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ON RECURRENCES IN GENERALIZED ARITHMETIC TRIANGLE

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ABSTRACT. In the present paper, we consider the generalized arithmetic triangle called GAT which is structurally identical to Pascal's triangle for which we keep the Pascal's rule of addition and we replace both legs by two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ with $a_0 = b_0 = \Omega$. Our goal is to describe the recurrence relation associated to the sum of elements lying along a finite ray in this triangle. As consequences, we obtain some combinatorial properties and we establish that the sum of elements lying along a main rising diagonal is a convolution of generalized Fibonacci sequence and another sequence which one will determine. We also precise the corresponding generating function. Further, we establish some nice identities by using the Morgan-Voyce phenomenon. Finally, we generalize the Golden ratio.

1. Introduction

Pascal's triangle appears in numerous situations and has many intriguing properties. Over the years, a number of Pascal-like triangular arrays have been developed. Koshy [20, 21] and Belbachir and Szalay [7] collected various generalizations. The generalization of Ensley [13] called *Generalized Arithmetic Triangle,* the Ensley's GAT for short, changed the legs called generator sequences to an arbitrary sequences of real numbers $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$. Then the GAT depends on the generator sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$. Ensley [13] studied the Fibonacci triangle, the case where $a_n = b_n = F_n$, the Fibonacci numbers. The cases, where $(a_n, b_n) = (0,\ F_n)$, $(a_n, b_n) = \left(0, \frac{1}{n}\right)$ and $(a_n, b_n) = (F_{2n-1}, F_{n-1})$, were treated by Dil and Mezo [12]. In [8], we can also find an interesting description of Fibonacci and Lucas triangles. As Fibonacci numbers can be generated from the rising diagonals of Pascal's triangle, Feinberg, see [14], was motivated to develop a similar triangle that would generate Lucas numbers from its rising diagonal. This triangle is Lucas triangle which is the particular case of Ensley's GAT when $a_n = 1$ and $b_n = 2$. For other recent works on the subject, we refer to references [2, 3]

In the present paper, we consider similar construction of Ensley's GAT and our aim is to determine the recurrence relation associated to the sum of elements lying along a finite ray. Then, we establish the corresponding generating function. Also, we study the Morgan-Voyce phenomenon that leads us to establish new identities. Finally, we provide some interesting applications, among

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which one will establish that the sum of elements lying along a main rising diagonal is a convolution of generalized Fibonacci sequence and another sequence which one will determine.

1.1. Ensley's Generalized Arithmetic Triangle

The following definition is given in [13].

Definition 1. Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be real sequences such that $a_0 = b_0 = \Omega$. The Ensley's *Generalized Arithmetic Triangle for* $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ contains elements $C(n,k)$ in the n^{th} row and k^{th} column defined as follows: $C(n,0) = a_n$, $C(n,n) = b_n$ and for $n > k > 0$,

$$C(n,k) = C(n-1,k) + C(n-1,k-1).$$

Belbachir and Szalay, see [7], considered the Ensley's GAT with suitable change of the rule of addition and the generator sequences. It is defined as follows.

Definition 2. Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two real sequences, A and B two real numbers. The GAT contains elements $\begin{bmatrix} n \\ k \end{bmatrix}$ in the n^{th} row and k^{th} column defined for $n \geq 2$, as follows: $\begin{bmatrix} n \\ 0 \end{bmatrix} = A^n a_n$, $\begin{bmatrix} n \\ n \end{bmatrix} = B^n b_n$ and for $1 \leq k \leq n-1$,

$$\begin{bmatrix} n \\ k \end{bmatrix} = A \begin{bmatrix} n-1 \\ k \end{bmatrix} + B \begin{bmatrix} n-1 \\ k-1 \end{bmatrix},$$

with the convention $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ whenever $k < 0$ or $k > n$ and $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is denoted by Ω.

The convention certainly works also for the binomial coefficients: $\binom{n}{k} = 0$ whenever $k < 0$ or $k > n$.

In this paper, we keep the rule of addition in the triangle defined by Belbachir and Szalay [7] and the generator sequences of Ensley's GAT. We call this triangle, generalized arithmetic triangle (GAT for short). It is defined as follows.

Definition 3. Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two real sequences, A and B two numbers. The GAT contains elements $\left\langle {n \atop k} \right\rangle$ in the n^{th} row and k^{th} column defined for $n \geq 2$, as follows: $\left\langle {n \atop 0} \right\rangle = a_n$, $\left\langle {n \atop n} \right\rangle = b_n$ and for $1 \leq k \leq n-1$,

$$\left\langle {n \atop k} \right\rangle = A \left\langle {n-1 \atop k} \right\rangle + B \left\langle {n-1 \atop k-1} \right\rangle,$$

with the convention $\left\langle {n \atop k} \right\rangle = 0$ whenever $k < 0$ or $k > n$ and $\left\langle {0 \atop 0} \right\rangle$ is denoted by Ω.

Both definitions given in [13] and [7] are special cases of the above definition.

Figure 1 shows the rows 0 to 4 of the GAT generated by the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ and the numbers A and B.

Ω				
a_1	b_1			
a_2	$Ba_1 + Ab_1$	b_2		
a_3	$B(Aa_1 + a_2) + A^2 b_1$	$B^2 a_1 + A(Bb_1 + b_2)$	b_3	
a_4	$B(A^2 a_1 + Aa_2 + a_3) + A^3 b_1$	$B^2(2Aa_1 + a_2) + A^2(2Bb_1 + b_2)$	$B^3 a_1 + A(B^2 b_1 + Bb_2 + b_3)$	

Figure 1. First rows of the GAT.

