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HARDY-LEINDLER TYPE INEQUALITIES FOR MULTIPLE INTEGRALS ON TIME SCALES

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ABSTRACT. Hardy-Leindler type inequalities and their converses for multiple integrals on time scales are proved by using Fubini's theorem and induction principle. Some generalized versions of Hardy, Wirtinger and Leindler inequalities in both continuous and discrete cases are also derived in seek of applications.

1. Introduction

Mathematics and different disciplines of science use the concept of inequalities to a great extent. The year 1934 came with a great introductory work on inequalities performed by Littlewood, Hardy and Pólya [8]. Since then many attempts are made for discovering new kinds of inequalities along with their applications in mathematical analysis. Copson [7] extended Hardy's inequalities (continuous as well as discrete). In 1970, Leindler [10] gave an analogous proof to Copson's discrete inequality given in [7].

The study of dynamical systems on time scales is currently receiving considerable attention. Although the basic aim of it is to unify the study of differential and difference systems, it also extends these classical cases to cases "in between". As an example, an insect populations model can be explore as a physical example in this field from [5]. The model can be taken as continuous function in summer season (and relatively may obey a difference design scheme with variation in step-size), seems expired out in (say) winter season, the eggs are appeared as incubating or inactive during this period, while they hatch in forthcoming new season and out comes is a non-overlapping new generation. For some other physical examples of time scales, the readers may consider [5: Example 1.39, Example 1.40] and [1].

The researchers working in the field of time scale calculus probe a lot and provided immense contribution in dynamic inequalities of Hardy types [11–15, 18]. Some results on integral type inequalities due to Hardy and the discrete inequalities due to Hardy and Littlewood have been studied in [2, 3, 16] for multiple integrals on time scales. In 2014, S. H. Saker [17] proved some Hardy Leindler type inequalities in time scales setting for function of one variable. Moreover, in [9], some Hardy type inequalities for nabla time scale calculus can be found for function of one variable.

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The objective of the paper is to extend some Hardy type inequalities for function of several variables including Leindler and Wirtinger type inequalities, which are applicable in partial differential and difference equations, harmonic analysis, approximations, number theory, optimization, convex geometry, spectral theory of differential and difference operators.

The paper is organized as follows: In the next section we introduce some basics on theory of time scales. In third section we prove some Hardy Leindler type inequalities for multiple integrals on time scales, whereas last section consists of converses to Hardy-Leindler type inequalities on time scales. Some concrete examples to check the validity of the proved results are given at the end of the paper.

2. Preliminaries

Definition 1 (Time scale). A non empty as well as closed subset of $\mathbb{R}$ is called a time scale, denoted by $\mathbb{T}$ [5,6]. The sets $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ are some examples of time scales.

Definition 2 (Classification of points, [5: Definition 1.1]). The operator $\sigma\colon \mathbb{T} \to \mathbb{T}$ defined by $\sigma(\gamma_1) := \inf\{z_1 \in \mathbb{T}\ ;\ z_1 > \gamma_1\}$ is called forward jump operator and the operator $\rho\colon \mathbb{T} \to \mathbb{T}$ defined by $\rho(\gamma_1) := \sup\{z_1 \in \mathbb{T}\ ;\ z_1 < \gamma_1\}$ is called backward jump operator, where $\gamma_1 \in \mathbb{T}$. A function $\mu\colon \mathbb{T} \to [0,\infty)$ is called the graininess function if $\mu(\gamma_1) := \sigma(\gamma_1) - \gamma_1$ for $\gamma_1 \in \mathbb{T}$.

The point γ_1 satisfying $\sigma(\gamma_1) > \gamma_1$ is right-scattered and called left-scattered if $\rho(\gamma_1) < \gamma_1$. The points are called isolated if they are left-scattered and right-scattered simultaneously. Also the point $\gamma_1 \in \mathbb{T}$ is called right-dense if $\gamma_1 < \sup \mathbb{T}$ and $\sigma(\gamma_1) = \gamma_1$ and is called left-dense if $\gamma_1 > \inf \mathbb{T}$ and $\rho(\gamma_1) = \gamma_1$. The points that are both left-dense and right-dense are dense points.

Definition 3 (rd-Continuous function, [5: Definition 1.58]). Suppose a function $h_1\colon \mathbb{T} \to \mathbb{R}$ which satisfies:

(a) h_1 is continuous at right dense points of $\mathbb{T}$,

(b) the left hand limits exist and are finite at left dense points in $\mathbb{T}$, then h_1 is called rd-continuous on $\mathbb{T}$.

Notation ([5]). We write the notation $h_1^\sigma(\cdot) = h_1(\sigma(\cdot))$ for any function $h_1\colon \mathbb{T} \to \mathbb{R}$.

Definition 4 (Delta differentiable function, [5: Definition 1.10]). Consider a function $h_1\colon \mathbb{T} \to \mathbb{R}$ and fix $\gamma_1 \in \mathbb{T}$ such that $h_1^\Delta(\gamma_1)$ exists and satisfies the property that, for given $\varepsilon > 0$, γ_1 has a neighborhood O with the condition,

$$|[h_1(\sigma(\gamma_1)) - h_1(\varsigma)] - h_1^\Delta(\gamma_1)[\sigma(\gamma_1) - \varsigma]| \le \varepsilon|\sigma(\gamma_1) - \varsigma|$$

for all $\varsigma \in O$, then h_1 is delta differentiable at γ_1 or $h^\Delta(\gamma_1)$ is the (delta) derivative of h_1 at γ_1.

Proposition 2.1 (Properties of Delta Derivative). *Assume $h_1, h_2\colon \mathbb{T} \to \mathbb{R}$ are delta differentiable at $\gamma_1 \in \mathbb{T}$, then the following properties hold:*

- *Product Rule* [5: Theorem 1.20 (iii)]: *The product $h_1h_2\colon \mathbb{T} \to \mathbb{R}$ is also differentiable at γ_1 with*

$$(h_1h_2)^\Delta(\gamma_1) = {h_1}^\Delta(\gamma_1)h_2(\gamma_1) + {h_1}^\sigma(\gamma_1){h_2}^\Delta(\gamma_1) = h_1(\gamma_1){h_2}^\Delta(\gamma_1) + {h_1}^\Delta(\gamma_1){h_2}^\sigma(\gamma_1). \tag{2.1}$$

- *Quotient Rule* [5: Theorem 1.20 (iv)]: *If $h_2(\gamma_1){h_2}^\sigma(\gamma_1) \neq 0$, then $\frac{h_1}{h_2}$ is differentiable at γ_1 and*

$$\left(\frac{h_1}{h_2}\right)^\Delta(\gamma_1) = \frac{{h_1}^\Delta(\gamma_1)h_2(\gamma_1) - h_1(\gamma_1){h_2}^\Delta(\gamma_1)}{h_2(\gamma_1){h_2}^\sigma(\gamma_1)}. \tag{2.2}$$

- *Chain Rule* [5: Theorem 1.87]: *Assume that* $\vartheta\colon \mathbb{T} \to \mathbb{R}$ *is delta differentiable on* $\mathbb{T}$*,* $\vartheta\colon \mathbb{R} \to \mathbb{R}$ *is continuous and* $\theta\colon \mathbb{R} \to \mathbb{R}$ *is continuous as well as differentiable such that the following holds*

$$[\theta(\vartheta(\gamma_1))]^{\Delta} = \theta'(\vartheta(c))\vartheta^{\Delta}(\gamma_1), \quad c \in [\gamma_1, \sigma(\gamma_1)], \qquad \gamma_1 \in \mathbb{T}. \tag{2.3}$$

Proposition 2.2 (Delta Integrable Function, [5: Theorem 1.74])**.** *Every rd-continuous function* $h_1\colon \mathbb{T} \to \mathbb{R}$ *has an anti-derivative. In particular if* $t_0 \in \mathbb{T}$ *then* $H_1(\cdot)$ *defined by*

$$H_1(t) = \int_{t_0}^{t} h_1(\tau)\Delta\tau, \qquad \tau \in \mathbb{T}$$

is antiderivative of h_1*.*

Proposition 2.3 (Properties of Integration)**.**

- *Scaler Multiplication,* [5: Theorem 1.77 (ii)]: *For* $c > 0$

$$\int_a^b c\theta(\gamma_1)\Delta\gamma_1 = c\int_a^b \theta(\gamma_1)\Delta\gamma_1, \qquad \gamma_1 \in \mathbb{T}. \tag{2.4}$$

- *Integration by Parts Formula,* [5: Theorem 1.77 (v)]: *For* $a, b \in \mathbb{T}$ *and* $\theta, \vartheta \in C_{rd}(\mathbb{T}, \mathbb{R})$*, the formula for integration by parts is given by*

$$\int_a^b \theta(\gamma_1)\vartheta^{\Delta}(\gamma_1)\Delta\gamma_1 \le |\theta(\gamma_1)\vartheta(\gamma_1)|_a^b - \int_a^b \theta^{\Delta}(\gamma_1)\vartheta^{\sigma}(\gamma_1)\Delta\gamma_1, \qquad \gamma_1 \in \mathbb{T}. \tag{2.5}$$

The following Hölder's Inequality on time scales is given in [5: Theorem 6.13].

