mathcal{H}_1 : \mathrm{e}^{\mathrm{i}\lambda} p(z) \prec \frac{1+z}{1-z} \right\}. \tag{1.1}$$

Here $\mathcal{P}_0 = \mathcal{P}$, the well-known Carathéodory class. A Janowski function is a bilinear transformation, which was first investigated in [3]. Author introduced the class $\mathcal{P}(A, B)$, where $-1 \leq B < A \leq 1$, which comprises of the set of all p in $\mathcal{H}_1$ such that

$$p(z) \prec \frac{1 + Az}{1 + Bz}.$$

The domain $p(\mathbb{D})$, where $p \in \mathcal{P}(A, B)$ is either a disk or a half plane, which is symmetric with respect to positive real axis and lying in the Carathéodory portion of the complex plane. In the present paper, we investigate much broader class, i.e., $\mathcal{P}(A, B, \alpha)$, where $A, B \in \mathbb{C}$ with $|B| \leq 1$, $A \neq B$ and $0 < \alpha \leq 1$, which includes the set of all $p \in \mathcal{H}_1$ satisfying

$$p(z) \prec \left(\frac{1 + Az}{1 + Bz} \right)^\alpha.$$

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Here the domain $p(\mathbb{D})$, where $p \in \mathcal{P}(A,B,1)$ is also either a disk or a half plane but is oblique, i.e., neither necessarily it is symmetric with respect to positive real axis nor necessarily it is lying in the Carathéodory portion of the complex plane, and whenever $p \in \mathcal{P}(A,B,\alpha)$, then $p(\mathbb{D})$ is either a squeezed disk forming a petal shaped domain or a sector. In particular, this class can give information about various well-known classes, like $\mathcal{SS}^*(A,B,\alpha)$, the class of Janowski strongly starlike functions of order α which consists all functions $f \in \mathcal{S}$ satisfying

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+Az}{1+Bz}\right)^{\alpha}. \tag{1.2}$$

Here $\mathcal{SS}^*(1,-1,\alpha) =: \mathcal{SS}^*(\alpha)$, the class of strongly starlike functions of order α. And for $\alpha = 1$ in (1.2), we obtain the class of Janowski starlike functions, denoted as $\mathcal{S}^*(A,B)$, which specifically reduces to many well-known classes such as the class of starlike functions, $\mathcal{S}^*(1,-1) =: \mathcal{S}^*$; the class of starlike functions of order α, $\mathcal{S}^*(1-2\alpha,-1) =: \mathcal{S}^*(\alpha)$ $(0 \le \alpha < 1)$. Note that analogous to all above classes, we can also study the classes of Janowski convex functions, convex functions, convex functions of order α, etc., by replacing $zf'(z)/f(z)$ by $1 + zf''(z)/f'(z)$. Furthermore, there are many more interesting classes, some of them are the class of starlike functions of reciprocal order α, $\mathcal{S}^*(1, 2\alpha-1) =: \mathcal{RS}^*(\alpha)(0 \le \alpha < 1)$, the class of Uralegaddi functions $\mathbb{M}(\beta) := \mathcal{S}^*(1-2\beta,-1)$ $(\beta > 1)$ and the class of α-spirallike functions of order β, $\mathcal{S}^*_{\alpha}(\beta) := \mathcal{S}^*(\mathrm{e}^{-\mathrm{i}\alpha}\left(\mathrm{e}^{-\mathrm{i}\alpha} - 2\beta\cos\alpha\right), -1)$ $(-\pi/2 < \alpha < \pi/2$ and $0 \le \beta < 1)$. From the above classes the range of A in $\mathcal{P}_{\lambda}$, $\mathbb{M}(\beta)$ and $\mathcal{S}^*_{\alpha}(\beta)$ is not as given by Janowski. This motivates us to extend and study Janowski function with complex parameters.

In the past, various authors have done subordination results for $(1+Az)/(1+Bz)$, where $A, B \in \mathbb{C}$ with $|B| \le 1$ and $A \ne B$ (see [1,4–6,8]). In this paper we found various bounds and properties of our class $\mathcal{P}(A,B,\alpha)$ and also establish several subordination, radius, argument problems involving Janowski function with complex parameters. Moreover, some results also generalize earlier well-known results pertaining to Janowski function.

2. Basic properties of the class $\mathcal{P}(A,B,\alpha)$

In the present section, we discuss various geometric properties concerned with functions belonging to $\mathcal{P}(A,B,\alpha)$. We begin with the following bound estimate result.

Theorem 2.1. *Let $h \in \mathcal{H}_1$ and $A, B \in \mathbb{C}$ with $A \ne B$ and $|B| \le 1$. Further if*

$$h(z) \prec (1-\gamma)\left(\frac{1+Az}{1+Bz}\right)^{\alpha} + \gamma \tag{2.1}$$

for some $0 < \alpha \le 1$, $\gamma \in \mathbb{C} \setminus \{1\}$, then for $|z| = r < 1$, we have

(i)

$$\left|\arg\left(\frac{h(z)-\gamma}{1-\gamma}\right) - \alpha\tan^{-1}\frac{\operatorname{Im}(A\overline{B})r^2}{\operatorname{Re}(A\overline{B})r^2 - 1}\right| < \alpha\sin^{-1}\frac{|A-B|r}{|1-A\overline{B}r^2|}, \quad \textit{whenever } |A| \le 1;$$

(ii)

$$\left(\frac{|1-A\overline{B}r^2| - |A-B|r}{1-|B|^2r^2}\right)^{\alpha} \le \left|\frac{h(z)-\gamma}{1-\gamma}\right| \le \left(\frac{|1-A\overline{B}r^2| + |A-B|r}{1-|B|^2r^2}\right)^{\alpha};$$

(iii)

$$\min\{M(t_1)\cos(N(t_1)), M(t_1+\pi)\cos(N(t_1+\pi))\} \leq \operatorname{Re}\left(\frac{h(z)-\gamma}{1-\gamma}\right)$$
$$\leq \max\{M(t_1)\cos(N(t_1)), M(t_1+\pi)\cos(N(t_1+\pi))\};$$

(iv)

$$\min\{M(t_2)\sin(N(t_2)), M(\tau-t_2)\sin(N(\tau-t_2))\} \leq \operatorname{Im}\left(\frac{h(z)-\gamma}{1-\gamma}\right)$$
$$\leq \max\{M(t_2)\sin(N(t_2)), M(\tau-t_2)\sin(N(\tau-t_2))\},$$

where

$$M(t) = \left(\sqrt{(u(t))^2+(v(t))^2}\right)^{\alpha} \quad \textit{and} \quad N(t) = \alpha\tan^{-1}\left(\frac{v(t)}{u(t)}\right),$$

with

$$u(t) = \frac{1-\operatorname{Re}(A\overline{B})r^2+|A-B|r\cos t}{1-|B|^2r^2} \quad \textit{and} \quad v(t) = \frac{|A-B|r\sin t-\operatorname{Im}(A\overline{B})r^2}{1-|B|^2r^2}.$$

Further t_1 and t_2 are the roots of

$$\frac{u(t)u'(t)+v(t)v'(t)}{u(t)v'(t)-v(t)u'(t)} = \tan\left(\alpha\tan^{-1}\frac{v(t)}{u(t)}\right) \tag{2.2}$$

and

$$\frac{v(t)u'(t)-u(t)v'(t)}{u(t)u'(t)+v(t)v'(t)} = \tan\left(\alpha\tan^{-1}\frac{v(t)}{u(t)}\right), \tag{2.3}$$

respectively. Further, all the above bounds are sharp.