Clearly, a GAT depends only on the generator sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ and the numbers A and B. Therefore, when necessary, we will denote by $\left\langle {n \atop k} \right\rangle_{(A,\, B,\, a_n,\, b_n)}$ ($\left\langle {n \atop k} \right\rangle$ for short) to emphasize the parameters of the triangle.

1.2. GAT'S elements and binomial coefficients

The aim of this section is to evaluate $\left\langle{n \atop k}\right\rangle$ the entry k of row n, of a GAT in terms of the sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, the values A and B and the binomial coefficients $\binom{n}{k}$ of Pascal's triangle. It is established in [7] with a slight modification, that:

Theorem 1.1. *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences and A and B two numbers. The corresponding GAT has as entry the number $\left\langle{n \atop k}\right\rangle$ given for $(n,k)\neq(0,0)$, by the expression*

$$\left\langle{n \atop k}\right\rangle = A^{n-k}B^k\left(\sum_{i=1}^{n-k}\binom{n-1-i}{k-1}\frac{a_i}{A^i}+\sum_{i=1}^{k}\binom{n-1-i}{k-i}\frac{b_i}{B^i}\right)+\left\lfloor\frac{n-k}{n}\right\rfloor a_n+\left\lfloor\frac{k}{n}\right\rfloor b_n.$$

As $\sum\limits_{i=1}^{n-k}\binom{n-1-i}{k-1}=\binom{n-1}{k}$ for $k\geq 1$, $n\geq 1$ and $\sum\limits_{i=1}^{k}\binom{n-1-i}{k-i}=\binom{n-1}{k-1}$ for $k<n$, $n\geq 1$, we get the following.

Corollary 1.1.1. *For $a_n=\Omega A^n$ and a given sequence $(b_n)_n$, the corresponding GAT has as entry the number $\left\langle{n \atop k}\right\rangle$ given for $(n,k)\neq(0,0)$, by the expression*

$$\left\langle{n \atop k}\right\rangle = A^{n-k}B^k\left(\Omega\binom{n-1}{k}+\sum_{i=1}^{k}\binom{n-1-i}{k-i}\frac{b_i}{B^i}\right)+\left\lfloor\frac{k}{n}\right\rfloor b_n.$$

Corollary 1.1.2. *For $b_n=\Omega B^n$ and a given sequence $(a_n)_n$, the corresponding GAT has as entry the number $\left\langle{n \atop k}\right\rangle$ given for $(n,k)\neq(0,0)$, by the expression*

$$\left\langle{n \atop k}\right\rangle = A^{n-k}B^k\left(\sum_{i=1}^{n-k}\binom{n-1-i}{k-1}\frac{a_i}{A^i}+\Omega\binom{n-1}{k-1}\right)+\left\lfloor\frac{n-k}{n}\right\rfloor a_n.$$

Corollary 1.1.3. *For $(a_n,\ b_n)=(\Omega A^n,\ \Omega B^n)$, the corresponding GAT has as entry the number $\left\langle{n \atop k}\right\rangle$ given for $(n,k)\neq(0,0)$, by the expression*

$$\left\langle{n \atop k}\right\rangle = \Omega A^{n-k}B^k\binom{n}{k}.$$

2. Quasi-symmetry and duality in a GAT

Note that in a GAT, we generally lose the symmetry. In fact, one observes that in general

$$\left\langle{n \atop k}\right\rangle_{(A,\ B,\ a_n,\ b_n)} \neq \left\langle{n \atop n-k}\right\rangle_{(A,\ B,\ a_n,\ b_n)}.$$

However one has the quasi-symmetry property.

Quasi-symmetry property. Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences, A and B two numbers. Then

$$\left\langle{n \atop k}\right\rangle_{(A,\ B,\ a_n,\ b_n)} = \left\langle{n \atop n-k}\right\rangle_{(B,\ A,\ b_n,\ a_n)}. \tag{2.1}$$

***Example* 1.** Let $A=B=1$. For $a_n=b_n=F_n$, where $(F_n)_n$ is the Fibonacci sequence, we obtain the symmetrical GAT given in Table 1. Whereas, for $a_n=F_n, b_n=J_n$, where $(J_n)_n$ is the Jacobsthal sequence, we obtain the asymmetrical GAT given in Table 2. The quasi-symmetry GAT are illustrated in Tables 3 and 4.

TABLE 1. The $(1, 1, F_n, F_n)$ GAT.

Ω						
1	1					
1	2	1				
2	3	3	2			
3	5	6	5	3		
5	8	11	11	8	5	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

TABLE 2. The $(1, 1, F_n, J_n)$ GAT.

Ω						
1	1					
1	2	1				
2	3	3	3			
3	5	6	6	5		
5	8	11	12	11	11	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

TABLE 3. The $(1, 1, F_{n+1}, 1)$ GAT.

Ω						
1	1					
2	2	1				
3	4	3	1			
5	7	7	4	1		
8	12	14	11	5	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

TABLE 4. The $(1, 1, 1, F_{n+1})$ GAT.