Theorem 2.4. *Suppose* $\ell_1, \ell_2 \in \mathbb{T}$ *and* $\theta, \vartheta : [\ell_1, \ell_2] \to \mathbb{R}$ *are rd-continuous functions, then*

$$\int_{\ell_1}^{\ell_2} |\theta(\gamma_1)\vartheta(\gamma_1)|\Delta\gamma_1 \le \left[\int_{\ell_1}^{\ell_2} |\theta(\gamma_1)|^q \Delta\gamma_1\right]^{\frac{1}{q}} \left[\int_{\ell_1}^{\ell_2} |\vartheta(\gamma_1)|^p \Delta\gamma_1\right]^{\frac{1}{p}}, \qquad \gamma_1 \in \mathbb{T}, \tag{2.6}$$

where $p > 1$ *and* $q = p/(p-1)$*.*

The following result is a time scales version of Fubini's Theorem [4].

Theorem 2.5. *Let* $\mathbb{T}_1$ *and* $\mathbb{T}_2$ *be two time scales,* $h_1\colon \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ *is integrable with respect to both time scales and if the function* $\varphi(y_1) = \int_{\mathbb{T}_1} h_1(x_1, y_1)\Delta x_1$ *exists for almost every* $y_1 \in \mathbb{T}_2$ *and* $\psi(x_1) = \int_{\mathbb{T}_2} h_1(x_1, y_1)\Delta y_1$ *exists for almost every* $x_1 \in \mathbb{T}_1$*, then*

$$\int_{\mathbb{T}_1} \Delta x_1 \int_{\mathbb{T}_2} h_1(x_1, y_1)\Delta y_1 = \int_{\mathbb{T}_2} \Delta y_1 \int_{\mathbb{T}_1} h_1(x_1, y_1)\Delta x_1. \tag{2.7}$$

Notation. The following notation is used in the paper for partial delta derivative

$$\frac{\partial}{\Delta_k \hbar_k} f(\hbar_1, \dots, \hbar_k, \dots, \hbar_n) = f^{\Delta_k}(\hbar_1, \dots, \hbar_k, \dots, \hbar_n),\ 1 \le k \le n.$$

3. Hardy-Leindler type inequalities for multiple integrals on time scales

In what follows, it is assumed that all the functions are non-negative and the involved integrals exist.

THEOREM 3.1. *Consider $\iota \in \{1, \dots, n\}$ and time scales $\mathbb{T}_\iota$ for $a_\iota \in [0, \infty)_{\mathbb{T}_\iota}$. Take $\xi_\iota \colon \mathbb{T}_\iota \to \mathbb{R}_+$, such that*

$$A_\iota(\hbar_\iota) := \int_{\hbar_\iota}^{\infty} \xi_\iota(\varsigma_\iota)\Delta\varsigma_\iota \tag{3.1}$$

exists, where $A_\iota(\infty) = 0$ for all ι.

Consider $g \colon \mathbb{T}_1 \times \cdots \times \mathbb{T}_n \to \mathbb{R}_+$ such that

$$\phi_n(\hbar_1, \dots, \hbar_n) := \int_{a_1}^{\hbar_1} \cdots \int_{a_n}^{\hbar_n} g(\varsigma_1, \dots, \varsigma_n)\Delta\varsigma_n \cdots \Delta\varsigma_1. \tag{3.2}$$

Then for $p > 1$,

$$\begin{aligned} &\int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} \xi_1(\hbar_1) \cdots \xi_n(\hbar_n)\phi_n^p(\sigma_1(\hbar_1), \dots, \sigma_n(\hbar_n))\Delta\hbar_n \cdots \Delta\hbar_1 \\ &\le p^{np} \int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} \prod_{\iota=1}^{n} A_\iota^p(\hbar_\iota)\xi_\iota^{1-p}(\hbar_\iota) g^p(\hbar_1, \dots, \hbar_n)\Delta\hbar_n \cdots \Delta\hbar_1, \end{aligned} \tag{3.3}$$

where $n \ge 1$.

P r o o f. Mathematical induction technique is used to prove this result. The statement is true for $n = 1$, see [17: Theorem 2.1].

Next assume (3.3) holds for $1 \le n \le k$. Now for $n = k + 1$, right hand side of (3.3) takes the form

$$\int_{a_1}^{\infty} \cdots \int_{a_k}^{\infty} \prod_{\iota=1}^{k} \xi_\iota(\hbar_\iota) \left\{ \int_{a_{k+1}}^{\infty} \xi_{k+1}(\hbar_{k+1})\phi_{k+1}^p(\sigma_1(\hbar_1), \dots, \sigma_{k+1}(\hbar_{k+1}))\Delta\hbar_{k+1} \right\} \Delta\hbar_k \cdots \Delta\hbar_1. \tag{3.4}$$

Denote $I_{k+1} = \int_{a_{k+1}}^{\infty} \xi_{k+1}(\hbar_{k+1})\phi_{k+1}^p(\sigma_1(\hbar_1), \dots, \sigma_{k+1}(\hbar_{k+1}))\Delta\hbar_{k+1}$.

Apply integration by parts (2.5) with $\theta_{k+1}^{\Delta_{k+1}}(\hbar_{k+1}) = \xi_{k+1}(\hbar_{k+1})$, $\vartheta_{k+1}^{\sigma_{k+1}}(\hbar_{k+1}) = \phi_{k+1}^p(\sigma_1(\hbar_1), \dots, \sigma_{k+1}(\hbar_{k+1}))$ and use the facts that $\phi_{k+1}(\hbar_1, \dots, \hbar_k, a_{k+1}) = 0$ and $A_{k+1}(\infty) = 0$ to get

$$I_{k+1} = \int_{a_{k+1}}^{\infty} -\theta_{k+1}(\hbar_{k+1})\big(\phi_{k+1}^p(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), \hbar_{k+1})\big)^{\Delta_{k+1}}\Delta\hbar_{k+1},$$

where $-\theta_{k+1}(\hbar_{k+1}) = A_{k+1}(\hbar_{k+1})$. Therefore,

$$I_{k+1} = \int_{a_{k+1}}^{\infty} A_{k+1}(\hbar_{k+1})\big(\phi_{k+1}^p(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), \hbar_{k+1})\big)^{\Delta_{k+1}}\Delta\hbar_{k+1}. \tag{3.5}$$

Use chain rule formula (2.3) to get

$$\begin{aligned} &[\phi_{k+1}^p(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), \hbar_{k+1})]^{\Delta_{k+1}} \\ &= p\phi_{k+1}^{p-1}(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), d_{k+1})\phi_{k+1}^{\Delta_{k+1}}(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), \hbar_{k+1}), \end{aligned} \tag{3.6}$$

where $d_{k+1} \in [\hbar_{k+1}, \sigma_{k+1}(\hbar_{k+1})]$. Since

$$\phi_{k+1}^{\Delta_{k+1}}(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), \hbar_{k+1}) = \int_{a_1}^{\sigma_1(\hbar_1)} \cdots \int_{a_k}^{\sigma_k(\hbar_k)} g(\varsigma_1, \dots, \varsigma_k, \hbar_{k+1})\varsigma_k \cdots \Delta\varsigma_1 \geq 0. \tag{3.7}$$

Therefore, (3.7) implies

$$\phi_{k+1}^{\Delta_{k+1}}(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), \hbar_{k+1}) = \phi_k(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), \hbar_{k+1}), \tag{3.8}$$

where

$$\phi_k(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), \hbar_{k+1}) = \int_{a_1}^{\sigma_1(\hbar_1)} \cdots \int_{a_k}^{\sigma_k(\hbar_k)} g(\varsigma_1, \dots, \varsigma_k, \hbar_{k+1})\varsigma_k \cdots \Delta\varsigma_1.$$

Use (3.8) and $d_{k+1} \leq \sigma_{k+1}(\hbar_{k+1})$ in (3.6) to get

$$\begin{aligned}&[\phi_{k+1}^p(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), \hbar_{k+1})]^{\Delta_{k+1}} \\ &\leq p\phi_{k+1}^{p-1}(\sigma_1(\hbar_1), \dots, \sigma_{k+1}(\hbar_{k+1})\phi_k(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), \hbar_{k+1}).\end{aligned} \tag{3.9}$$

Substitute (3.9) in (3.5) to obtain,

$$\begin{aligned}I_{k+1} \leq p \int_{a_{k+1}}^{\infty} &\frac{A_{k+1}(\hbar_{k+1})\phi_k(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), \hbar_{k+1})\xi_{k+1}^{\frac{p-1}{p}}(\hbar_{k+1})}{\xi_{k+1}^{\frac{p-1}{p}}(\hbar_{k+1})} \\ &\times \phi_{k+1}^{p-1}(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), \sigma_{k+1}(\hbar_{k+1}))\Delta\hbar_{k+1}.\end{aligned} \tag{3.10}$$