P r o o f. Let $H(z) := ((h(z)-\gamma)/(1-\gamma))^{\frac{1}{\alpha}}$. Since $h(z) \neq 0$ and $h(0)=1$, we have $H \in \mathcal{H}_1$ and

$$H(z) \prec \frac{1+Az}{1+Bz}. \tag{2.4}$$

As $A, B \in \mathbb{C}$, clearly $H(z)$ is contained in the oblique circle shown in the Figure 1, whose radius and center are given by $R := |A-B|r/(1-|B|^2r^2)$ and $C := 1-A\overline{B}r^2/(1-|B|^2r^2)$, respectively, with angles $\tau(r) := \tan^{-1}(\operatorname{Im}(A\overline{B})r^2/\ (\operatorname{Re}(A\overline{B})r^2-1))$ and $\zeta(r) := \sin^{-1}\left(|A-B|r^2/(|1-A\overline{B}|r^2)\right)$. By taking argument estimate of (2.4) and using the Figure 1 with the fact that circle is symmetric about the line passing through origin and center, we obtain (i). Let $|z|=r<1$ and $t \in [0,2\pi)$, we have $(h(re^{it})-\gamma)/(1-\gamma) \in \Omega := ((1+Az)/(1+Bz))^{\alpha}$, which implies $\partial\Omega : (C+Re^{it})^{\alpha} := M(t)e^{iN(t)}$. To find modulus, real and imaginary parts estimate, we need the critical points. By a simple computation, we obtain $M'(t)=0$ at $t=\tau(1)=\tau$ and $\tau+\pi$, $(M(t)\cos N(t))'=0$ at the roots t_1 and $t_1+\pi$ of the equation (2.2) and $(M(t)\sin N(t))'=0$ at the roots t_2 and $\tau-t_2$ of the equation (2.3), all these values eventually yield (ii), (iii) and (iv), respectively. □

Remark 1. 1) When $\gamma = 0$, we can obtain from Theorem 2.1 various bound estimates for functions in $\mathcal{P}(A,B,\alpha)$.

2) By taking $\alpha = 1/2$, $\gamma = 0$, $A = 1$ and $B = 0$ in Theorem 2.1, we have $h(z) \prec \sqrt{1+z}$. Therefore the estimates are $|\arg h(z)| \leq \sin^{-1} r/2$ and $\sqrt{1-r} \leq |h(z)| \leq \sqrt{1+r}$. If $r=1$, we have $t_1 = 0$ and $t_2 = 2\pi/3$, which implies $0 < \operatorname{Re} h(z) < \sqrt{2}$ and $-0.5 < \operatorname{Im} h(z) < 0.5$.

3) Note that if $w(z) = (1+Az)/(1+Bz)$, then $w(0) = 1 \in w(\mathbb{D})$. Thus the image domain $w(\mathbb{D})$ will always intersect real axis even if it is an oblique domain, non-symmetric with respect to real axis.

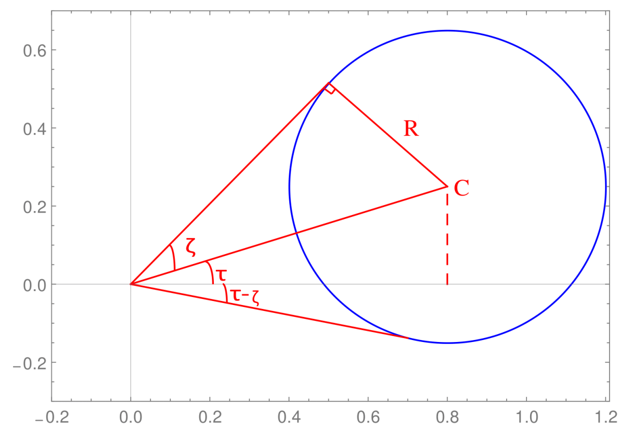

Figure 1. The image of $\partial\mathbb{D}$ under $(1+Az)/(1+Bz)$.

For $\alpha = 1$ and $\gamma = 0$, Theorem 2.1 reduces to the following sharp bounds:

Corollary 2.1.1. *Let $h \in \mathcal{H}_1$ and $A, B \in \mathbb{C}$ with $A \neq B$, $|A| \leq 1$ and $|B| \leq 1$. Further if*

$$h(z) \prec \frac{1+Az}{1+Bz},$$

then for $|z| = r < 1$, we have

(i)
$$\left|\arg h(z) - \tan^{-1}\frac{\operatorname{Im}(A\overline{B})r^2}{\operatorname{Re}(A\overline{B})r^2 - 1}\right| < \sin^{-1}\frac{|A-B|r}{|1 - A\overline{B}r^2|};$$

(ii)
$$\frac{|1-A\overline{B}r^2| - |A-B|r}{1-|B|^2r^2} \leq |h(z)| \leq \frac{|1-A\overline{B}r^2| + |A-B|r}{1-|B|^2r^2};$$

(iii)
$$\frac{1-\operatorname{Re}(A\overline{B})r^2 - |A-B|r}{1-|B|^2r^2} \leq \operatorname{Re} h(z) \leq \frac{1-\operatorname{Re}(A\overline{B})r^2 + |A-B|r}{1-|B|^2r^2};$$

(iv)
$$\frac{\operatorname{Im}(A\overline{B})r^2 + |A-B|r}{|B|^2r^2 - 1} \leq \operatorname{Im} h(z) \leq \frac{\operatorname{Im}(A\overline{B})r^2 - |A-B|r}{|B|^2r^2 - 1}.$$

Note that if $A = ae^{\mathrm{i}m\pi}$ and $B = be^{\mathrm{i}n\pi}$, where $a \geq 0$, $0 \leq b < 1$, $-1 \leq m, n \leq 1$ and $A \neq B$, then $(1+Az)/(1+Bz)$ maps the unit disk onto

$$H(\mathbb{D}) := \left\{w \in \mathbb{C} : \left|w - \frac{1-A\overline{B}}{1-|B|^2}\right| < \frac{|A-B|}{1-|B|^2}\right\}. \tag{2.5}$$

Clearly, $H(\mathbb{D})$ represents a disk when $b < 1$ and a half plane when $b = 1$. We see that $w = 0$ is an exterior or interior or boundary point of $H(\mathbb{D})$ is decided by the value of a as $a < 1$ or $a > 1$ or $a = 1$, respectively. Therefore to investigate argument related problems of a Janowski function, we take $0 \leq a \leq 1$ or $|A-B| \leq |1-A\overline{B}|$ ($|B| < 1$) so that $w = 0$ is not an interior point of $H(\mathbb{D})$.

Following are the radius and center of the disk (2.5):

$$R := \frac{|A-B|}{1-|B|^2} = \frac{\sqrt{a^2+b^2-2ab\cos((n-m)\pi)}}{1-b^2},$$
$$C := \frac{1-A\overline{B}}{1-|B|^2} = \frac{1-ab\cos((n-m)\pi)+\mathrm{i}ab\sin((n-m)\pi)}{1-b^2}.$$

We observe that whenever the difference of m and n is same, then the corresponding R and C also remain same. Therefore, without lose of generality, we can fix $n = 1$. Accordingly, we confine our study of Janowski function by considering

$$\frac{1+Az}{1-bz}, \tag{2.6}$$

where $A+b \neq 0$, $A \in \mathbb{C}$, $0 \le b \le 1$ and the class $\tilde{P}(A,b) = \{p \in \mathcal{H}_1 : p(z) \prec (1+Az)/(1-bz)\}$. The corresponding class of Janowski strongly starlike functions of order α is denoted by $\widetilde{SS}^*(A,b,\alpha) = \{f \in \mathcal{S} : zf'(z)/f(z) \prec ((1+Az)/(1-bz))^\alpha\}$. As a consequence of Theorem 2.1, the following result yields an equivalence relation between half plane Janowski sector whose boundary passes through origin and its argument bounds.

Theorem 2.2. *For $0 < \alpha \le 1$ and $-1 < m < 1$, then the function*

$$h(z) = \left(\frac{1+\mathrm{e}^{\mathrm{i}m\pi}z}{1-z}\right)^\alpha \tag{2.7}$$

is analytic, univalent and convex in $\mathbb{D}$ with

$$h(\mathbb{D}) = \left\{w \in \mathbb{C} : -\alpha(1-m)\frac{\pi}{2} \le \arg w \le \alpha(1+m)\frac{\pi}{2}\right\}. \tag{2.8}$$

Proof. By using Theorem 2.1, the function $h(z)$ given in (2.7) satisfy

$$\left|\arg h(z) - \alpha\tan^{-1}\left(\tan\frac{m\pi}{2}\right)\right| < \frac{\alpha\pi}{2},$$

which gives the desired result. $\square$

Remark 2. 1) From the domain (2.8) we have:

$$(h(\mathbb{D}))^{1/\alpha} = \{w \in \mathbb{C} : \operatorname{Re} \mathrm{e}^{-\mathrm{i}m\pi/2}w > 0\}. \tag{2.9}$$

2) For $0 < \alpha_1 \le 1$ and $0 < \alpha_2 \le 1$, if $m = (\alpha_1-\alpha_2)/(\alpha_1+\alpha_2)$ and $\alpha = (\alpha_1+\alpha_2)/2$ in (2.7), then Theorem 2.2 reduces to [9: Lemma 3].