Ω						
1	1					
1	2	2				
1	3	4	3			
1	4	7	7	5		
1	5	11	14	12	8	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Duality. Let $n \in \mathbb{N}$, $r \in \mathbb{N}$, $q \in \mathbb{Z}$, $p = 0, \dots, r-1$ such that $r + q > 0$. Then the grid point (n, p) and the direction (r, q) define a transversal ray of the GAT which contains the elements

$$\left\langle {n - qk \atop p + rk} \right\rangle, \qquad k = 0, 1, \dots, \left\lfloor \frac{n-p}{r+q} \right\rfloor.$$

For a fixed direction (r, q) and a fixed value of p, the sequence $\left(G_n^{(r,q,p)}\right)_{n \geq 0}$ defined by

$$G_{n+1}^{(r,q,p)} := \sum_{k=0}^{\lfloor (n-p)/(r+q) \rfloor} \left\langle {n - qk \atop p + rk} \right\rangle,$$

and $G_0^{(r,q,p)} = 0$, constitutes the sum of elements lying on the corresponding ray in the GAT. Note that (because a sum over empty set is zero)

$$G_1^{(r,q,p)} = \cdots = G_p^{(r,q,p)} = 0 \quad \text{and} \quad G_{n+1}^{(r,q,p)} = \left\langle {n \atop p} \right\rangle \qquad \text{for } p \leq n < p + r + q. \tag{2.2}$$

For more details concerning directions in Pascal's triangle, see [6] and [7]. In the following, using the quasi-symmetry property, we prove that there exists a grid point (m, p') and a direction $(r', -q)$, $q > 0$ which define a transversal ray of GAT containing the same elements in reverse order. More precisely, we have the result.

PROPOSITION 2.1. *Let $r \in \mathbb{N}$, $q < 0$, $p = 0, \dots, r-1$ such that $r + q > 0$. Then*

$$\sum_{k=0}^{\lfloor (n-p)/(r+q) \rfloor} \left\langle {n - qk \atop p + rk} \right\rangle_{(A,\ B,\ a_n,\ b_n)} = \sum_{k=0}^{\left\lfloor (n-p')/(r'+q') \right\rfloor} \left\langle {m - q'k \atop p' + r'k} \right\rangle_{(B,\ A,\ b_n,\ a_n)},$$

with $m = n - q \lfloor (n-p)/(r+q) \rfloor$, $q' = -q$, $r' = r + q$, $p' = n - p - (r+q) \lfloor (n-p)/(r+q) \rfloor$.

P r o o f. We have $r' + q' > 0$ and $p' < r'$. It is easy to see that

$$\left\lfloor \frac{(m-p')}{(r'+q')} \right\rfloor = \left\lfloor \frac{1}{r}\left(p + r\left\lfloor \frac{(n-p)}{(r+q)} \right\rfloor\right) \right\rfloor = \left\lfloor (p/r) + \left\lfloor \frac{(n-p)}{(r+q)} \right\rfloor \right\rfloor .$$

Since $p < r$, $\lfloor (m-p')\,/\,(r'+q') \rfloor = \lfloor (n-p)/\,(r+q) \rfloor$ and

$$\begin{aligned}\sum_{k=0}^{\lfloor (n-p')/(r'+q') \rfloor} \left\langle {m-q'k \atop p'+r'k} \right\rangle_{(B,A,b_n,a_n)} &= \sum_{k=0}^{\lfloor (n-p)/(r+q) \rfloor} \left\langle {n-q(\lfloor (n-p)/(r+q) \rfloor - k) \atop n-p-(r+q)(\lfloor (n-p)\,/(r+q) \rfloor - k)} \right\rangle_{(B,A,b_n,a_n)} \\ &= \sum_{l=0}^{\lfloor (n-p)/(r+q) \rfloor} \left\langle {n-ql \atop n-p-(r+q)l} \right\rangle_{(B,A,b_n,a_n)},\end{aligned}$$

with $l = \lfloor (n-p)/(r+q) \rfloor - k$.

We conclude by (2.1). □

According to the quasi-symmetry property in the GAT, we will consider in the sequel only directions (r, q) with $q \geq 0$.

3. Sum of elements lying along a finite ray in GAT

In what follows, we shall use the following notation given in [15]. Let S be a statement that can be true or false. The bracketed notation $[S]$ stands for 1 if S is true, 0 otherwise.

Ensley [13] discussed some GAT's properties generated by the Fibonacci sequence and $A = B = 1$. He showed that the sum s_n of the elements in the n^{th} row

$$s_n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{(1,\,1,\,F_n,\,F_n)}$$

satisfy the recurrence relation

$$s_n = s_{n-1} + s_{n-2} + 2^{n-1}.$$

Moreover (see for instance, [7]), the ascending diagonal sum

$$d_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \left\langle {n-k \atop k} \right\rangle_{(1,\,1,\,F_n,\,F_n)}$$

satisfies the recursion formula

$$d_n = d_{n-1} + d_{n-2} + F_{n-2} + F_{n/2-2}\,[n \text{ even}]\,.$$

In this section, we consider the GAT as defined in Definition 3. Our aim is to determine the sum of elements located in a finite ray of the GAT generated by any sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$. The case $a_n = A^n$ and $b_n = B^n$ was treated in [7]. One can easily show the following.

Lemma 3.1. *For $p \geq 0$ and $\alpha = (n-p)/(r+q)$, we have*

$$\sum_{k=0}^{\lfloor \alpha \rfloor} \left\lfloor \frac{n-p-(r+q)k}{n-qk} \right\rfloor a_{n-qk} = a_n\,[p=0]\,,$$

$$\sum_{k=0}^{\lfloor \alpha \rfloor} \left\lfloor \frac{p+rk}{n-qk} \right\rfloor b_{n-qk} = b_{p+\alpha r}\,[\alpha \in \mathbb{N}]\,.$$

Recall that the bracketed notation $[S]$ stands for 1 if S is true, 0 otherwise.
Theorem 1.1 and Lemma 3.1 allow us to state the following result which gives the expression of $G_{n+1}^{(r,p,q)}$ in terms of $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, A, B and binomial coefficients.