Using Hölder's inequality (2.6) to (3.10), one's get

$$I_{k+1} \leq p\Bigg[\int_{a_{k+1}}^{\infty} \left[\frac{A_{k+1}(\hbar_{k+1})\phi_k(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), \hbar_{k+1})}{\xi_{k+1}^{\frac{p-1}{p}}(\hbar_{k+1})}\right]^p \Delta\hbar_{k+1}\Bigg]^{\frac{1}{p}} \times [I_{k+1}]^{\frac{p-1}{p}}. \tag{3.11}$$

Simplification in (3.11) gives

$$I_{k+1} \leq p^p \int_{a_{k+1}}^{\infty} A_{k+1}^p(\hbar_{k+1})\phi_k^p(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), \hbar_{k+1})\xi_{k+1}^{1-p}(\hbar_{k+1})\Delta\hbar_{k+1}. \tag{3.12}$$

Put (3.12) in (3.4) to have

$$\begin{aligned}&\int_{a_1}^{\infty} \cdots \int_{a_{k+1}}^{\infty} \xi_1(\hbar_1) \cdots \xi_{k+1}(\hbar_{k+1})\phi_{k+1}^p(\sigma_1(\hbar_1), \dots, \sigma_{k+1}(\hbar_{k+1}))\Delta\hbar_{k+1} \cdots \Delta\hbar_1 \\ &\leq p^p \int_{a_1}^{\infty} \cdots \int_{a_k}^{\infty} \xi_1(\hbar_1) \cdots \xi_k(\hbar_k) \int_{a_{k+1}}^{\infty} (A_{k+1}(\hbar_{k+1}))^p \phi_k^p(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), \hbar_{k+1}) \\ &\quad \times \xi_{k+1}^{1-p}(\hbar_{k+1})\Delta\hbar_{k+1} \cdots \Delta\hbar_1.\end{aligned} \tag{3.13}$$

Exchange the integrals k times by using Fubini's Theorem on right hand side of (3.13)

$$\begin{aligned}&= p^p \int_{a_{k+1}}^{\infty} A_{k+1}^p(\hbar_{k+1})\xi_{k+1}^{1-p}(\hbar_{k+1}) \\ &\quad \times \Bigg(\int_{a_1}^{\infty} \cdots \int_{a_k}^{\infty} \xi_1(\hbar_1) \cdots \xi_k(\hbar_k)\phi_k^p(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), \hbar_{k+1})\Delta\hbar_k \cdots \Delta\hbar_1\Bigg)\Delta\hbar_{k+1}.\end{aligned} \tag{3.14}$$

Use induction hypothesis for $\phi_k(\sigma_1(\hbar_1), \ldots, \sigma_k(\hbar_k), \hbar_{k+1})$ instead of $\phi_k(\sigma_1(\hbar_1), \ldots, \sigma_k(\hbar_k))$ for fixed $\hbar_{k+1} \in \mathbb{T}_{k+1}$ on right hand side of (3.14) and once again apply the Fubini's Theorem k times to obtain

$$\int_{a_1}^{\infty} \cdots \int_{a_{k+1}}^{\infty} \xi_1(\hbar_1) \cdots \xi_{k+1}(\hbar_{k+1}) \phi_{k+1}^p(\sigma_1(\hbar_1), \ldots, \sigma_{k+1}(\hbar_{k+1})) \Delta\hbar_{k+1} \cdots \Delta\hbar_1$$
$$\leq p^{(k+1)p} \int_{a_1}^{\infty} \cdots \int_{a_{k+1}}^{\infty} \prod_{\iota=1}^{k+1} A_\iota^p(\hbar_\iota) \xi_\iota^{1-p}(\hbar_\iota) g^p(\hbar_1, \ldots, \hbar_{k+1}) \Delta\hbar_{k+1} \cdots \Delta\hbar_1.$$

Hence by the principle of mathematical induction, the result is true for all positive integers n. □

Remark 1. As a special case of Theorem 3.1, when $\mathbb{T}_i = \mathbb{R}$, (note that in this case, we have $\sigma_i(\hbar_i) = \hbar_i$ and $\hbar_i \in T_i$, for all $i \in \{1, \ldots, n\}$). We have the Wirtinger type inequality,

$$\int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} \prod_{\iota=1}^{n} \xi_i^{1-p}(\hbar_\iota) A_\iota^p(\hbar_\iota) \left(\frac{\partial^n}{\partial(\hbar_1) \cdots \partial(\hbar_n)} \phi_n(\hbar_1, \ldots, \hbar_n) \right)^p \mathrm{d}\hbar_n \cdots \mathrm{d}\hbar_1$$
$$\geq \frac{1}{p^{np}} \int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} \prod_{\iota=1}^{n} \xi_\iota(\hbar_\iota) \phi_n^p(\hbar_1, \ldots, \hbar_n) \mathrm{d}\hbar_n \cdots \mathrm{d}\hbar_1,$$

where $p > 1$ and $\frac{\partial^n}{\partial(\hbar_1) \cdots \partial(\hbar_n)} \phi(\hbar_1, \ldots, \hbar_n) = g(\hbar_1, \ldots, \hbar_n)$. Moreover $\phi(\hbar_1, \ldots, \hbar_n)$ is differentiable function with $\phi(a_1, \ldots, a_n) = 0$.

As a particular case if $\xi_i(\hbar_i) = \frac{1}{\hbar_i}$ and $p = 2$, then replace upper limit of the integral by 1 to get the following famous inequality due to Hardy (in several variables),

$$\int_0^1 \cdots \int_0^1 \left(\frac{\partial^n}{\partial(\hbar_1) \cdots \partial(\hbar_n)} \phi(\hbar_1, \ldots, \hbar_n) \right)^2 \mathrm{d}\hbar_n \cdots \mathrm{d}\hbar_1$$
$$\geq \frac{1}{4^n} \int_0^1 \cdots \int_0^1 \frac{1}{(\hbar_1)^2} \cdots \frac{1}{(\hbar_n)^2} \phi^2(\hbar_1, \ldots, \hbar_n) \mathrm{d}\hbar_n \cdots \mathrm{d}\hbar_1,$$

where $\phi(0, \ldots, 0) = 0$, with the constant $\frac{1}{4^n}$ as best possible.

Remark 2. Assume that $\mathbb{T}_i = \mathbb{N}$, $p > 1$ and $a_i = 1$ for all $i = 1, \ldots, n$ in Theorem 3.1. Furthermore assume that,

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \prod_{\iota=1}^{n} \xi_\iota^{1-p}(m_\iota) A_\iota^p(m_\iota) g^p(m_1, \ldots, m_n)$$

is convergent, then (3.3) becomes the following discrete Leindler's inequality for $p > 1$.

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \prod_{\iota=1}^{n} \xi_\iota^{1-p}(m_\iota) A_\iota^p(m_\iota) (\Delta_1 \cdots \Delta_n \phi_n(m_1, \ldots, m_n))^p$$
$$\geq \frac{1}{p^{np}} \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \prod_{\iota=1}^{n} \xi_\iota(m_\iota) (\phi_n(m_1, \ldots, m_n))^p,$$

where $\phi_n(m_1, \ldots, m_n)$ is a positive sequences with $\phi_n(1, \ldots, 1) = 0$ and Δ_ι represents forward difference operator with respect to m_ι.

THEOREM 3.2. *Consider $\iota = \{1, \ldots, n\}$ and $\mathbb{T}_i$ be a time scales with $a_\iota \in [0, \infty)_{\mathbb{T}_\iota}$. Take $\xi_\iota \colon \mathbb{T}_i \to \mathbb{R}_+$ such that*

$$B_\iota(\hbar_\iota) := \int_{a_\iota}^{\hbar_\iota} \xi_\iota(\varsigma_\iota) \Delta\varsigma_\iota \tag{3.15}$$

exists and $B_\iota(\infty)=0$ *for all* ι. *Furthermore let*

$$\varphi_n(\hbar_1,\dots,\hbar_n) := \int_{\hbar_1}^{\infty}\cdots\int_{\hbar_n}^{\infty} g(\varsigma_1,\dots,\varsigma_n)\Delta\varsigma_n\cdots\Delta\varsigma_1 \tag{3.16}$$

for any $\hbar_\iota\in[a_\iota,\infty)_{\mathbb{T}_\iota}$ *and* $g\colon \mathbb{T}_1\times\cdots\times\mathbb{T}_n\to\mathbb{R}_+$. *Then for* $p>1$,

$$\begin{aligned}&\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\prod_{\iota=1}^{n}\xi_\iota(\hbar_\iota)\varphi_n^p(\hbar_1,\dots,\hbar_n)\Delta\hbar_n\cdots\Delta\hbar_1\\ &\le p^{np}\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\prod_{\iota=1}^{n}[B_\iota^{\sigma_\iota}(\hbar_\iota)]^p\xi_\iota^{1-p}(\hbar_\iota)g^p(\hbar_1,\dots,\hbar_n)\Delta\hbar_n\cdots\Delta\hbar_1.\end{aligned} \tag{3.17}$$

P r o o f. We use mathematical induction method to prove the required result. For $n=1$, the result is obvious, see [17: Theorem 2.2]. Let (3.17) holds for $1\le n\le k$, then for $n=k+1$ left hand side of (3.17) takes the form,

$$\int_{a_1}^{\infty}\cdots\int_{a_k}^{\infty}\prod_{\iota=1}^{k}\xi_\iota(\hbar_\iota)\Big(\int_{a_{k+1}}^{\infty}\xi_{k+1}(\hbar_{k+1})\varphi_{k+1}^p(\hbar_1,\dots,\hbar_{k+1})\Delta\hbar_{k+1}\Big)\Delta\hbar_k\cdots\Delta\hbar_1. \tag{3.18}$$

Denote $I_{k+1}=\int_{a_{k+1}}^{\infty}\xi_{k+1}(\hbar_{k+1})\varphi_{k+1}^p(\hbar_1,\dots,\hbar_{k+1})\Delta\hbar_{k+1}$.