The following result generalizes Theorem 2.2 with $\alpha = 1$.

Theorem 2.3. *Let $h \in \mathcal{H}_1$ and $0 \le b \le 1$ with $b + \mathrm{e}^{\mathrm{i}m\pi} \neq 0$, where $-1 \le m \le 1$. Also, if*

$$h(z) \prec \frac{1+\mathrm{e}^{\mathrm{i}m\pi}z}{1-bz}, \tag{2.10}$$

then

$$\operatorname{Re} \mathrm{e}^{-\mathrm{i}\lambda}h(z) > 0, \tag{2.11}$$

where $\lambda = \tan^{-1}\left(\dfrac{b\sin(m\pi)}{b\cos(m\pi)+1}\right)$.

P r o o f. To obtain (2.11), it suffices to show that $|\arg(\mathrm{e}^{-\mathrm{i}\lambda}w)| < \pi/2$ or $|\arg w - \lambda| < \pi/2$, where $w = h(z)$. By using Theorem 2.1 with $\alpha = 1$, the function $h(z)$ given in (2.10) satisfies

$$\left|\arg h(z) - \tan^{-1}\frac{b\sin(m\pi)}{b\cos(m\pi)+1}\right| < \sin^{-1}\frac{|\mathrm{e}^{im\pi}+b|}{|1+b\mathrm{e}^{im\pi}|} = \frac{\pi}{2},$$

which leads to the desired result. □

Remark 3. 1) When $b = 1$, then Theorem 2.3 provides sufficient condition for functions to be in the class $\mathcal{P}_{-\lambda}$.

2) Let $|A| \le 1$ and $|B| \le 1$ with $A \ne B$. Assume $a(\alpha) := \arg((1+Az)/(1+Bz))^{\alpha}$. Then from Theorem 2.1, we observe that $\max a(\alpha_1) \le \max a(\alpha_2)$ and $\min a(\alpha_1) \ge \min a(\alpha_2)$, whenever $0 < \alpha_1 \le \alpha_2 \le 1$. Therefore we have

$$\left(\frac{1+Az}{1+Bz}\right)^{\alpha_1} \prec \left(\frac{1+Az}{1+Bz}\right)^{\alpha_2} \quad (0 < \alpha_1 \le \alpha_2 \le 1).$$

Theorem 2.4. *Let $|A_j| \le 1$ and $|B_j| \le 1$ $(j = 1, 2)$ with $A_1 \ne B_1$ and $A_2 \ne B_2$. Let C_j and R_j be the centres and radii of $(1 + A_jz)/(1 + B_jz) = \phi_j(z)$ $(j = 1, 2)$, respectively. Then*

(i) *$\mathcal{P}(A_1, B_1) \subseteq \mathcal{P}(A_2, B_2)$ if and only if the line segment C_1C_2 lies entirely in the domain $\phi_1(\mathbb{D})$, whenever $|C_1 - C_2| \le R_1$.*

(ii) *$\mathcal{P}(A_1, B_1) \subseteq \mathcal{P}(A_2, B_2)$ if and only if the line segment C_1C_2 does not lie entirely in the domain $\phi_1(\mathbb{D})$, whenever $|C_1 - C_2| \ge R_1$.*

The proof is skipped here, as the result is evident from the Figure 2.

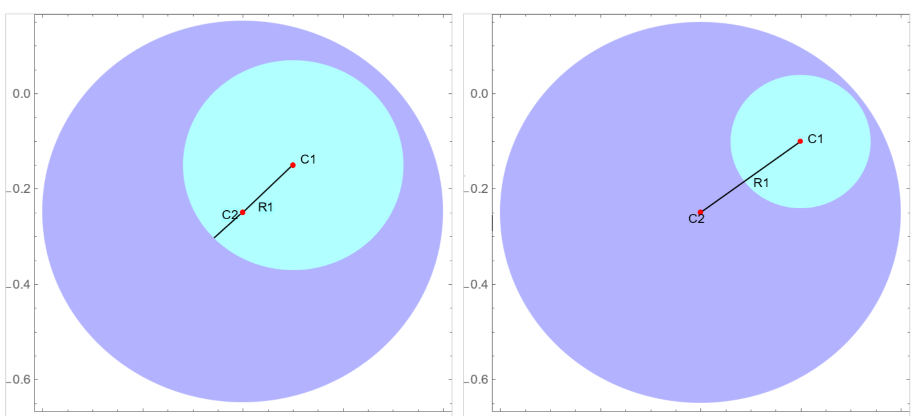

Figure 2. The image of $\mathbb{D}$ under $\phi_i(z)$ $(i = 1, 2)$.

We note that if $\mathcal{P}(A_1, B_1) \subseteq \mathcal{P}(A_2, B_2)$, then $(1 + A_1z)/(1 + B_1z) \prec (1 + A_2z)/(1 + B_2z)$, thus from Theorem 2.4, we obtain the following result:

Corollary 2.4.1. *If $\mathcal{P}(A_1, B_1) \subseteq \mathcal{P}(A_2, B_2)$, then for $0 < \alpha \le 1$, we also have*

$$\left(\frac{1+A_1z}{1+B_1z}\right)^{\alpha} \prec \left(\frac{1+A_2z}{1+B_2z}\right)^{\alpha}.$$

Note that the observations made in [19] are generalized in part (2) of Remark 3 and Corollary 2.4.1.

3. Argument related results

In this section, we obtain certain subordination results using Theorem 2.2 and the following versions of the Jack's Lemma:

Lemma 3.1 ([11]). *Let $h \in \mathcal{H}[1,n]$. If there exists a point $z_0 \in \mathbb{D}$ such that*

$$|\arg p(z)| < |\arg p(z_0)| = \frac{\pi\beta}{2} \quad (|z| < |z_0|)$$

for some $\beta > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2\mathrm{i}k \arg p(z_0)}{\pi}$$

for some $k \geq n(a + a^{-1})/2 > n$, where $p(z_0)^{1/\beta} = \pm \mathrm{i}a$, and $a > 0$.

Lemma 3.2 ([13]). *Let $h(z)$ be analytic in $\mathbb{D}$ with $h(0) = 1$ and $h(z) \neq 0$. If there exist two points $z_1, z_2 \in \mathbb{D}$ such that*

$$-\frac{\alpha_1\pi}{2} = \arg h(z_1) < \arg h(z) < \arg h(z_2) = \frac{\alpha_2\pi}{2}$$

for $\alpha_1, \alpha_2 \in (0,2]$ and $|z| < |z_1| = |z_2|$, then we have

$$\frac{z_1 h'(z_1)}{h(z_1)} = -\mathrm{i}\frac{\alpha_1 + \alpha_2}{2}k \quad \textit{and} \quad \frac{z_2 h'(z_2)}{h(z_2)} = \mathrm{i}\frac{\alpha_1 + \alpha_2}{2}k,$$

where

$$k \geq \frac{1 - |a|}{1 + |a|} \quad \textit{and} \quad a = \mathrm{i}\tan\frac{\pi}{4}\left(\frac{\alpha_2 - \alpha_1}{\alpha_2 + \alpha_1}\right).$$

In [18], authors have considered Janowski function with complex parameters and therefore the corresponding Janowski disk is non-symmetric with respect to real axis. However, their findings were based on the Janowski disk that is symmetric with respect to real axis, which is possible only if the parameters are real in Janowski function. Therefore by eliminating this limitation, we obtain the following result as an extension of [18: Lemma 2.17].