Proposition 3.1. *For $p \geq 0$ and $\alpha = (n-p)/(r+q)$, we have*

$$G_{n+1}^{(r,p,q)} = \sum_{k=0}^{\lfloor \alpha \rfloor} A^{n-p-(r+q)k} B^{p+rk} \left(\sum_{i=1}^{n-p-(r+q)k} \binom{n-qk-1-i}{p+rk-1} \frac{a_i}{A^i} + \sum_{i=1}^{p+rk} \binom{n-qk-1-i}{p+rk-i} \frac{b_i}{B^i} \right) + a_n \left\lfloor \frac{n-p}{n} \right\rfloor + b_{p+\alpha r} [\alpha \in \mathbb{N}].$$

Remark 1. Note that for $p = 0$, the summation can starts at $k = 1$ as $\binom{n}{-1} = 0$.

Remark 2. Formula (3) in [6] is a special case of Proposition 3.1. Indeed, when $a_n = A^n$ and $b_n = B^n$ for $n \geq 0$, we get

$$G_{n+1}^{(r,p,q)} = \sum_{k=0}^{\lfloor \alpha \rfloor} \binom{n-qk}{p+rk} A^{n-p-(r+q)k} B^{p+rk}. \tag{3.1}$$

To avoid some complication, we agree that $a_{-n} = b_{-n} = 0$ for $n \geq 1$.

4. Recurrence relation associated to rays in the GAT

We deal now with the main result, establishing a recurrence relation associated to the sum of elements located in a finite ray of the GAT generated by any sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$. According to Proposition 2.1, we have to consider only the case $q > 0$. First, we set, for $m \geq 1$,

$$\lambda_0^{(m)} := (-1)^m \binom{r-1}{m},$$

and for $s \geq 1$,

$$\lambda_s^{(m)} := (-1)^{m-1} \binom{r-s-1}{m-1}.$$

Note that the $\lambda_s^{(m)}$'s, $1 \leq s \leq r-m$, are the binomial coefficients given by the extension of Pascal's triangle to negative rows. By induction, we have the following.

Lemma 4.1. *For $r \geq 2$ and $1 \leq m \leq r-1$,*

$$\lambda_0^{(m)} = -\sum_{s=1}^{r-m} \lambda_s^{(m)}.$$

Let now $\alpha_j := \frac{n-1-j-p}{r+q}$, $0 \leq j \leq r-1$. For $\alpha_j \in \mathbb{N}$ and $r \geq 1$ we define

$$M_0 := -b_{p+r\alpha_0} + B^r b_{p+r\alpha_0-r}, \tag{4.1}$$

and for $j \geq 1$,

$$M_j := -A^j \left(\lambda_0^{(j)} b_{p+r\alpha_j} + \lambda_1^{(j)} b_{p+r\alpha_j-1} B + \cdots + \lambda_{r-j}^{(j)} b_{p+r\alpha_j-r+j} B^{r-j} \right). \tag{4.2}$$

Let us consider the expression of $G_n^{(r,q,p)}$ as given in Proposition 3.1. We denote by $\left(G_n^{(p)}\right)_n$ the sequence $\left(G_n^{(r,q,p)}\right)_n$. We are now able to give our main result.

Theorem 4.1. *The terms of the sequence $\left(G_n^{(p)}\right)_n$ satisfy the recurrence relation*

$$\sum_{j=0}^{r}(-A)^j\binom{r}{j}G_{n-j}^{(p)}-B^p\sum_{j=0}^{r}(-A)^j\binom{r-p}{j}a_{n-p-j-1}+\sum_{j=0}^{r-1}M_j\left[\alpha_j\in\mathbb{N}\right]=B^rG_{n-q-r}^{(p)}.$$

Remark 3. For the sake of clarity, the proof of Theorem 4.1 will be given at the end of this section in the case $p=0$. A similar proof can easily be established in the case $p\geq 1$.

From Lemma 4.1 and Theorem 4.1, we can deduce the following.

Corollary 4.1.1. *For $b_n=B^nb$, $b\in\mathbb{R}$, the terms of the sequence $\left(G_n^{(p)}\right)_n$ satisfy the recurrence relation*

$$\sum_{j=0}^{r}(-A)^j\binom{r}{j}G_{n-j}^{(p)}-B^p\sum_{j=0}^{r-p}(-A)^j\binom{r-p}{j}a_{n-p-j-1}=B^rG_{n-q-r}^{(p)}.$$

Corollary 4.1.2. *For $a_n=A^na$, $a\in\mathbb{R}$, the terms of the sequence $\left(G_n^{(p)}\right)_n$ satisfy the recurrence relation*

$$\sum_{j=0}^{r}(-A)^j\binom{r}{j}G_{n-j}^{(p)}+\sum_{j=0}^{r-1}M_j\left[\alpha_j\in\mathbb{N}\right]=B^rG_{n-q-r}^{(p)}.$$

Remark 4. [6: Theorem 1] is a special case of Theorem 4.1 with $a_n=A^n$ and $b_n=B^n$, $n\geq 0$.

As the last result of this section, we express the sequence $\left(G_n^{(0)}\right)_n$ as the convolution of two well determined sequences.