Apply integration by parts (2.5) and use $\varphi_{k+1}(\hbar_1,\dots,\hbar_k,\infty)=0$, $B_{k+1}(a_{k+1})=0$ to get

$$I_{k+1}=\int_{a_{k+1}}^{\infty} -\frac{\partial}{\Delta_{k+1}\hbar_{k+1}}\varphi_{k+1}^p(\hbar_1,\dots,\hbar_{k+1})B_{k+1}^{\sigma_{k+1}}(\hbar_{k+1})\Delta\hbar_{k+1}. \tag{3.19}$$

Apply chain rule (2.3) to find

$$-\frac{\partial}{\Delta_{k+1}\hbar_{k+1}}\varphi_{k+1}^p(\hbar_1,\dots,\hbar_{k+1}) = -p\varphi_{k+1}^{p-1}(\hbar_1,\dots,\hbar_k,d_{k+1})\varphi_{k+1}^{\Delta_{k+1}}(\hbar_1,\dots,\hbar_{k+1}), \tag{3.20}$$

where $d_{k+1}\in[\hbar_{k+1},\sigma_{k+1}(\hbar_{k+1})]$. Since

$$\begin{aligned}\frac{\partial}{\Delta_{k+1}\hbar_{k+1}}\varphi_{k+1}(\hbar_1,\dots,\hbar_{k+1}) &= \int_{\hbar_1}^{\infty}\cdots\int_{\hbar_k}^{\infty}\Big\{\frac{\partial}{\Delta_{k+1}\hbar_{k+1}}\int_{\hbar_{k+1}}^{\infty}g(\varsigma_1,\dots,\varsigma_{k+1})\Delta\varsigma_{k+1}\Big\}\Delta\varsigma_k\cdots\Delta\varsigma_1\\ &= -\int_{\hbar_1}^{\infty}\cdots\int_{\hbar_k}^{\infty}g(\varsigma_1,\dots,\varsigma_k,\hbar_{k+1})\Delta\varsigma_k\cdots\Delta\varsigma_1\le 0,\end{aligned}$$

and $d_{k+1}\ge\hbar_{k+1}$. Therefore (3.20) implies,

$$-[\varphi_{k+1}^p(\hbar_1,\dots,\hbar_{k+1})]^{\Delta_{k+1}}\le p\varphi_k^{p-1}(\hbar_1,\dots,\hbar_{k+1})\varphi_{k+1}(\hbar_1,\dots,\hbar_{k+1}), \tag{3.21}$$

where $\varphi_k(\hbar_1,\dots,\hbar_{k+1})=\int_{\hbar_1}^{\infty}\cdots\int_{\hbar_k}^{\infty}g(\varsigma_1,\dots,\varsigma_k,\hbar_{k+1})\Delta\varsigma_k\cdots\Delta\varsigma_1$.

Use (3.21) in (3.19) to get

$$I_{k+1}\le p\int_{a_{k+1}}^{\infty}\frac{B_{k+1}^{\sigma_{k+1}}(\hbar_{k+1})\varphi_k(\hbar_1,\dots,\hbar_{k+1})[\xi_{k+1}(\hbar_{k+1})]^{\frac{p-1}{p}}}{\xi_{k+1}^{\frac{p-1}{p}}(\hbar_{k+1})}\varphi_k^{p-1}(\hbar_1,\dots,\hbar_{k+1})\Delta\hbar_{k+1}. \tag{3.22}$$

Apply Hölder's inequality to (3.22) to get

$$I_{k+1} \leq p\Big[\int_{a_{k+1}}^{\infty}\Big[\frac{B_{k+1}^{\sigma_{k+1}}(\hbar_{k+1})\varphi_k(\hbar_1,\dots,\hbar_{k+1})}{\xi_{k+1}^{\frac{p-1}{p}}(\hbar_{k+1})}\Big]^p\Delta\hbar_{k+1}\Big]^{\frac{1}{p}}\Big[\int_{a_{k+1}}^{\infty}\xi_{k+1}(\hbar_{k+1})\varphi_k^p(\hbar_1,\dots,\hbar_{k+1})\Delta\hbar_{k+1}\Big]^{\frac{p-1}{p}},$$

which results in

$$I_{k+1} \leq p^p\int_{a_{k+1}}^{\infty}[B_{k+1}^{\sigma_{k+1}}(\hbar_{k+1})]^p\varphi_{k+1}^p(\hbar_1,\dots,\hbar_{k+1})\xi_{k+1}^{1-p}(\hbar_{k+1})\Delta\hbar_{k+1}. \tag{3.23}$$

Put (3.23) in (3.18) to get

$$\int_{a_1}^{\infty}\cdots\int_{a_{k+1}}^{\infty}\xi_1(\hbar_1)\cdots\xi_{k+1}(\hbar_{k+1})\varphi_{k+1}^p(\hbar_1,\dots,\hbar_{k+1})\Delta\hbar_{k+1}\cdots\Delta\hbar_1 \tag{3.24}$$

$$\leq p^p\int_{a_1}^{\infty}\cdots\int_{a_k}^{\infty}\xi_1(\hbar_1)\cdots\xi_k(\hbar_k)\Delta\hbar_k,\dots,\Delta\hbar_1\int_{a_{k+1}}^{\infty}[B_{k+1}^{\sigma_{k+1}}(\hbar_{k+1})]^p\xi_{k+1}^{1-p}(\hbar_{k+1})\varphi_k^p(\hbar_1,\dots,\hbar_k,\hbar_{k+1})\Delta\hbar_{k+1}. \tag{3.25}$$

Exchange the integrals k times by using Fubini's Theorem on right hand side of (3.24) to get

$$\begin{aligned}&\int_{a_1}^{\infty}\cdots\int_{a_{k+1}}^{\infty}\xi_1(\hbar_1)\cdots\xi_{k+1}(\hbar_{k+1})\varphi_k^p(\hbar_1,\dots,\hbar_{k+1})\Delta\hbar_{k+1}\cdots\Delta\hbar_1\\ &\leq p^p\int_{a_{k+1}}^{\infty}[B_{k+1}^{\sigma_{k+1}}(\hbar_{k+1})]^p\xi_{k+1}^{1-p}(\hbar_{k+1})\\ &\quad\times\Big(\int_{a_1}^{\infty}\cdots\int_{a_k}^{\infty}\xi_1(\hbar_1)\cdots\xi_k(\hbar_k)\varphi_k^p(\hbar_1,\dots,\hbar_k,\hbar_{k+1})\Delta\hbar_k,\dots,\Delta\hbar_1\Big)\Delta\hbar_{k+1}.\end{aligned} \tag{3.26}$$

Use induction hypothesis for $\varphi_k(\hbar_1,\dots,\hbar_k,\hbar_{k+1})$ instead of $\varphi_k(\hbar_1,\dots,\hbar_k)$ for some fixed $\hbar_{k+1}\in\mathbb{T}_{k+1}$ on right hand side of (3.26) and again apply Fubini's Theorem k times to obtain

$$\begin{aligned}&\int_{a_1}^{\infty}\cdots\int_{a_{k+1}}^{\infty}\prod_{\tau=1}^{k+1}\xi_\iota(\hbar_\iota)\varphi^p(\hbar_1,\dots,\hbar_k,\hbar_{k+1})\Delta\hbar_{k+1}\cdots\Delta\hbar_1\\ &\leq p^{(k+1)p}\int_{a_1}^{\infty}\cdots\int_{a_{k+1}}^{\infty}\prod_{\iota=1}^{k+1}(B_\iota{}^{\sigma_\iota}(\hbar_\iota))^p\xi_\iota^{1-p}(\hbar_\iota)g^p(\hbar_1,\dots,\hbar_{k+1})\Delta\hbar_{k+1}\cdots\Delta\hbar_1.\end{aligned}$$