Theorem 3.1. *Let $\alpha \in (0,1]$, $l, m \in [-1,1]$ and $a, b, c, d \in [0,1]$ such that $a\mathrm{e}^{\mathrm{i}l\pi} + b \neq 0$ and $c\mathrm{e}^{\mathrm{i}m\pi} + d \neq 0$. Let $Q \in \mathcal{H}[1,n]$ satisfy*

$$Q(z) \prec \frac{1 + a\mathrm{e}^{\mathrm{i}l\pi}z}{1 - bz} \tag{3.1}$$

and

$$Q(z)p^{\alpha}(z) \prec \frac{1 + c\mathrm{e}^{\mathrm{i}m\pi}z}{1 - dz} \tag{3.2}$$

for $p \in \mathcal{H}_1$. If

$$\mu := \sin^{-1}\sqrt{\frac{c^2 + d^2 + 2cd\cos(m\pi)}{1 + c^2d^2 + 2cd\cos(m\pi)}} + \sin^{-1}\sqrt{\frac{a^2 + b^2 + 2ab\cos(l\pi)}{1 + a^2b^2 + 2ab\cos(l\pi)}} \leq \frac{\alpha\pi}{2}, \tag{3.3}$$

then

$$\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\gamma\pi/2}p(z)) > 0,$$

where $\gamma = (\tan^{-1} A - \tan^{-1} B)/\mu$ with $A = \dfrac{cd\sin(m\pi)}{cd\cos(m\pi) + 1}$ and $B = \dfrac{ab\sin(l\pi)}{ab\cos(l\pi) + 1}$.

P r o o f. From Theorem 2.1, (3.1) yields

$$\left|\arg Q(z) - \tan^{-1}\left(\frac{ab\sin(l\pi)}{ab\cos(l\pi)+1}\right)\right| < \sin^{-1}\sqrt{\frac{a^2+b^2+2ab\cos(l\pi)}{1+a^2b^2+2ab\cos(l\pi)}}. \tag{3.4}$$

Similarly, from (3.2), we obtain

$$\left|\arg Q(z) + \alpha\arg p(z) - \tan^{-1}\left(\frac{cd\sin(m\pi)}{cd\cos(m\pi)+1}\right)\right| < \sin^{-1}\sqrt{\frac{c^2+d^2+2cd\cos(m\pi)}{1+c^2d^2+2cd\cos(m\pi)}}. \tag{3.5}$$

After some computations using (3.4) and (3.5), we obtain

$$-\frac{\pi}{2}\left(\frac{2}{\alpha\pi}(\mu-\gamma\mu)\right) \le \arg p(z) \le \frac{\pi}{2}\left(\frac{2}{\alpha\pi}(\mu+\gamma\mu)\right) \tag{3.6}$$

which eventually yields

$$p(z) \prec \frac{1+\mathrm{e}^{\mathrm{i}\gamma\pi}z}{1-z},$$

that completes the proof. □

The next result of this section produces various corollaries and also generalizes Pommerenke's result [17]: Let $f \in \mathcal{A}$, $g \in \mathcal{C}$ and $0 < \alpha \le 1$, then

$$\left|\arg\left(\frac{f'(z)}{g'(z)}\right)\right| < \frac{\alpha\pi}{2} \quad\Longrightarrow\quad \left|\arg\left(\frac{f(z)}{g(z)}\right)\right| < \frac{\beta(\alpha)\pi}{2}.$$

Theorem 3.2. *Let $f, g \in \mathcal{A}$ and $0 < \alpha \le 1$. For some $m \in [-1, 1)$ and $\beta \in (0, 1)$, let $a = \mathrm{i}\tan\frac{m\pi}{4}$ and $|g(z)/(zg'(z))| > \beta$. If*

$$\frac{f'(z)}{g'(z)} \prec \left(\frac{1+\mathrm{e}^{\mathrm{i}\mu\pi}z}{1-z}\right)^{\alpha(\mu_1+\mu_2)/2}, \tag{3.7}$$

then

$$\frac{f(z)}{g(z)} \prec \left(\frac{1+\mathrm{e}^{\mathrm{i}m\pi}z}{1-z}\right)^{\alpha}, \tag{3.8}$$

where

$$\mu_j = 1 + (-1)^j m + \frac{2}{\alpha\pi}\tan^{-1}\frac{\alpha\beta(1-|a|)\cos\left(\arg\frac{g(z)}{zg'(z)}\right)}{1+|a|+(-1)^{j+1}\alpha\beta(1-|a|)\sin\left(\arg\frac{g(z)}{zg'(z)}\right)} \;\; (j=1,2) \text{ and } \mu = \frac{\mu_2-\mu_1}{\mu_1+\mu_2}.$$

P r o o f. Let $p(z) := f(z)/g(z)$. Then in view of Theorem 2.2, to prove (3.8), it is sufficient to show $-\alpha(1-m)\pi/2 \le \arg p(z) \le \alpha(1+m)\pi/2$. On the contrary, if there exist two points $z_1, z_2 \in \mathbb{D}$ such that

$$-\alpha(1-m)\frac{\pi}{2} = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \alpha(1+m)\frac{\pi}{2}$$

for $|z| < |z_1| = |z_2|$, then by Lemma 3.2, we have

$$\frac{z_1p'(z_1)}{p(z_1)} = -\mathrm{i}\alpha k \quad\text{and}\quad \frac{z_2p'(z_2)}{p(z_2)} = \mathrm{i}\alpha k.$$

Since $p(z) = f(z)/g(z)$, thus we have

$$\frac{f'(z)}{g'(z)} = p(z)\left(1 + \frac{zp'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right)$$

and

$$\arg\left(\frac{f'(z)}{g'(z)}\right) = \arg p(z) + \arg\left(1 + \frac{zp'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right).$$

For $z = z_1$, we have

$$\arg\left(\frac{f'(z_1)}{g'(z_1)}\right) \leq -\alpha(1-m)\frac{\pi}{2} + \arg\left(1 - \mathrm{i}\alpha k\beta\left(\cos\left(\arg\frac{g(z_1)}{z_1 g'(z_1)}\right) + \mathrm{i}\sin\left(\arg\frac{g(z_1)}{z_1 g'(z_1)}\right)\right)\right)$$

$$\leq -\alpha(1-m)\frac{\pi}{2} + \tan^{-1}\left(\frac{\alpha\beta(|a|-1)\cos\left(\arg\frac{g(z_1)}{z_1 g'(z_1)}\right)}{1+|a|+\alpha\beta(1-|a|)\sin\left(\arg\frac{g(z_1)}{z_1 g'(z_1)}\right)}\right),$$

which contradicts (3.7). Similarly, for $z = z_2$, we have

$$\arg\left(\frac{f'(z_2)}{g'(z_2)}\right) \geq \alpha(1+m)\frac{\pi}{2} + \arg\left(1 + \mathrm{i}\alpha m\beta\left(\cos\left(\arg\frac{g(z_2)}{z_2 g'(z_2)}\right) + \mathrm{i}\sin\left(\arg\frac{g(z_2)}{z_2 g'(z_2)}\right)\right)\right)$$

$$\geq \alpha(1+m)\frac{\pi}{2} + \tan^{-1}\left(\frac{\alpha\beta(1-|a|)\cos\left(\arg\frac{g(z_2)}{z_2 g'(z_2)}\right)}{1+|a|-\alpha\beta(1-|a|)\sin\left(\arg\frac{g(z_2)}{z_2 g'(z_2)}\right)}\right),$$

which again contradicts (3.7), that completes the proof. □

If we choose $g(z) = z$ in Theorem 3.2, it reduces to the following corollary:

Corollary 3.2.1. *Let $f \in \mathcal{A}$ and $0 < \alpha \leq 1$. For some $m \in [-1, 1)$, let $a = \mathrm{i}\tan\frac{m\pi}{4}$. If*

$$f'(z) \prec \left(\frac{1+\mathrm{e}^{\mathrm{i}\mu\pi}z}{1-z}\right)^{\alpha(\mu_1+\mu_2)/2},$$

then

$$\frac{f(z)}{z} \prec \left(\frac{1+\mathrm{e}^{\mathrm{i}m\pi}z}{1-z}\right)^{\alpha}, \tag{3.9}$$

where $\mu_j = 1 + (-1)^j m + \dfrac{2}{\alpha\pi}\tan^{-1}\dfrac{\alpha(1-|a|)}{1+|a|}$ $(j = 1, 2)$ *and* $\mu = \dfrac{\mu_2-\mu_1}{\mu_1+\mu_2}$.