Let $(X_n)_n$ be a sequence defined for $n\geq 0$ by

$$X_n:=\sum_{k=0}^{\lfloor n/(r+q)\rfloor}A^{n-(r+q)k}B^{rk}\binom{n+r-1-qk}{r-1+rk},$$

and $(Y_n)_n$ a sequence defined by: $Y_0:=0$ and

$$Y_n:=\begin{cases}\sum\limits_{j=0}^{n-1}(-A)^j\binom{r}{j}a_{n-j-1}, & \text{for } 1\leq n<r+q,\\ \sum\limits_{j=0}^{r}(-A)^j\binom{r}{j}a_{n-j-1}-\sum\limits_{j=0}^{r-1}M_j\left[\alpha_j\in\mathbb{N}\right], & \text{for } n\geq r+q.\end{cases}$$

Theorem 4.2. *For $n\geq 1$, we have*

$$G_n^{(0)}=\sum_{k=0}^{n-1}X_kY_{n-k}.$$

P r o o f. We use the induction on the non negative integer n.
For $n\geq r+q$, assume that $G_m^{(0)}=\sum\limits_{k=0}^{m-1}X_kY_{m-k}$, $m\leq n-1$. We have from Theorem 4.1,

$$\begin{aligned}G_n^{(0)}&=-\sum_{j=1}^{r}(-A)^j\binom{r}{j}G_{n-j}^{(0)}+B^rG_{n-q-r}^{(0)}+Y_n\\&=-\sum_{j=1}^{r}(-A)^j\binom{r}{j}\sum_{k=0}^{n-j-1}X_kY_{n-j-k}+B^r\sum_{k=0}^{n-r-q-1}X_kY_{n-r-q-k}+Y_n.\end{aligned}$$

Letting $s = k + j$ and $t = k + r + q$, we get

$$G_n^{(0)} = -\sum_{j=1}^{r} (-A)^j \binom{r}{j} \sum_{s=j}^{n-1} X_{s-j} Y_{n-s} + B^r \sum_{t=r+q}^{n-1} X_{t-r-q} Y_{n-t} + Y_n.$$

Since $X_n = 0$ for $n < 0$, we can write

$$\begin{aligned} G_n^{(0)} &= -\sum_{j=1}^{r} (-A)^j \binom{r}{j} \sum_{s=1}^{n-1} X_{s-j} Y_{n-s} + B^r \sum_{t=1}^{n-1} X_{t-r-q} Y_{n-t} + Y_n \\ &= \sum_{s=1}^{n-1} Y_{n-s} \Big(-\sum_{j=1}^{r} (-A)^j \binom{r}{j} X_{s-j} \Big) + B^r \sum_{t=1}^{n-1} X_{t-r-q} Y_{n-t} + Y_n \\ &= \sum_{s=1}^{n-1} Y_{n-s} \Big(-\sum_{j=1}^{r} (-A)^j \binom{r}{j} X_{s-j} + B^r X_{s-r-q} \Big) + Y_n. \end{aligned}$$

We know that $(X_n)_n$ satisfies the recurrence relation, (see [6])

$$\sum_{j=0}^{r} (-A)^j \binom{r}{j} X_{n-j} = B^r X_{n-q-r}.$$

Therefore, $G_n^{(0)} = \sum\limits_{s=1}^{n-1} Y_{n-s} X_s + Y_n = \sum\limits_{s=0}^{n-1} Y_{n-s} X_s$ as $X_0 = 1$.

We now assume that $1 \leq n \leq r + q - 1$. So, we have $X_n = \binom{n+r-1}{r-1} A^n$ and from (2.2), $G_n^{(0)} = \left\langle {n-1 \atop 0} \right\rangle = a_{n-1}$.

Then, we have to prove that $\sum\limits_{k=0}^{n-1} X_k Y_{n-k} = a_{n-1}$.

We have

$$\sum_{k=0}^{n-1} X_k Y_{n-k} = X_0 Y_n + \sum_{k=1}^{n-1} X_k Y_{n-k}.$$

Since $X_0 = 1$ and $Y_n = \sum\limits_{j=0}^{n-1} (-A)^j \binom{r}{j} a_{n-j-1}$, we obtain that

$$\sum_{k=0}^{n-1} X_k Y_{n-k} = a_{n-1} + \sum_{j=1}^{n-1} (-A)^j \binom{r}{j} a_{n-j-1} + \sum_{k=1}^{n-1} X_k Y_{n-k}.$$

Therefore, we aim to prove that for $1 \leq n \leq r + q - 1$,

$$\sum_{k=1}^{n-1} X_k Y_{n-k} = -\sum_{j=1}^{n-1} (-A)^j \binom{r}{j} a_{n-j-1}, \tag{4.3}$$

by considering two cases.

1. If $1 \leq n \leq r + 1$.

$$\sum_{k=1}^{n-1} X_k Y_{n-k} = \sum_{k=1}^{n-1} A^k \binom{k-1+r}{r-1} \sum_{j=0}^{n-k-1} (-A)^j \binom{r}{j} a_{n-k-j-1},$$

$$\sum_{k=1}^{n-1} X_k Y_{n-k} = \sum_{k=1}^{n-1} \sum_{i=0}^{n-k-1} (-1)^{n-1-k-i} A^{n-k} \binom{r+i}{r-1} \binom{r}{n-1-k-i} a_{k-1}.$$

For each $1 \leq k \leq n-1$, we set $j = n-1-k-i$. Then, we get

$$\sum_{k=1}^{n-1} X_k Y_{n-k} = \sum_{k=1}^{n-1} A^{n-k} \sum_{j=0}^{n-k-1} (-1)^j \binom{r}{j} \binom{r+n-1-k-j}{r-1} a_{k-1}.$$

By putting in (9.3), $t = n-1-k$ for $k \geq 1$, $t \leq r-1$ since $n \leq r+1$, one has

$$\sum_{k=1}^{n-1} X_k Y_{n-k} = \sum_{k=1}^{n-1} (-1)^{n-1-k} A^{n-k} \binom{r}{n-k} a_{k-1}.$$

If we put $j = n-k$, we get (4.3).