Hence statement is true for all integers n. □

Remark 3. As a special case of Theorem 3.2, when $\mathbb{T}_i=\mathbb{R}_i$ and $p>1$, we have the following integral inequality of Leindler type (for functions of several variables), note that in this case we have

$$\varphi(\sigma_1(\hbar_1),\dots,\sigma_n(\hbar_n))=\varphi(\hbar_1,\dots,\hbar_n).$$

Therefore,

$$\begin{aligned}&\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\prod_{\iota=1}^{n}\xi_\iota(\hbar_\iota)\Big(\int_{\hbar_1}^{\infty}\cdots\int_{\hbar_n}^{\infty}g(\varsigma_1,\dots,\varsigma_n)\mathrm{d}\varsigma_n\cdots\mathrm{d}\varsigma_1\Big)^p\mathrm{d}\hbar_n\cdots\mathrm{d}\hbar_1\\ &\leq p^{np}\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\prod_{\iota=1}^{n}\xi_i^{1-p}(\hbar_\iota)B_\iota{}^p(\hbar_\iota)g^p(\hbar_1\cdots\hbar_n)\mathrm{d}\hbar_n\cdots\mathrm{d}\hbar_1.\end{aligned}$$

Remark 4. Assume that $\mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n = \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$ in Theorem 3.2, $p > 1$ and $a_i = 1$. In this case (3.17) becomes the following discrete Leindler's inequality

$$\begin{aligned}&\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \prod_{\iota=1}^{n} \xi_\iota(m_\iota)\Big(\sum_{k_1=m}^{\infty} \cdots \sum_{k_n=m_n}^{\infty} g(k_1,\dots,k_n)\Big)^p \\ &\le p^{np} \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \prod_{\iota=1}^{n} \xi_\iota^{1-p}(m_\iota) B_\iota{}^p(m_\iota) g^p(m_1,\dots,m_n), \qquad p>1,\end{aligned}$$

where $B_\iota(m_\iota) = \sum\limits_{k_\iota=1}^{m_\iota-1} \xi_\iota(k_\iota)$.

4. Converses to Hardy-Leindler type inequalities for multiple integrals on time scales

Theorem 4.1. *Consider $\iota = \{1,\dots,n\}$ and $\mathbb{T}_\iota$ are time scales with $a_\iota \in [0,\infty)_{\mathbb{T}_\iota}$. Take $\xi_\iota \colon \mathbb{T}_\iota \to \mathbb{R}_i$ such that*

$$C_\iota(\hbar_\iota) = \int_{\hbar_\iota}^{\infty} \xi_\iota(\varsigma_\iota)\Delta\varsigma_\iota,$$

where $C_\iota(\infty) = 0$. Furthermore assume

$$\psi_n(\hbar_1,\dots,\hbar_n) := \int_{a_1}^{\hbar_1} \cdots \int_{a_n}^{\hbar_n} g(\varsigma_1,\dots,\varsigma_n)\Delta\varsigma_n \cdots \Delta\varsigma_1,$$

where $g \colon \mathbb{T}_1 \times \cdots \times \mathbb{T}_n \to \mathbb{R}_+$. Then for $0 < p \le 1$,

$$\begin{aligned}&\int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} \prod_{\iota=1}^{n} \xi_\iota(\hbar_\iota)\psi_n^p(\sigma_1(\hbar_1),\dots,\sigma_n(\hbar_n))\Delta\hbar_n \cdots \Delta\hbar_1 \\ &\ge p^{np} \int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} \prod_{\iota=1}^{n} C_\iota^p(\hbar_\iota)\xi_\iota^{1-p}(\hbar_\iota) g^p(\hbar_1,\dots,\hbar_n)\Delta\hbar_n \cdots \Delta\hbar_1.\end{aligned} \tag{4.1}$$

Proof. For $n = 1$, the statement is true by [17: Theorem 2.3]. Next suppose, for $1 \le n \le k$, the statement holds. To observe the statement for $n = k+1$, the left hand side of the inequality (4.1) takes the form

$$\int_{a_1}^{\infty} \cdots \int_{a_k}^{\infty} \prod_{\iota=1}^{k} \xi_i(\hbar_i)\Big\{\int_{a_{k+1}}^{\infty} \xi_{k+1}(\hbar_{k+1})\psi_{k+1}^p(\sigma_1(\hbar_1),\dots,\sigma_{k+1}(\hbar_{k+1}))\Delta\hbar_{k+1}\Big\}\Delta\hbar_k \cdots \Delta\hbar_1. \tag{4.2}$$

Denote $I_{k+1} = \int\limits_{a_{k+1}}^{\infty} \xi_{k+1}(\hbar_{k+1})\psi_{k+1}^p(\sigma_1(\hbar_1),\dots,\sigma_{k+1}(\hbar_{k+1}))\Delta\hbar_{k+1}$. Apply integration by parts (2.5) with $\theta^{\Delta_{k+1}}(\hbar_{k+1}) = \xi_{k+1}(\hbar_{k+1})$ and $\vartheta(\sigma_{k+1}(\hbar_{k+1})) = \psi_{k+1}(\sigma_1(\hbar_1),\dots,\sigma_{k+1}(\hbar_{k+1}))$ and use the fact $\psi_{k+1}(\sigma_1(\hbar_1),\dots,\sigma_k(\hbar_k),a_{k+1}) = 0 = \theta_{k+1}(\infty)$ to get

$$I_{k+1} = \int_{a_{k+1}}^{\infty} C_{k+1}(\hbar_{k+1})[\psi_{k+1}^p(\sigma_1(\hbar_1),\dots,\sigma_k(\hbar_k),\hbar_{k+1})]^{\Delta_{k+1}}\Delta\hbar_{k+1}, \tag{4.3}$$

where $\theta_{k+1}(\hbar_{k+1}) = -\int_{\hbar_{k+1}}^{\infty} \xi_{k+1}(\varsigma_{k+1})\Delta\varsigma_{k+1} = -C_{k+1}(\hbar_{k+1})$. Also by chain rule (2.3) we have

$$\begin{aligned} &[\psi_{k+1}^{p}(\sigma_1(\hbar_1),\dots,\sigma_k(\hbar_k),\hbar_{k+1})]^{\Delta_{k+1}} \\ &= p\psi_{k+1}^{p-1}(\sigma_1(\hbar_1),\dots,\sigma_k(\hbar_k),d_{k+1})\psi_{k+1}^{\Delta_{k+1}}(\sigma_1(\hbar_1),\dots,\sigma_k(\hbar_k),\hbar_{k+1}), \end{aligned} \tag{4.4}$$

where $d_{k+1} \in [\hbar_{k+1}, \sigma_{k+1}(\hbar_{k+1})]$. Since

$$\begin{aligned} &\psi_{k+1}^{\Delta_{k+1}}(\sigma_1(\hbar_1),\dots,\sigma_k(\hbar_k),\hbar_{k+1}) \\ &= \int_{a_1}^{\sigma_1(\hbar_1)} \cdots \int_{a_k}^{\sigma_k(\hbar_k)} \left\{ \frac{\partial}{\Delta_{k+1}\hbar_{k+1}} \int_{a_{k+1}}^{\hbar_{k+1}} g(\varsigma_1,\dots,\varsigma_{k+1})\Delta\varsigma_{k+1} \right\} \Delta\varsigma_k \cdots \Delta\varsigma_1, \end{aligned} \tag{4.5}$$

and $\sigma_{k+1}(\hbar_{k+1}) \geq d_{k+1}$. Therefore (4.5) implies

$$\begin{aligned} &\psi_{k+1}^{\Delta_{k+1}}(\sigma_1(\hbar_1),\dots,\sigma_k(\hbar_k),\hbar_{k+1}) = \int_{a_1}^{\sigma_1(\hbar_1)} \cdots \int_{a_k}^{\sigma_k(\hbar_k)} g(\varsigma_1,\dots,\varsigma_k,\hbar_{k+1})\Delta\varsigma_k \cdots \Delta\varsigma_1 \\ \Longrightarrow\ &\psi_{k+1}^{\Delta_{k+1}}(\sigma_1(\hbar_1),\dots,\sigma_k(\hbar_k),\hbar_{k+1}) = \psi_k(\sigma_1(\hbar_1),\dots,\sigma_k(\hbar_k),\hbar_{k+1}), \end{aligned} \tag{4.6}$$

where $\psi_k(\sigma_1(\hbar_1),\dots,\sigma_k(\hbar_k),\hbar_{k+1}) = \int_{a_1}^{\sigma_1(\hbar_1)} \cdots \int_{a_k}^{\sigma_k(\hbar_k)} g(\varsigma_1,\dots,\varsigma_k,\hbar_{k+1})\Delta\varsigma_k \cdots \Delta\varsigma_1$.
Use (4.6) in (4.4) together with the facts $d_{k+1} \leq \sigma_{k+1}(\hbar_{k+1})$, $0 < p \leq 1$ to get

$$\begin{aligned} &[\psi_{k+1}^{p}(\sigma_1(\hbar_1),\dots,\sigma_k(\hbar_k),\hbar_{k+1})]^{\Delta_{k+1}} \\ &\geq p\psi_{k+1}^{p-1}(\sigma_1(\hbar_1),\dots,\sigma_{k+1}(\hbar_{k+1}))\psi_k(\sigma_1(\hbar_1),\dots,\sigma_k(\hbar_k),\hbar_{k+1}. \end{aligned} \tag{4.7}$$