Further, $f \in \mathcal{SS}^*(\beta)$, *where* $\beta = \alpha\pi + \tan^{-1}\dfrac{\alpha(1-|a|)}{1+|a|}$.

P r o o f. The subordination (3.9) holds, by using Theorem 3.2. Now just to prove $f \in \mathcal{SS}^*(\beta)$. Since

$$\arg\frac{zf'(z)}{f(z)} = \arg\frac{z}{f(z)} + \arg f'(z),$$

using Theorem 2.2, we obtain

$$-\alpha(1+m+\mu_1)\frac{\pi}{2} < \arg\frac{zf'(z)}{f(z)} < \alpha(1-m+\mu_2)\frac{\pi}{2},$$

$$-\alpha\pi - \tan^{-1}\frac{\alpha(1-|a|)}{1+|a|} < \arg\frac{zf'(z)}{f(z)} < \alpha\pi + \tan^{-1}\frac{\alpha(1-|a|)}{1+|a|},$$

which implies $f \in \mathcal{SS}^*(\beta)$. □

Remark 4. If $m = 0$, then Corollary 3.2.1 reduces to [14: Theorem 1.7].

The next corollary is a generalization of [12: Theorem 1].

Corollary 3.2.2. *Let $f \in \mathcal{A}$, $g \in \mathcal{C}$ and $g \in \mathcal{RS}^*(\beta)$, where $0 \leq \beta < 1$. Suppose $0 < \alpha \leq 1$ and for some $m \in [-1, 1)$, let $a = \mathrm{i}\tan\frac{m\pi}{4}$. If*

$$\frac{f'(z)}{g'(z)} \prec \left(\frac{1+\mathrm{e}^{\mathrm{i}\mu\pi}z}{1-z}\right)^{(\alpha(\mu_1+\mu_2))/2},$$

then

$$\frac{f(z)}{g(z)} \prec \left(\frac{1+\mathrm{e}^{\mathrm{i}m\pi}z}{1-z}\right)^{\alpha},$$

where $\mu_j = 1 - m + \dfrac{2}{\alpha\pi}\tan^{-1}\dfrac{\alpha\beta(1-|a|)}{1+|a|+\alpha(1-|a|)}$ $(j=1,2)$ *and* $\mu = \dfrac{\mu_2-\mu_1}{\mu_1+\mu_2}$.

P r o o f. Since $g \in \mathcal{C}$, from Marx-Strohhäcker's theorem, we have $\mathrm{Re}(zg'(z)/g(z)) > 1/2$, which implies that $|g(z)/(zg'(z)) - 1| < 1$. Thus we obtain

$$\left|\mathrm{Im}\left(\frac{g(z)}{zg'(z)}\right)\right| < 1 \quad \text{and} \quad \mathrm{Re}\left(\frac{g(z)}{zg'(z)}\right) > \beta. \tag{3.10}$$

Now using (3.10) and the methodology of Theorem 3.2, the result follows at once. □

Remark 5. When we take $m = 0$ in Corollary 3.2.2, then the result reduces to [12: Theorem 1].

Next result also have a wide range of applications and the result generalizes [21: Theorem 1].

Theorem 3.3. *Let* $p \in \mathcal{H}_1$ *and* $m \in [-1,1)$. *Also, for a fixed* $\gamma \in [0,1]$ *and* $\alpha > 0$, *let* $\beta > \beta_0 (\geq 0)$, *where* β_0 *is the solution of the equation*

$$\alpha\beta(1-m) + \frac{2\gamma}{\pi}\tan^{-1}\eta = 0, \tag{3.11}$$

and for a suitable fixed $\eta \geq 0$, *let* $\lambda(z): \mathbb{D} \to \mathbb{C}$ *be a function satisfying*

$$\frac{\beta\,\mathrm{Re}\,\lambda(z)}{1+\beta|\mathrm{Im}\,\lambda(z)|} \geq \eta. \tag{3.12}$$

If

$$(p(z))^{\alpha}\left(1+\lambda(z)\frac{zp'(z)}{p(z)}\right)^{\gamma} \prec \left(\frac{1+\mathrm{e}^{\mathrm{i}\mu\pi}z}{1-z}\right)^{\delta}, \tag{3.13}$$

then

$$p(z) \prec \left(\frac{1+\mathrm{e}^{\mathrm{i}m\pi}z}{1-z}\right)^{\beta}, \tag{3.14}$$

where $\mu_j = \alpha\beta(1+(-1)^j m) + \dfrac{2\gamma}{\pi}\tan^{-1}\eta$ $(j=1,2)$, $\delta = \dfrac{\mu_1+\mu_2}{2}$ *and* $\mu = \dfrac{\mu_2-\mu_1}{\mu_1+\mu_2}$.

P r o o f. According to Theorem 2.2, to prove (3.14), it is sufficient to show that $-\beta(1-m)\frac{\pi}{2} < \arg p(z) < \beta(1+m)\frac{\pi}{2}$. On the contrary if there exists two points $z_1, z_2 \in \mathbb{D}$ such that

$$-\beta(1-m)\frac{\pi}{2} = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \beta(1+m)\frac{\pi}{2}$$

for $|z| < |z_1| = |z_2|$, then from Lemma 3.2, we have

$$\frac{z_1p'(z_1)}{p(z_1)} = -\mathrm{i}k\beta \quad \text{and} \quad \frac{z_2p'(z_2)}{p(z_2)} = \mathrm{i}k\beta,$$

where $k \geq \frac{1-|a|}{1+|a|}$ and $a = \mathrm{i}\tan\frac{m\pi}{4}$. For $z = z_1$, using (3.12) we have

$$\begin{aligned}\arg\left((p(z_1))^{\alpha}\left(1+\lambda(z_1)\frac{z_1p'(z_1)}{p(z_1)}\right)^{\gamma}\right) &\leq -\alpha\beta(1-m)\frac{\pi}{2} - \gamma\tan^{-1}\frac{\beta m\,\mathrm{Re}\,\lambda(z_1)}{1+\beta m|\mathrm{Im}\,\lambda(z_1)|}\\ &\leq -\left(\alpha\beta(1-m)\frac{\pi}{2} + \gamma\tan^{-1}\eta\right),\end{aligned}$$

which contradicts (3.13). Similarly, for $z = z_2$, we have

$$\begin{aligned}\arg\left((p(z_2))^\alpha\left(1+\lambda(z_2)\frac{z_2p'(z_2)}{p(z_2)}\right)^\gamma\right) &\geq \alpha\beta(1+m)\frac{\pi}{2}+k\tan^{-1}\frac{\beta\gamma\operatorname{Re}\lambda(z_2)}{1+\beta\gamma|\operatorname{Im}\lambda(z_2)|}\\ &\geq \alpha\beta(1+m)\frac{\pi}{2}+\gamma\tan^{-1}\eta,\end{aligned}$$

which also contradicts (3.13) and this completes the proof of the theorem. $\square$

Corollary 3.3.1. *Let $f \in \mathcal{A}$, $g \in \mathcal{A}$ and $\lambda(z) = g(z)/(zg'(z))$, which satisfies* (3.12). *Also, for a fixed $\gamma \in [0,1]$, $m \in [-1,1)$, $\alpha > 0$ and β be as defined in* (3.11). *If*

$$\left(\frac{f(z)}{g(z)}\right)^{\alpha-\gamma}\left(\frac{f'(z)}{g'(z)}\right)^\gamma \prec \left(\frac{1+\mathrm{e}^{\mathrm{i}\mu\pi}z}{1-z}\right)^\delta,$$

then

$$\frac{f(z)}{g(z)} \prec \left(\frac{1+\mathrm{e}^{im\pi}z}{1-z}\right)^\beta,$$

where μ and δ are same as defined in Theorem 3.3.