2. If $r+2 \leq n \leq r+q-1$. In this case, $Y_n = \sum_{j=0}^{r} (-1)^j A^j \binom{r}{j} a_{n-1-j}$ since $n-1 > r$. So,

$$\sum_{k=1}^{n-1} X_k Y_{n-k} = \sum_{k=1}^{n-1} A^k \binom{k-1+r}{r-1} \sum_{j=0}^{r} (-A)^j \binom{r}{j} a_{n-k-j-1}.$$

Now, we separate the sum following the index k in three parts; $1 \leq k \leq r$, $k = r+1$ and $r+2 \leq k \leq n-1$ to obtain

$$\begin{aligned}
\sum_{k=1}^{n-1} X_k Y_{n-k} = & \sum_{k=1}^{r} \sum_{j=0}^{k-1} (-1)^{k-j-1} A^k \binom{r}{k-1-j} \binom{r+j}{r-1} a_{n-k-1} \\
& + \sum_{j=0}^{r} (-1)^{r-j} A^{r+1} \binom{r}{r-j} \binom{r+j}{r-1} a_{n-r-2} \\
& + \sum_{k=r+2}^{n-1} \sum_{j=0}^{n-2} (-1)^{k-j-2} A^k \binom{r}{r-j} \binom{k-1+j}{r-1} a_{n-k-1}.
\end{aligned}$$

We set $s = k-1-j$, $t = r-j$ and we get

$$\begin{aligned}
\sum_{k=1}^{n-1} X_k Y_{n-k} = & \sum_{k=1}^{r} A^k \sum_{s=0}^{k-1} (-1)^s \binom{r}{s} \binom{r+k-1-s}{r-1} a_{n-k-1} \\
& + A^{r+1} \sum_{t=0}^{r} (-1)^t \binom{r}{t} \binom{2r-t}{r-1} a_{n-r-2} \\
& + \sum_{k=r+2}^{n-1} (-1)^{k-2-r} A^k \sum_{t=0}^{r} (-1)^t \binom{r}{t} \binom{k-1+r-t}{r-1} a_{n-k-1}.
\end{aligned}$$

By putting in (9.3), $t = k-1 \leq r-1$ in the first sum of the right hand side, one has

$$\sum_{k=1}^{r} A^k \sum_{s=0}^{k-1} (-1)^s \binom{r}{s} \binom{r+k-1-s}{r-1} a_{n-k-1} = \sum_{k=1}^{r} (-1)^{k-1} A^k \binom{r}{k} a_{n-k-1}.$$

From Lemma 9.3,

$$\sum_{t=0}^{r} (-1)^t \binom{r}{t} \binom{2r-t}{r-1} = 0 \qquad \text{and} \qquad \sum_{t=0}^{r} (-1)^t \binom{r}{t} \binom{k-1+r-t}{r-1} = 0.$$

Therefore, we get (4.3). The proof is complete. □

5. The generating function

In this section, we give the generating function associated to the sum of GAT's elements lying along a finite ray. But first, we will need the following lemmas and theorem.

Lemma 5.1. *For $a \geq r \geq 1$,*

$$\sum_{j=0}^{r}(-A)^j\binom{r}{j}\left\langle{n-j \atop a}\right\rangle = B^r\left\langle{n-r \atop a-r}\right\rangle.$$

Proof. Let $I_r = \sum\limits_{j=0}^{r}(-A)^j\binom{r}{j}\left\langle{n-j \atop a}\right\rangle$. We use induction on the positive integer r. From Definition 3,

$$I_1 = \left\langle{n \atop a}\right\rangle - A\left\langle{n-1 \atop a}\right\rangle = B\left\langle{n-1 \atop a-1}\right\rangle.$$

Assume now that $I_r = B^r\left\langle{n-r \atop a-r}\right\rangle$. We have

$$\begin{aligned} I_{r+1} &= \sum_{j=0}^{r}(-A)^j\binom{r}{j}\left\langle{n-j \atop a}\right\rangle + \sum_{j=0}^{r}(-A)^j\binom{r}{j-1}\left\langle{n-j \atop a}\right\rangle \\ &= \sum_{j=0}^{r}(-A)^j\binom{r}{j}\left\langle{n-j \atop a}\right\rangle - A\sum_{j=0}^{r}(-A)^j\binom{r}{j}\left\langle{n-j-1 \atop a}\right\rangle = B^r\left\langle{n-r \atop a-r}\right\rangle - AB^r\left\langle{n-r-1 \atop a-r}\right\rangle. \end{aligned}$$

We conclude by Definition 3. □

Lemma 5.2. *Let $Z_n := \sum\limits_{k=0}^{p}\binom{n+p-k}{p-k}b_k$. The following identity holds true*

$$\sum_{n\geq 0} Z_n t^n = (1-t)^{-p-1}\sum_{k=0}^{p}(1-t)^k b_k.$$

Proof.