Use (4.7) in (4.3) and take power p on both sides to have

$$\begin{aligned} &I_{k+1}^{p} \\ &\geq p^p\left[\int_{a_{k+1}}^{\infty} C_{k+1}(\hbar_{k+1})\psi_{k+1}^{p-1}(\sigma_1(\hbar_1),\dots,\sigma_{k+1}(\hbar_{k+1}))\psi_k(\sigma_1(\hbar_1),\dots,\sigma_k(\hbar_k),\hbar_{k+1}))\Delta\hbar_{k+1}\right]^p. \end{aligned} \tag{4.8}$$

Apply Hölder's inequality (2.6) on the term

$$\left[\int_{a_{k+1}}^{\infty} (C_{k+1}^{p}(\hbar_{k+1})\psi_{k+1}^{p(p-1)}(\sigma_1(\hbar_1),\dots,\sigma_{k+1}(\hbar_{k+1}))\psi_k^{p}(\sigma_1(\hbar_1),\dots,\sigma_k(\hbar_k),\hbar_{k+1}))^{\frac{1}{p}}\Delta\hbar_{k+1}\right]^p$$

with indices $q = \frac{1}{p} > 1$ and $h = \frac{1}{1-p}$, and note that $\frac{1}{q} + \frac{1}{h} = 1$, where $q > 1$. Choose

$$\theta(\hbar) = \frac{C_{k+1}^{p}(\hbar_{k+1})\psi_k^{p}(\sigma_1(\hbar_1),\dots,\sigma_k(\hbar_k),\hbar_{k+1})}{\psi_{k+1}^{p(1-p)}(\sigma_1(\hbar_1),\dots,\sigma_{k+1}(\hbar_{k+1}))}$$

and

$$\vartheta(\hbar) = \xi_{k+1}^{1-p}(\hbar_{k+1})[\psi_{k+1}^{p(1-p)}(\sigma_1(\hbar_1),\dots,\sigma_k(\hbar_k),\sigma_{k+1}(\hbar_{k+1})].$$

$$\begin{aligned} &\left[\int_{a_{k+1}}^{\infty} \left(C_{k+1}^{p}(\hbar_{k+1})\psi_{k+1}^{p(p-1)}(\sigma_1(\hbar_1),\dots,\sigma_{k+1}(\hbar_{k+1}))\psi_k^{p}(\sigma_1(\hbar_1),\dots,\sigma_k(\hbar_k),\hbar_{k+1})\right)^{\frac{1}{p}}\Delta\hbar_{k+1}\right]^p \\ &\geq \frac{\int_{a_{k+1}}^{\infty} \psi_k^{p}(\sigma_1(\hbar_1),\dots,\sigma_k(\hbar_k),\hbar_{k+1})\xi_{k+1}^{1-p}(\hbar_{k+1})C_{k+1}^{p}(\hbar_{k+1})\Delta\hbar_{k+1}}{[I_{k+1}]^{1-p}}. \end{aligned} \tag{4.9}$$

Use (4.9) in (4.8) to get

$$[I_{k+1}]^p \geq p^p \frac{\int\limits_{a_{k+1}}^{\infty} \psi_k^p(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), \hbar_{k+1})\xi_{k+1}^{1-p}(\hbar_{k+1})C_{k+1}^p(\hbar_{k+1})\Delta\hbar_{k+1}}{[I_{k+1}]^{1-p}}.$$

After simplification, we get

$$I_{k+1} \geq p^p \int\limits_{a_{k+1}}^{\infty} \psi_k^p(\sigma_1(\hbar_1), \dots, \sigma_k(\hbar_k), \hbar_{k+1})\xi_{k+1}^{1-p}(\hbar_{k+1})C_{k+1}^p(\hbar_{k+1})\Delta\hbar_{k+1}. \tag{4.10}$$

Use (4.10) in (4.2) and apply Fubini's Theorem k times,

$$\begin{aligned} &\prod_{\iota=1}^{k+1} \int\limits_{a_\iota}^{\infty} \xi_\iota(\hbar_\iota)\psi_{k+1}^p(\sigma_1(\hbar_1), \dots, \sigma_{k+1}(\hbar_{k+1})\Delta\hbar_{k+1} \cdots \Delta\hbar_1 \\ &\geq p^p \int\limits_{a_{k+1}}^{\infty} \xi_{k+1}^{1-p}(\hbar_{k+1})C_{k+1}^p(\hbar_{k+1})\bigg\{\int\limits_{a_1}^{\infty} \cdots \int\limits_{a_k}^{\infty} \prod_{\iota=1}^{k} \xi_i(\hbar_i)\psi_k^p(\hbar_1, \dots, \hbar_{k+1})\Delta\hbar_k \cdots \Delta\hbar_1\bigg\}\Delta\hbar_{k+1}. \end{aligned} \tag{4.11}$$

Use induction hypothesis for $\psi_k(\hbar_1, \dots, \hbar_{k+1})$ instead of $\psi_k(\hbar_1, \dots, \hbar_k)$ for fixed $\hbar_{k+1} \in \mathbb{T}_{k+1}$ on right hand side of (4.11) and again exchange integrals k times to obtain

$$\begin{aligned} &\prod_{\iota=1}^{k+1} \int\limits_{a_\iota}^{\infty} \xi_\iota(\hbar_\iota)\psi_{k+1}^p(\sigma_1(\hbar_1), \dots, \sigma_{k+1}(\hbar_{k+1})\Delta\hbar_{k+1} \cdots \Delta\hbar_1 \\ &\geq p^{(k+1)p} \int\limits_{a_1}^{\infty} \cdots \int\limits_{a_{k+1}}^{\infty} \prod_{\iota=1}^{k+1} C_\iota^p(\hbar_\iota)\xi_\iota^{1-p}(\hbar_\iota)g^p(\hbar_1, \dots, \hbar_{k+1})\Delta\hbar_{k+1} \cdots \Delta\hbar_1. \end{aligned}$$

Hence the result is true for all positive integers n. □

Remark 5. As a special case of (4.1), when $\mathbb{T}_1 \times \cdots \times \mathbb{T}_n = \mathbb{R} \times \cdots \times \mathbb{R}$ in Theorem 4.1, we have the following integral inequality of Leindler type

$$\begin{aligned} &\int\limits_{a_1}^{\infty} \cdots \int\limits_{a_n}^{\infty} \prod_{\iota=1}^{n} \xi_\iota(\hbar_\iota)\psi_n^p(\hbar_1, \dots, \hbar_n)\mathrm{d}\hbar_n \cdots \mathrm{d}\hbar_1 \\ &\geq p^{np} \int\limits_{a_1}^{\infty} \cdots \int\limits_{a_n}^{\infty} \prod_{\iota=1}^{n} C_\iota^p(\hbar_\iota)\xi_\iota^{1-p}(\hbar_\iota)g^p(\hbar_1, \dots, \hbar_n)\mathrm{d}\hbar_n \cdots \mathrm{d}\hbar_1, \end{aligned}$$

where $p < 1$.

Remark 6. Assume that $\mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n = \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$, $p \leq 1$ and $a_1 = \cdots = a_n = 1$ in Theorem 4.1. Furthermore assume that

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \prod_{\iota=1}^{n} \xi_\iota^{1-p}(m_\iota)C_\iota^p(m_\iota)a_\iota^p(m_\iota)$$

is convergent. In this case the inequality (4.1) becomes the following discrete Leindler type inequality

$$\begin{aligned} &\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \prod_{\iota=1}^{n} \xi_\iota(m_\iota)\bigg(\sum_{k_\iota=1}^{m_\iota-1} g(k_1, \dots, k_n)\bigg)^p \\ &\geq p^{np} \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \prod_{\iota=1}^{n} \xi_\iota^{1-p}(m_\iota)C_\iota^p(m_\iota)g^p(m_1, \dots, m_n), \end{aligned}$$

where $C_\iota(m_\iota) = \sum\limits_{k_\iota=m_\iota}^{\infty} \xi_\iota(k_\iota)$.