Proof. Let $p(z) = f(z)/g(z)$. By simple computation we have

$$\left(\frac{f(z)}{g(z)}\right)^{\alpha-\gamma}\left(\frac{f'(z)}{g'(z)}\right)^\gamma = (p(z))^\alpha\left(1+\frac{g(z)}{zg'(z)}\frac{zp'(z)}{p(z)}\right)^\gamma.$$

Hence $p \in \mathcal{H}_1$, thus result follows from Theorem 3.3. $\square$

Corollary 3.3.2. *Let $f \in \mathcal{A}$, $g \in \mathcal{C}$ and $g \in \mathcal{RS}^*(\zeta)$, where $0 \leq \zeta < 1$. Also, for a fixed $\gamma \in [0,1]$, $m \in [-1,1)$ and $\alpha > 0$. If*

$$\left(\frac{f(z)}{g(z)}\right)^{\alpha-\gamma}\left(\frac{f'(z)}{g'(z)}\right)^\gamma \prec \left(\frac{1+\mathrm{e}^{\mathrm{i}\mu\pi}z}{1-z}\right)^\delta,$$

then

$$\frac{f(z)}{g(z)} \prec \left(\frac{1+\mathrm{e}^{im\pi}z}{1-z}\right)^\beta,$$

where μ and δ are same as defined in Theorem 3.3 *and $\beta > \beta_0(\geq 0)$ is the solution of the equation*

$$\alpha\beta(1-m)+\frac{2\gamma}{\pi}\tan^{-1}\frac{\beta\zeta}{1+\beta} = 0.$$

Proof. Since $g \in \mathcal{C}$, from Marx-Strohhäcker's theorem, we have $\operatorname{Re}(zg'(z)/g(z)) > 1/2$, which implies that $|g(z)/(zg'(z))-1| < 1$. Thus we obtain

$$|\operatorname{Im}\lambda(z)| = \left|\operatorname{Im}\left(\frac{g(z)}{zg'(z)}\right)\right| < 1 \quad \text{and} \quad \operatorname{Re}\lambda(z) = \operatorname{Re}\left(\frac{g(z)}{zg'(z)}\right) > \beta,$$

which satisfies (3.12) with $\eta = \beta\zeta/(1+\beta)$. Now letting $p(z) = f(z)/g(z)$, the result follows at once, by following the proof of Corollary 3.3.1. $\square$

4. Subordination results

In this section we discuss certain differential subordination implications to obtain sufficient conditions for functions to be in $\widetilde{\mathcal{SS}}^*(A,b,\gamma)$. The book [10] provides numerous results pertaining to differential subordination and motivate authors (see [2,7]) to establish various generalised results on differential subordination. Here are some of the results which we need in context of our study.

Lemma 4.1 ([10: Theorem 3.1d, p. 76]). *Let h be analytic and starlike univalent in $\mathbb{D}$ with $h(0)=0$. If g is analytic in $\mathbb{D}$ and $zg'(z) \prec h(z)$, then*

$$g(z) \prec g(0) + \int_0^z \frac{h(t)}{t}\mathrm{d}t.$$

Lemma 4.2 ([10: Theorem 3.4h, p. 132]). *Let $g(z)$ be univalent in $\mathbb{D}$, Φ and Θ be analytic in a domain Ω containing $g(\mathbb{D})$ such that $\Phi(w) \neq 0$, when $w \in g(\mathbb{D})$. Now letting $G(z) = zg'(z)\cdot\Phi(g(z))$, $h(z) = \Theta(g(z)) + G(z)$ and either h or g is convex. Further, if*

$$\operatorname{Re}\frac{zh'(z)}{G(z)} = \operatorname{Re}\left(\frac{\Theta'(g(z))}{\Phi(g(z))} + \frac{zG'(z)}{G(z)}\right) > 0$$

as well as p is analytic in $\mathbb{D}$, with $p(0)=g(0)$, $p(\mathbb{D}) \subset \Omega$ and

$$\Theta(p(z)) + zp'(z)\cdot\Phi(p(z)) \prec \Theta(g(z)) + zg'(z)\cdot\Phi(g(z)) := h(z),$$

then $p \prec g$, and g is the best dominant.

The following result gives us the sufficient condition for a function $p(z)\in\mathcal{H}_1$ to be in $\widetilde{\mathcal{SS}}^*(A,b,\gamma)$.

Theorem 4.1. *Let $p(z) \in \mathcal{H}_1$, $A \in \mathbb{C}$ and $0 \le b \le 1$ with $|A| \le 1$, $A+b \neq 0$ and $\operatorname{Re}(1+Ab) \ge |A+b|$. Further let α, γ are two real parameters lying in $[0,1]$ and μ, δ, ρ and η are complex parameters such that $\operatorname{Re}(\mu/\eta) > 0$, $\operatorname{Re}\delta > 0$ and $\operatorname{Re}\rho \ge 0$. If*

$$\mu(p(z))^{\alpha}(\delta + \rho p(z)) + \eta z p'(z)(p(z))^{\alpha-1} \prec h(z),$$

then $p \in \widetilde{\mathcal{SS}}^*(A,b,\gamma)$, where

$$h(z) = \left(\frac{1+Az}{1-bz}\right)^{\alpha\gamma}\left(\mu\delta + \mu\rho\left(\frac{1+Az}{1-bz}\right)^{\gamma} + \eta\gamma\frac{(A+b)z}{(1+Az)(1-bz)}\right).$$

P r o o f. Let us choose $g(z) = ((1+Az)/(1-bz))^{\gamma}$, $\Phi(w) = \eta w^{\alpha-1}$ and $\Theta(w) = \mu w^{\alpha}(\delta+\rho w)$, then clearly $g \in \mathcal{P}$, univalent and convex in $\mathbb{D}$. Φ, Θ are analytic in a domain Ω containing $g(\mathbb{D})$, with $\Phi(w) \neq 0$ when $w \in g(\mathbb{D})$. If $G(z) = zg'(z)\Phi(g(z))$, then

$$\operatorname{Re}\frac{zG'(z)}{G(z)} = \operatorname{Re}\left(\frac{\alpha+1}{1-bz} - \frac{\alpha-1}{1+Az} - 1\right) \ge \frac{\alpha+1}{1+b} - \frac{\alpha-1}{1+|A|} - 1 \ge 0$$

and

$$\operatorname{Re}\frac{zh'(z)}{G(z)} = \operatorname{Re}\left(\frac{\mu\alpha\delta}{\eta} + \frac{\mu(\alpha+1)\rho g(z)}{\eta} + \frac{zG'(z)}{G(z)}\right) > 0.$$

Thus by Lemma 4.2, we obtain that $p \prec g$ and g is the best dominant. □

Above result have a wide range of applications such as by taking $\mu = 1, \delta = 1-\lambda, \rho = \lambda, \eta = \lambda$ and $p(z) = zf'(z)/f(z)$ in Theorem 4.1, where $\lambda \in [0,1]$, we obtain the following more modified and simplified form of results in [5].

Corollary 4.1.1. *Let $f(z) \in \mathcal{A}$ such that $f(z)/z \neq 0$ in $\mathbb{D}$, $A \in \mathbb{C}$ and $0 \leq b \leq 1$ with $|A| \leq 1$, $A+b \neq 0$ and $\operatorname{Re}(1+Ab) \geq |A+b|$. Suppose also that the real parameters λ, α and γ are such that they lies in $[0,1]$. If*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\lambda\frac{zf''(z)}{f'(z)}\right) \prec h(z),$$

then $f \in \widetilde{\mathcal{SS}}^{*}(A,b,\gamma)$, where

$$h(z) = \left(\frac{1+Az}{1-bz}\right)^{\alpha\gamma-1}\left((1-\lambda)\frac{1+Az}{1-bz} + \frac{\lambda(1+Az)^{1+\gamma}(1-bz)^{1-\gamma}+\lambda\gamma(A+b)z}{(1-bz)^2}\right).$$

By taking $\delta = 1$ and $\rho = 0$ in Theorem 4.1, we obtain the following result.