$$\sum_{n\geq 0}\left(\sum_{k=0}^{p}\binom{n+p-k}{p-k}b_k\right)t^n = \sum_{k=0}^{p}\sum_{n\geq 0}\binom{n+p-k}{p-k}t^n b_k = \sum_{k=0}^{p}(1-t)^{k-p-1}\, b_k$$

as

$$\sum_{m\geq 0}\binom{m+s}{s}t^m = (1-t)^{-s-1}\,. \qquad \square$$

For fixed r and p, we now set

$$C_k\,(t) := \sum_{n\geq p+rk}\left\langle{n \atop p+rk}\right\rangle t^n \qquad \text{and} \qquad H\,(t) := \sum_{n\geq 0} a_n t^n. \tag{5.1}$$

Lemma 5.3. *For $r \in \mathbb{N}$, $0 \leq p < r$, $q \in \mathbb{Z}$ and the sequences $(a_n)_n$ and $(b_n)_n$, the generating function $C_k\,(t)$ verifies the relation, for $k \geq 0$,*

$$C_k\,(t) = \left(\frac{Bt}{1-At}\right)^{rk} C_0(t), \tag{5.2}$$

with

$$C_0\,(t) = b_p t^p + \left(\frac{Bt}{1-At}\right)^p\left(H(t) - \Omega + \frac{At}{1-At}\sum_{k=1}^{p}\left(\frac{1-At}{B}\right)^k b_k\right). \tag{5.3}$$

Proof. For $k \geq 0$,

$$(1-At)^r C_{k+1}(t) = \sum_{j=0}^{r} (-A)^j \binom{r}{j} \sum_{n\geq p+rk+r} \left\langle {n \atop p+rk+r} \right\rangle t^{n+j}$$
$$= \sum_{j=0}^{r} (-A)^j \binom{r}{j} \sum_{n\geq p+rk+r+j} \left\langle {n-j \atop p+rk+r} \right\rangle t^{n}.$$

So,

$$(1-At)^r C_{k+1}(t) = \sum_{j=0}^{r} (-A)^j \binom{r}{j} \Bigg(\sum_{n\geq p+rk+r} \left\langle {n-j \atop p+rk+r} \right\rangle t^{n} \Bigg),$$

as $\left\langle {n-j \atop p+rk+r} \right\rangle = 0$ for $p+rk+r \leq n < p+rk+r+j$.
From Lemma 5.1, we get

$$(1-At)^r C_{k+1}(t) = \sum_{n\geq p+rk+r} B^r \left\langle {n-r \atop p+rk} \right\rangle t^n = B^r t^r \sum_{n\geq p+rk} \left\langle {n \atop p+rk} \right\rangle t^n.$$

Thus from (5.1), $(1-At)^r C_{k+1}(t) = B^r t^r C_k(t)$. It follows that $C_k(t) = \left(\frac{Bt}{1-At}\right)^{rk} C_0(t)$.

For $p=0$, $C_0(t) = \sum_{n\geq 0} \left\langle {n \atop 0} \right\rangle t^n = \sum_{n\geq 0} a_n t^n = H(t)$.
For $p \geq 1$ and using Theorem 1.1,

$$C_0(t) = \left\langle {p \atop p} \right\rangle t^p + \sum_{n\geq p+1} A^{n-p} B^p \Bigg(\sum_{i=1}^{n-p} \binom{n-1-i}{p-1} \frac{a_i}{A^i} + \sum_{i=1}^{p} \binom{n-1-i}{p-i} \frac{b_i}{B^i} \Bigg) t^n,$$

as $\left\lfloor \frac{n-p}{n} \right\rfloor = \left\lfloor \frac{p}{n} \right\rfloor = 0$.
Let us set $C_0(t) = b_p t^p + T_1 + T_2$ with

$$T_1 := \sum_{n\geq p+1} A^{n-p} B^p \sum_{i=1}^{n-p} \binom{n-1-i}{p-1} \frac{a_i}{A^i} t^n$$

and

$$T_2 := \sum_{n\geq p+1} A^{n-p} B^p \sum_{i=1}^{p} \binom{n-1-i}{p-i} \frac{b_i}{B^i} t^n.$$

We have

$$T_1 = t^p B^p \sum_{m\geq 1} \Bigg[\sum_{i=1}^{m} \binom{m-i+p-1}{p-1} A^{m-i} a_i \Bigg] t^m, \text{ with } m = n-p,$$
$$= t^p B^p \Bigg(\sum_{m\geq 1} a_m t^m \Bigg) \Bigg(\sum_{m\geq 0} \binom{m+p-1}{p-1} (At)^m \Bigg)$$
$$= t^p B^p (H(t) - \Omega)(1-At)^{-p}.$$
$$T_2 = t^p B^p \sum_{m\geq 0} \Bigg[\sum_{i=1}^{p} \binom{m+p-i}{p-i} \frac{b_i}{B^i} \Bigg] (At)^{m+1}, \text{ with } m = n-p-1,$$
$$= At^{p+1} B^p (1-At)^{-p-1} \sum_{i=1}^{p} \left(\frac{1-At}{B} \right)^i b_i.$$

By replacing, the proof is complete. □

Remark 5. We can easily deduce the following special cases:

- For $b_n = \Omega B^n$, $C_0(t) = \left(\frac{Bt}{1-At}\right)^p H(t)$.
- For $a_n = \Omega A^n$, $C_0(t) = b_p t^p + At^{p+1} \sum_{k=0}^{p} (1-At)^{k-p-1} B^{p-k} b_k$.
- For $a_n = \Omega A^n$ and $b_n = \Omega B^n$, $C_0(t) = \Omega (Bt)^p (1-At)^{-p-1}$.