Theorem 4.2. *Let $\iota = \{1, \dots, n\}$ and $\mathbb{T}_i$ be time scales with $a_\iota \in [0, \infty)_{\mathbb{T}_\iota}$. Define $\xi_\iota \colon \mathbb{T}_\iota \to \mathbb{R}_+$ such that*

$$D_\iota(\hbar_\iota) := \int\limits_{a_\iota}^{\hbar_\iota} \xi_\iota(\varsigma_\iota)\Delta\varsigma_\iota$$

exists and $D_\iota(\infty) = 0$. Furthermore let

$$\chi_n(\hbar_1, \dots, \hbar_n) := \int\limits_{\hbar_1}^{\infty} \cdots \int\limits_{\hbar_n}^{\infty} g(\varsigma_1, \dots, \varsigma_n)\Delta\varsigma_n \cdots \Delta\varsigma_1$$

for any $\hbar_\iota \in [a_\iota, \infty)_{\mathbb{T}_\iota}$ and $g \colon \mathbb{T}_1 \times \dots \times \mathbb{T}_n \to \mathbb{R}_+$. Then for $0 < p \le 1$,

$$\begin{aligned} &\prod_{\iota=1}^{n} \int\limits_{a_\iota}^{\infty} \xi_\iota(\hbar_\iota)\chi_n^p(\hbar_1, \dots, \hbar_n)\Delta\hbar_n \cdots \Delta\hbar_1 \\ &\ge p^{np} \prod_{\iota=1}^{n} \int\limits_{a_\iota}^{\infty} [D_\iota^{\sigma_\iota}(\hbar_\iota)]^p \xi_\iota^{1-p}(\hbar_\iota) g^p(\hbar_1, \dots, \hbar_n)\Delta\hbar_n \cdots \Delta\hbar_1. \end{aligned} \tag{4.12}$$

P r o o f. By using the mathematical induction method the result is true for $n = 1$ by [17: Theorem 2.4]. Let the inequality (4.12) holds for $1 \le n \le k$. For $n = k + 1$ left hand side of (4.12) becomes

$$\int\limits_{a_1}^{\infty} \cdots \int\limits_{a_k}^{\infty} \xi_1(\hbar_1) \cdots \xi_k(\hbar_k) \Bigg\{ \int\limits_{a_{k+1}}^{\infty} \xi_{k+1}(\hbar_{k+1})\chi_{k+1}^p(\hbar_1, \dots, \hbar_{k+1})\Delta\hbar_{k+1} \Bigg\} \Delta\hbar_k \cdots \Delta\hbar_1. \tag{4.13}$$

Denote $I_{k+1} = \int\limits_{a_{k+1}}^{\infty} \xi_{k+1}(\hbar_{k+1})\chi_{k+1}^p(\hbar_1, \dots, \hbar_{k+1})\Delta\hbar_{k+1}$.

Apply integration by parts (2.5), $\chi_{k+1}(\hbar_1, \dots, \hbar_k, \infty) = 0$ and $D_{k+1}(a_{k+1}) = 0$ to get

$$I_{k+1} = \int\limits_{a_{k+1}}^{\infty} -\frac{\partial}{\Delta_{k+1}\hbar_{k+1}} \chi_{k+1}^p(\hbar_1, \dots, \hbar_{k+1}) D_{k+1}^{\sigma_{k+1}}(\hbar_{k+1})\Delta\hbar_{k+1}. \tag{4.14}$$

Apply chain rule (2.3) to have

$$-\frac{\partial}{\Delta_{k+1}\hbar_{k+1}} \chi_{k+1}^p(\hbar_1, \dots, \hbar_{k+1}) = -p\chi_{k+1}^{p-1}(\hbar_1, \dots, \hbar_k, d_{k+1})[\chi_{k+1}(\hbar_1, \dots, \hbar_{k+1})]^{\Delta_{k+1}}, \tag{4.15}$$

where $d_{k+1} \in [\hbar_{k+1}, \sigma_{k+1}(\hbar_{k+1})]$. Since

$$\begin{aligned} \frac{\partial}{\Delta_{k+1}\hbar_{k+1}} \chi_{k+1}(\hbar_1, \dots, \hbar_{k+1}) &= \int\limits_{\hbar_1}^{\infty} \cdots \int\limits_{\hbar_k}^{\infty} \Bigg\{ \frac{\partial}{\Delta_{k+1}\hbar_{k+1}} \int\limits_{\hbar_{k+1}}^{\infty} g(\varsigma_1, \dots, \varsigma_{k+1})\Delta\varsigma_{k+1} \Bigg\} \Delta\varsigma_k \cdots \Delta\varsigma_1 \\ &= -\int\limits_{\hbar_1}^{\infty} \cdots \int\limits_{\hbar_k}^{\infty} g(\varsigma_1, \dots, \varsigma_k, \hbar_{k+1})\Delta\varsigma_k \cdots \Delta\varsigma_1, \end{aligned} \tag{4.16}$$

and $d_{k+1} \ge \hbar_{k+1}$. Hence (4.15) together with (4.16) implies

$$-[\chi_{k+1}^p(\hbar_1, \dots, \hbar_{k+1})]^{\Delta_{k+1}} \ge p \frac{\chi_k(\hbar_1, \dots, \hbar_{k+1})}{\chi_{k+1}^{1-p}(\hbar_1, \dots, \hbar_{k+1})}, \tag{4.17}$$

where $\chi_k(\hbar_1,\dots,\hbar_{k+1}) = \int_{\hbar_1}^{\infty}\cdots\int_{\hbar_k}^{\infty} g(\varsigma_1,\dots,\varsigma_k,\hbar_{k+1})\Delta\varsigma_k\cdots\Delta\varsigma_1$ and $0<p\le 1$. Therefore use (4.17) in (4.14) and take power$p$ on both sides to get

$$I_{k+1}^p \ge p^p\left(\int_{a_{k+1}}^{\infty}\left([(D_{k+1})^{\sigma_{k+1}}(\hbar_{k+1})]^p\frac{\chi_k^p(\hbar_1,\dots,\hbar_{k+1})}{\chi^{p(1-p)}(\hbar_1,\dots,\hbar_{k+1})}\right)^{\frac{1}{p}}\Delta\hbar_{k+1}\right)^p. \tag{4.18}$$

By applying Hölder's inequality and after making simplifications, it can be viewed that

$$I_{k+1} \ge p^p\int_{a_{k+1}}^{\infty}\chi_k^p(\hbar_1,\dots,\hbar_{k+1})\xi_{k+1}^{1-p}(\hbar_{k+1})[(D_{k+1})^{\sigma_{k+1}}(\hbar_{k+1})]^p\Delta\hbar_{k+1}. \tag{4.19}$$

Substitute (4.19) in (4.13) and exchange integrals k times by using Fubini's Theorem on right hand side of (4.13),

$$\begin{aligned}&\int_{a_1}^{\infty}\cdots\int_{a_{k+1}}^{\infty}\prod_{\iota=1}^{k+1}\xi_\iota(\hbar_\iota)\chi_{k+1}^p(\hbar_1,\dots,\hbar_{k+1})\Delta\hbar_{k+1}\cdots\Delta\hbar_1 \\ &\ge p^p\int_{a_{k+1}}^{\infty}\xi_{k+1}^{1-p}(\hbar_{k+1})[D_{k+1}^{\sigma_{k+1}}(\hbar_{k+1})]^p\left(\int_{a_1}^{\infty}\cdots\int_{a_k}^{\infty}\prod_{\iota=1}^{k}\xi_\iota(\hbar_\iota)\chi_k^p(\hbar_1,\dots,\hbar_{k+1})\Delta\hbar_k\cdots\Delta\hbar_1\right)\Delta\hbar_{k+1}.\end{aligned} \tag{4.20}$$

Use the induction hypothesis for $\chi_k(\hbar_1,\dots,\hbar_{k+1})$ instead of $\chi_k(\hbar_1,\dots,\hbar_k)$ for fixed $\hbar_{k+1}\in\mathbb{T}_{k+1}$ on right hand side of (4.20) and apply Fubini's Theorem k times to get

$$\begin{aligned}&\prod_{\iota=1}^{k+1}\int_{a_\iota}^{\infty}\xi_\iota(\hbar_\iota)\chi_{k+1}^p(\hbar_1,\dots,\hbar_{k+1})\Delta\hbar_{k+1}\cdots\Delta\hbar_1 \\ &\ge p^{(k+1)p}\prod_{\iota=1}^{k+1}\int_{a_\iota}^{\infty}[D_\iota^{\sigma_\iota}(\hbar_\iota)]^p\xi_\iota^{1-p}(\hbar_\iota)g^p(\hbar_1,\dots,\hbar_{k+1})\Delta\hbar_{k+1}\cdots\Delta\hbar_1.\end{aligned}$$ □