Corollary 4.1.2. *Let $p(z) \in \mathcal{H}_1$ and $0 \leq b \leq 1$ with $b + \mathrm{e}^{\mathrm{i}m\pi} \neq 0$, where $-1 \leq m \leq 1$. Also let $\alpha > -1$ and if*

$$\mu(p(z))^{\alpha} + \eta zp'(z)(p(z))^{\alpha-1} \prec \left(\frac{1+\mathrm{e}^{\mathrm{i}m\pi}z}{1-bz}\right)^{\alpha\gamma}\left(\mu + \eta\gamma\frac{(b+\mathrm{e}^{\mathrm{i}m\pi})z}{(1+\mathrm{e}^{\mathrm{i}m\pi}z)(1-bz)}\right) := h(z)$$

for some $\mu, \eta \in \mathbb{C}$ such that $\operatorname{Re}(\mu/\eta) \geq 0$, then

$$\operatorname{Re}\mathrm{e}^{-\mathrm{i}\beta}(p(z))^{1/\gamma} > 0,$$

where $\beta = \tan^{-1}\dfrac{b\sin(m\pi)}{b\cos(m\pi)+1}$. And the inequality is sharp for the function $p(z)$ defined by

$$p(z) = \left(\frac{1+\mathrm{e}^{\mathrm{i}m\pi}z}{1-bz}\right)^{\gamma}.$$

Proof. By Theorem 4.1 we have $p(z) \prec ((1+\mathrm{e}^{\mathrm{i}m\pi}z)/(1-bz))^{\gamma} := g(z)$ and g is the best dominant. Further, by Theorem 2.3 we obtain the desired conclusion. □

By taking $\mu = 1-\lambda$, $\alpha = 1$ and $\eta = \lambda$ in Corollary 4.1.2, we obtain the following result.

Corollary 4.1.3. *Let $p(z) \in \mathcal{H}_1$ and $0 \leq b \leq 1$ with $b + \mathrm{e}^{\mathrm{i}m\pi} \neq 0$, where $-1 \leq m \leq 1$. Now if*

$$(1-\lambda)p(z) + \lambda zp'(z) \prec \left(\frac{1+\mathrm{e}^{\mathrm{i}m\pi}z}{1-bz}\right)^{\gamma}\left(1-\lambda+\lambda\gamma\frac{(b+\mathrm{e}^{\mathrm{i}m\pi})z}{(1+\mathrm{e}^{\mathrm{i}m\pi}z)(1-bz)}\right) := h(z)$$

for some $0 < \lambda \leq 1$ and $0 < \gamma \leq 1$, then

$$\operatorname{Re}\mathrm{e}^{-\mathrm{i}\beta}(p(z))^{1/\gamma} > 0,$$

where $\beta = \tan^{-1}\dfrac{b\sin(m\pi)}{b\cos(m\pi)+1}$. Further the inequality is sharp for the function $p(z)$ given by

$$p(z) = \left(\frac{1+\mathrm{e}^{\mathrm{i}m\pi}z}{1-bz}\right)^{\gamma}.$$

Remark 6. When $m = 0$ and $b = 1$, then Corollary 4.1.3 reduces to the [16: Theorem 1].

In case when $m = 0$, $\gamma = 1$ and $\lambda = 0.5$, Corollary 4.1.3 yields:

Corollary 4.1.4. *Let $p(z) \in \mathcal{H}_1$ and $0 \leq b \leq 1$. Suppose*

$$p(z) + zp'(z) \prec \frac{1+2z-bz^2}{(1-bz)^2},$$

then

$$p(z) \prec \frac{1+z}{1-bz}.$$

Now the following theorem gives us the sufficient condition for a function $f(z) \in \mathcal{G}_\beta$ to be in $\widetilde{\mathcal{SS}}^*(A,b,\alpha)$, where $\mathcal{G}_\beta$ is the Silverman class first investigated in [20]. Silverman considered the following class for $\beta \in (0,1]$,

$$\mathcal{G}_\beta = \left\{ f \in \mathcal{A} : \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \prec \beta z \right\}.$$

Theorem 4.2. *Let $|A| \leq 1$ and $0 \leq b \leq 1$ with $A + b \neq 0$ and $\beta(1+b)^{\alpha-1}(1+|A|)^{\alpha+1} \leq \alpha|A+b|$. If $f \in \mathcal{G}_\beta$, then $f \in \widetilde{\mathcal{SS}}^*(A,b,\alpha)$.*

P r o o f. For $f \in \mathcal{G}_\beta$, we define the function $p(z) = zf'(z)/f(z)$. By standard calculations we obtain that

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 = \frac{zp'(z)}{p^2(z)}.$$

For $f \in \widetilde{\mathcal{SS}}^*(A,b,\alpha)$, it is enough to show $p(z) \prec ((1+Az)/(1-bz))^\alpha := q(z)$. For if, there exist points $z_0 \in \mathbb{D}$, $\varsigma_0 \in \partial\mathbb{D} \setminus \{-1/A, 1/b\}$ and $m \geq 1$ such that $p(|z| < |z_0|) \subset q(\mathbb{D})$ with $p(z_0) = q(\varsigma_0)$ and $z_0 p'(z_0) = m\varsigma_0 q'(\varsigma_0)$. Since

$$\frac{\alpha|A+b|}{(1+|A|)^{\alpha+1}(1+b)^{\alpha-1}} \geq \beta \implies \left| \frac{\alpha\varsigma_0(A+b)}{(1+A\varsigma_0)^{\alpha+1}(1-b\varsigma_0)^{\alpha-1}} \right| \geq \beta.$$

Thus,

$$\left| \frac{m\varsigma_0 q'(\varsigma_0)}{q^2(\varsigma_0)} \right| \geq \beta \quad \text{for all } m \geq 1,$$

or equivalently,

$$\left| \frac{z_0 p'(z_0)}{p^2(z_0)} \right| \geq \beta,$$

which contradicts $f \in \mathcal{G}_\beta$. Thus $p(z) \prec q(z)$ and that completes proof. □

Remark 7. If $\alpha = 1$ and $-1 \leq B < A \leq 1$ in Theorem 4.2, then the result reduces to [22: Theorem 3.2, p. 9].

Theorem 4.3. *If $f \in \mathcal{G}_\beta$, then f is starlike of reciprocal order α.*

P r o o f. For $f \in \mathcal{G}_\beta$, we define the function $p(z)$ by $f(z)/(zf'(z)) = \alpha + (1-\alpha)p(z)$. Clearly, $p \in \mathcal{H}_1$ and $\alpha + (1-\alpha)p(z) \neq 0$ for $z \in \mathbb{D}$. Now by a simple computation we obtain that

$$1 - \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} = (1-\alpha)zp'(z).$$

Since $f \in \mathcal{G}_\beta$, therefore we have

$$(1-\alpha)zp'(z) \prec \beta z. \tag{4.1}$$

Now using Lemma 4.1, (4.1) leads to

$$p(z) \prec 1 + \frac{\beta}{1-\alpha} z,$$

which shows that

$$|p(z) - 1| < \frac{\beta}{1-\alpha}.$$

Now using Theorem 2.1, we obtain that

$$\left| \operatorname{Arg}\left(\frac{f(z)}{zf'(z)} - \alpha \right) \right| = |\operatorname{Arg} p(z)| < \sin^{-1} \frac{\beta}{1-\alpha} = \frac{\delta\pi}{2}.$$

Since $\alpha \in [0,1)$, and $\beta \in (0,1]$. Therefore $\delta \in (0,1]$, which completes our result. □

5. Radius results

This section aims to find the largest radius R of a property $\mathfrak{P}$ such that every function of a set $\mathcal{M}$ has the property $\mathfrak{P}$ in the disk $\mathbb{D}_r$, where $r \le R$. First result of this section generalizes [1: Theorems 2.1 and 2.2].