Remark 7. Assume that $\mathbb{T}_i=\mathbb{R}$ for all $\iota=\{1,2,\dots,n\}$ in Theorem 4.2 and $p\le 1$. In this case, (4.12) becomes the integral inequality of Leindler type as follows

$$\begin{aligned}&\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\prod_{\iota=1}^{n}\xi_\iota(\hbar_\iota)\left(\int_{\hbar_1}^{\infty}\cdots\int_{\hbar_n}^{\infty}g(\varsigma_1,\dots,\varsigma_\iota)\mathrm{d}\varsigma_n\cdots\mathrm{d}\varsigma_1\right)^p\mathrm{d}\hbar_n\cdots\mathrm{d}\hbar_1 \\ &\ge p^{np}\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\prod_{\iota=1}^{n}\xi_i^{1-p}(\hbar_\iota)(D_\iota(\hbar_\iota))^p g^p(\hbar_1,\dots,\hbar_n)\mathrm{d}\hbar_n\cdots\mathrm{d}\hbar_1,\end{aligned}$$

where

$$D_\iota(\hbar_\iota)=\int_{a_1}^{\hbar_1}\cdots\int_{a_n}^{\hbar_n}\xi_\iota(\varsigma_\iota)\Delta\varsigma_n\cdots\Delta\varsigma_1.$$

Remark 8. Assume that $\mathbb{T}_\iota=\mathbb{N}$, $a_i=1$ for all $\iota=\{1,\dots,n\}$ in Theorem 4.2, furthermore assume that

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\prod_{\iota=1}^{n}\xi_\iota^{1-p}(m_\iota)\left(\sum_{k_\iota=1}^{m_\iota-1}\xi_\iota(k_\iota)\right)^p a_\iota^p(m_\iota)$$

is convergent. In this case (4.12) becomes the discrete Leindler type inequality

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\prod_{\iota=1}^{n}\xi_\iota(m_\iota)\Big(\sum_{k_\iota=m_\iota}^{\infty} g(k_1,\dots,k_n)\Big)^p$$
$$\geq p^{np}\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\prod_{\iota=1}^{n}\xi_\iota^{1-p}(m_\iota)\Big(\sum_{k_\iota=1}^{m_\iota-1}\xi_\iota(k_\iota)\Big)^p g^p(m_1,\dots,m_n).$$

Some special cases

***Example* 1.** If we choose $\xi_\iota(\hbar_\iota)=\frac{1}{\hbar_\iota(\sigma(\hbar_\iota))}$, $a_\iota<p$ for all $\iota\in\{1,\dots,n\}$ and $g(\hbar_1,\dots,\hbar_n)=\prod_{\iota=1}^{n}\frac{1}{\sigma(\hbar_\iota)}$ in Theorem 3.1, then (3.3) takes the form

$$\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\prod_{\iota=1}^{n}\frac{1}{\hbar_\iota(\sigma(\hbar_\iota))}\Big(\int_{a_1}^{\hbar_1}\cdots\int_{a_n}^{\hbar_n}\prod_{\iota=1}^{n}\frac{1}{\sigma(\varsigma_\iota)}\Delta\varsigma_n\cdots\Delta\varsigma_1\Big)^p\Delta\hbar_n\cdots\Delta\hbar_1\leq p^{np}\prod_{\iota=1}^{n}\frac{1}{a_\iota}. \tag{4.21}$$

Additionally, if $\mathbb{T}_\iota=\mathbb{R}$, then (4.21) is reduced to the following:

$$\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\prod_{\iota=1}^{n}\frac{1}{{\hbar_\iota}^2}\Big(\prod_{\iota=1}^{n}\ln\Big(\frac{\hbar_\iota}{a_\iota}\Big)\Big)^p\Delta\hbar_n\cdots\Delta\hbar_1\leq p^{np}\prod_{\iota=1}^{n}\frac{1}{a_\iota}.$$

If $\mathbb{T}_\iota=\mathbb{N}$ then (4.21) reduces to the following:

$$\sum_{\hbar_1=a_1}^{\infty}\cdots\sum_{\hbar_n=a_n}^{\infty}\prod_{\iota=1}^{n}\frac{1}{\hbar_\iota(\hbar_\iota+1)}\Big(\sum_{\varsigma_1=a_1}^{\hbar_1}\cdots\sum_{\varsigma_n=a_n}^{\hbar_n}\prod_{\iota=1}^{n}\frac{1}{(\varsigma_\iota+1)}\Big)^p\leq p^{np}\prod_{\iota=1}^{n}\frac{1}{a_\iota}.$$

***Example* 2.** If we choose $\xi_\iota(\hbar_\iota)=1$ for all $\iota\in\{1,\dots,n\}$ and $g(\hbar_1,\dots,\hbar_n)=\prod_{\iota=1}^{n}\frac{1}{(\sigma(\hbar_\iota))^{\frac{p+1}{p}}\hbar_\iota^{\frac{1}{p}}}$ in Theorem 3.2, then (3.17) takes the form

$$\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\Big(\int_{\hbar_1}^{\infty}\cdots\int_{\hbar_n}^{\infty}\prod_{\iota=1}^{n}\frac{1}{(\sigma(\varsigma_\iota))^{\frac{p+1}{p}}\varsigma_\iota^{\frac{1}{p}}}\Delta\varsigma_n\cdots\Delta\varsigma_1\Big)^p\Delta\hbar_n\cdots\Delta\hbar_1\leq p^{np}\prod_{\iota=1}^{n}\frac{1}{a_\iota}. \tag{4.22}$$

If $\mathbb{T}_\iota=\mathbb{R}$, then (4.22) is reduced to the following:

$$\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\Big(\int_{\hbar_1}^{\infty}\cdots\int_{\hbar_n}^{\infty}\prod_{\iota=1}^{n}\frac{1}{(\varsigma_\iota+1)^{\frac{p+1}{p}}\varsigma_\iota^{\frac{1}{p}}}\mathrm{d}\varsigma_n\cdots\mathrm{d}\varsigma_1\Big)^p\mathrm{d}\hbar_n\cdots\mathrm{d}\hbar_1\leq p^{np}\prod_{\iota=1}^{n}\frac{1}{a_\iota}. \tag{4.23}$$

If $\mathbb{T}_\iota=\mathbb{N}$, then (4.22) is reduced to the following:

$$\sum_{a_1}^{\infty}\cdots\sum_{a_n}^{\infty}\Big(\sum_{\hbar_1}^{\infty}\cdots\sum_{\hbar_n}^{\infty}\prod_{\iota=1}^{n}\frac{1}{(\varsigma_\iota+1)^{\frac{p+1}{p}}\varsigma_\iota^{\frac{1}{p}}}\Big)^p\leq p^{np}\prod_{\iota=1}^{n}\frac{1}{a_\iota}. \tag{4.24}$$

Remark 9. If $0\leq p<1$ in Example 1 and Example 2, then these are converted to examples for Theorem 4.1 and Theorem 4.2, respectively.

It is also possible to prove these inequalities for nabla integrals. For example, for nabla integrals, Theorem 3.1 takes the following form:

Theorem 4.3. *Consider $\iota\in\{1,\dots,n\}$ and time scales $\mathbb{T}_\iota$ for $a_\iota\in[0,\infty)_{\mathbb{T}_\iota}$. Take $\xi_\iota\colon\mathbb{T}_\iota\to\mathbb{R}_+$, such that*

$$A_\iota(\hbar_\iota):=\int_{\hbar_\iota}^{\infty}\xi_\iota(\varsigma_\iota)\nabla\varsigma_\iota, \tag{4.25}$$

exists, where $A_\iota(\infty) = 0$ *for all* ι.
Consider $g\colon \mathbb{T}_1 \times \cdots \times \mathbb{T}_n \to \mathbb{R}_+$ *such that*

$$\phi_n(\hbar_1, \ldots, \hbar_n) := \int_{a_1}^{\hbar_1} \cdots \int_{a_n}^{\hbar_n} g(\varsigma_1, \ldots, \varsigma_n) \nabla\varsigma_n \cdots \nabla\varsigma_1. \tag{4.26}$$

Then for $p > 1$,

$$\begin{aligned} &\int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} \xi_1(\hbar_1) \cdots \xi_n(\hbar_n) \phi_n^p((\hbar_1), \ldots, (\hbar_n)) \nabla\hbar_n \cdots \nabla\hbar_1 \\ &\le p^{np} \int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} \prod_{\iota=1}^{n} A_\iota^p(\rho(\hbar_\iota)) \xi_\iota^{1-p}(\hbar_\iota) g^p(\hbar_1, \ldots, \hbar_n) \nabla\hbar_n \cdots \nabla\hbar_1, \end{aligned} \tag{4.27}$$

where $n \ge 1$.

It can be proved for $n = 1$ by using integration by parts formula, chain rule, and Holder's inequality for nabla integrals. Further, applying the same technique of induction principle, we are able to get the required result for multiple integrals.

5. Concluding remark

Some Hardy Leindler inequalities and their converses for multiple delta integrals on time scales are proved in the paper. The proved results are also discussed in continuous and discrete calculus. Some particular cases include existing inequalities of Hardy time from literature. Further to check validity of proved results some examples are provided by choosing special functions and special time scales. Finally, it is discussed the existence of main inequalities by using nabla calculus instead of delta calculus.

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