Theorem 5.1. *Let $\gamma, \delta \in \mathbb{C} \setminus \{1\}$ and $A, B, C, D \in \mathbb{C}$ with $A \neq B$, $|B| \le 1$, $D \neq C$ and $|D| \le 1$. If*

$$f(z) \prec (1-\delta)\left(\frac{1+Cz}{1+Dz}\right)^{\beta} + \delta,$$

then

$$f(z) \prec (1-\gamma)\left(\frac{1+Az}{1+Bz}\right)^{\alpha} + \gamma,$$

in $|z| \le R$, where

$$R = \min\left\{ \frac{\alpha|(A-B)(1-\gamma)^{\frac{1}{\alpha}}|}{|(C-D)\beta(1-\gamma)^{\frac{1}{\alpha}-1}(\delta-1)| + \beta|(1-\gamma)^{\frac{1}{\alpha}-1}(AD(\gamma-1)+B(C(1-\delta)+D(\delta-\gamma)))|}, 1 \right\}.$$

P r o o f. Let P and Q be functions defined as

$$P(z) = (1-\gamma)\left(\frac{1+Az}{1+Bz}\right)^{\alpha} + \gamma \quad \text{and} \quad Q(z) = (1-\delta)\left(\frac{1+Cz}{1+Dz}\right)^{\beta} + \delta.$$

Further, define the function M as

$$M(z) = P^{-1}(Q(z)) = \frac{\left((1-\delta)(1+Cz)^{\beta} + (\delta-\gamma)(1+Dz)^{\beta}\right)^{\frac{1}{\alpha}} - (1+Dz)^{\frac{\beta}{\alpha}}(1-\gamma)^{\frac{1}{\alpha}}}{A(1+Dz)^{\frac{\beta}{\alpha}}(1-\gamma)^{\frac{1}{\alpha}} - B\left((1-\delta)(1+Cz)^{\beta} + (\delta-\gamma)(1+Dz)^{\beta}\right)^{\frac{1}{\alpha}}}.$$

Replacing z by Rz so that $|z| = R \le 1$, we obtain

$$|M(Rz)| \le \frac{|(C-D)\beta(1-\gamma)^{\frac{1}{\alpha}-1}(\delta-1)|R}{\alpha|(A-B)(1-\gamma)^{\frac{1}{\alpha}}-\beta|(1-\gamma)^{\frac{1}{\alpha}-1}(AD(\gamma-1)+B(C-C\delta+D(\delta-\gamma)))|R} \le 1$$

for

$$R \le \frac{\alpha|(A-B)(1-\gamma)^{\frac{1}{\alpha}}|}{|(C-D)\beta(1-\gamma)^{\frac{1}{\alpha}-1}(\delta-1)| + \beta|(1-\gamma)^{\frac{1}{\alpha}-1}(AD(\gamma-1)+B(C(1-\delta)+D(\delta-\gamma)))|}.$$

Thus $f(z) \prec Q(z)$ for $|z| \le \min\{R, 1\}$ and this completes the proof. $\square$

Taking $\gamma = \delta = 0$ in the Theorem 5.1, we obtain the following corollary.

Corollary 5.1.1. *Let $A, B, C, D \in \mathbb{C}$ with $A \neq B$, $|B| \le 1$, $D \neq C$ and $|D| \le 1$. Then $\mathcal{S}^*(C, D, \beta) \subseteq \mathcal{S}^*(A, B, \alpha)$ if and only if*

$$\beta(|C-D| + |AD-BC|) \le \alpha|A-B|.$$

In particular,

(1) *for $\beta_1 \in (0,1]$, we have $\mathcal{SS}^*(\beta_1) \subseteq \mathcal{S}^*(A, B, \alpha)$ if and only if*

$$\beta(2 + |A+B|) \le \alpha|A-B|;$$

(2) *for $\alpha_1 \in (0,1]$, we have $\mathcal{S}^*(C, D, \beta) \subseteq \mathcal{SS}^*(\alpha_1)$ if and only if*

$$\beta(|C+D| + |C-D|) \le 2\alpha.$$

Remark 8. Let $\alpha_1, \alpha_2 \in [0,1)$, $A = 1-2\alpha_1$, $C = 1-2\alpha_2$, $B = D = -1$ and $\alpha = \beta = 1$, then the Corollary 5.1.1 reduces to the well-known fact that $\mathcal{S}^*(\alpha_2) \subseteq \mathcal{S}^*(\alpha_1)$ if and only if $\alpha_2 \le \alpha_1$.

Corollary 5.1.2. *Let $A, B \in \mathbb{C}$ with $A \neq B$, $|B| \leq 1$ and $\beta_2 > 0$. If $f \in \mathcal{S}^*(A, B, \alpha)$, then*

(1) *$f \in \mathbb{M}(\beta_2)$ in $|z| \leq R_{\mathbb{M}}(\beta_2)$ for $\beta > 1$, where*

$$R_{\mathbb{M}}(\beta_2) = \min\left\{\frac{a|A-B|}{2(\beta_2+1)+|A-B(2\beta-1)|}, 1\right\}; \tag{5.1}$$

(2) *$f \in \mathcal{RS}^*(\beta_2)$ in $|z| \leq R_{\mathcal{RS}^*}(\beta_2)$ for $0 \leq \beta < 1$, where*

$$R_{\mathcal{RS}^*}(\beta_2) = \min\left\{\frac{a|A-B|}{2\beta_2+|A-B(2\beta-1)|}, 1\right\}. \tag{5.2}$$

Theorem 5.2. *Let $f(z) \in \mathcal{A}$ with $f'(z) \neq 0$ in $\mathbb{D}$. Also let $0 < A \leq 1$, $-1 \leq B < 0$, $\alpha = 0.38344486\ldots$ and $\beta \geq 0.61655\ldots$ which satisfy the conditions $\tan^{-1}\alpha = (1-2\alpha)/2$ and $\tan^{-1}\alpha \leq (\beta-\alpha)\pi/2$, respectively. If*

$$f'(z) \prec \left(\frac{1+Az}{1+Bz}\right)^{\beta}, \tag{5.3}$$

then $f(z)$ is starlike in $|z| < r_0$, where

$$r_0 = \frac{-(A-B)+\sqrt{(A-B)^2+4AB\left(\sin(\alpha+\frac{2}{\pi}\tan^{-1}\alpha)\frac{\pi}{2\beta}\right)^2}}{2AB\sin\left((\alpha+\frac{2}{\pi}\tan^{-1}\alpha)\frac{\pi}{2\beta}\right)} \tag{5.4}$$

is the smallest positive root of the equation

$$\left(\alpha+\frac{2}{\pi}\tan^{-1}\alpha\right)\frac{\pi}{2} = \beta\sin^{-1}\left(\frac{(A-B)x}{1-ABx^2}\right). \tag{5.5}$$

P r o o f. By Theorem 2.1, the subordination (5.3) yields that

$$|\arg f'(z)| \leq \beta\sin^{-1}\frac{(A-B)|z|}{1-AB|z|^2}. \tag{5.6}$$

Let $p(z) = f(z)/z$. We suppose that there exists a point z_0, $|z_0| = r_0 < 1$ such that $|\arg p(z)| < \alpha\pi/2$ for $|z| < r_0$ and $|\arg p(z_0)| = \alpha\pi/2$. Then by Lemma 3.1 we have

$$\frac{z_0p'(z_0)}{p(z_0)} = \frac{2\mathrm{i}k\arg p(z_0)}{\pi}, \tag{5.7}$$

where $k \geq m(a+a^{-1})/2 \geq 1$, $p(z_0) = \pm\mathrm{i}a$ and $a > 0$. Also if $\arg p(z_0) = \alpha\pi/2$, then using (5.5), we have

$$\begin{aligned}
\arg f'(z_0) &= \arg(p(z_0)+zp'(z_0))\\
&= \arg p(z_0) + \arg\left(1+\frac{z_0p'(z_0)}{p(z_0)}\right)\\
&= \frac{\alpha\pi}{2} + \arg(1+\mathrm{i}\alpha k)\\
&\geq \frac{\alpha\pi}{2} + \tan^{-1}\alpha\\
&= \beta\sin^{-1}\left(\frac{(A-B)r_0}{1-ABr_0^2}\right),
\end{aligned}$$

which contradicts (5.6). Similar contrary conclusion will arrive for the case when $\arg p(z_0) = -\alpha\pi/2$. Therefore we have

$$|\arg p(z_0)| = \left|\arg\left(\frac{f(z)}{z}\right)\right| < \frac{\alpha\pi}{2}. \tag{5.8}$$

Now using (5.5), (5.6) and (5.8), we have

$$\begin{aligned}\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| &\le |\arg f'(z)| + \left|\arg\left(\frac{z}{f(z)}\right)\right| \\ &< \left(2\alpha + \frac{2}{\pi}\tan^{-1}\alpha\right)\frac{\pi}{2}.\end{aligned}$$

For f to be starlike, we must have $2\alpha + (2\tan^{-1}\alpha)/\pi = 1$, which gives $\alpha = 0.38344486\ldots$ and since

$$\left(\alpha + \frac{2}{\pi}\tan^{-1}\alpha\right)\frac{\pi}{2} = \beta\sin^{-1}\left(\frac{(A-B)r_0}{1-ABr_0^2}\right),$$

we have $f(z)$ is starlike in $|z| < r_0$, where r_0 is given in (5.4), that completes the proof. $\square$

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