Now, we are able to give the generating function of $\left(G_n^{(p)}\right)_n$.

Theorem 5.1. *For $r \in \mathbb{N}$, $0 \le p < r$, $q \in \mathbb{Z}$ and $B \neq 0$, the generating function $G(t) := \sum_{n \ge 0} G_{n+1}^{(p)} t^n$ associated to $\left(G_n^{(p)}\right)_n$ is given by*

$$G(t) = \frac{(1-At)^r}{(1-At)^r - B^r t^{r+q}} C_0(t),$$

with $C_0(t)$ as given by (5.3).

Proof. We have

$$G(t) = \sum_{n \ge 0} G_{n+1}^{(p)} t^n = \sum_{k \ge 0} \sum_{n \ge 0} \left\langle {n-qk \atop p+rk} \right\rangle t^n,$$

as $\left\langle {n \atop k} \right\rangle = 0$ for $n < k$. According to (5.1) and (5.2),

$$G(t) = \sum_{k \ge 0} t^{qk} C_k(t) = C_0(t) \sum_{k \ge 0} \left(\frac{B^r t^{r+q}}{(1-At)^r} \right)^k = \frac{(1-At)^r}{(1-At)^r - B^r t^{r+q}} C_0(t). \qquad \square$$

Corollary 5.1.1. *For $r \in \mathbb{N}$, $0 \le p < r$, $q \in \mathbb{Z}$ and $B \neq 0$, the generating function associated to $\left(G_n^{(p)}\right)_n$ is given for $b_n = \Omega B^n$ by*

$$G(t) = \frac{(Bt)^p H(t) (1-At)^{r-p}}{(1-At)^r - B^r t^{r+q}}, \tag{5.4}$$

and for $a_n = \Omega A^n$ by

$$G(t) = \frac{(1-At)^r b_p t^p + At^{p+1} \sum_{k=0}^{p} (1-At)^{r+k-p-1} B^{p-k} b_k}{(1-At)^r - B^r t^{r+q}},$$

and for $a_n = \Omega A^n$ and $b_n = \Omega B^n$ by

$$G(t) = \frac{\Omega (Bt)^p (1-At)^{r-p-1}}{(1-At)^r - B^r t^{r+q}}.$$

Remark 6. For $a_n = A^n$ and $b_n = B^n$, we get Theorem 2 given in [6] which holds not only for $q > 0$, but for $q \le 0$ as well.

6. Morgan-Voyce and Fibonacci numbers

In [6], the authors established that the sequence (v_n) associated to different directions of the ray in Pascal's triangle, defined for $r \in \mathbb{N}$, $q \in \mathbb{Z}$, $p \le r-1$ such that $r+q > 0$, by

$$v_{n+1} = \sum_{k=0}^{\lfloor (n-p)/(r+q) \rfloor} \binom{n-qk}{p+rk} x^{n-p-(r+q)} y^{p+rk},$$

satisfies the linear recurrence relation

$$v_n - x\binom{r}{1}v_{n-1} + \cdots + (-x)^r\binom{r}{r}v_{n-r} = y^r v_{n-r-q}.$$

Thus, the sequence (v_n) is of order r for $q \leq 0$ and we can write

$$v_n - x\binom{r}{1}v_{n-1} + \cdots + \left((-x)^{r+q}\binom{r}{r+q} - y^r\right)v_{n-r-q} + \cdots + (-x)^r\binom{r}{r}v_{n-r} = 0.$$

This is what we call the Morgan-Yoyce phenomenon. For more details, see [22, 24, 25]. Using this phenomenon, we can infer remarkable identities for Fibonacci numbers.

Let $\delta_0 = \Omega$, δ_1 be two parameters and $(\delta_n)_n$ be a sequence defined by

$$\delta_n = 2A\delta_{n-1} - A^2\delta_{n-2}, \qquad n \geq 2. \tag{6.1}$$

In this section, we consider the GAT generated by the sequences $(a_n)_n$ and $(b_n)_n$ such that

$$\begin{aligned} a_1 &= A\delta_1, \quad a_n = \delta_n, && n \geq 2,\\ b_n &= B^n, && n \geq 1. \end{aligned} \tag{6.2}$$

By setting in Corollary 4.1.1, $r = 2$, $p = 0$ and $q = -1$, we get for $n \geq 3$,

$$G_n - (2A + B^2)G_{n-1} + A^2G_{n-2} = a_{n-1} - 2Aa_{n-2} + A^2a_{n-3}.$$

Assume that $A = \mathrm{i}$, $B^2 = 1 - 2\mathrm{i}$, with $\mathrm{i}^2 = -1$. It follows from (6.1) and (6.2) that

$$G_n = G_{n-1} + G_{n-2}.$$

However, $G_{n+1} = \sum_{k=0}^{n}\left\langle{n+k \atop 2k}\right\rangle$, so $G_1 = \left\langle{0 \atop 0}\right\rangle = \Omega$, $G_2 = 1 + i\,(\delta_1 - 2)$. If we write $G_n = \alpha F_n + \beta F_{n+1}$, then from G_1 and G_2 it follows that

$$G_n = (2\Omega - 1)\,F_n + (1 - \Omega)\,F_{n+1} + \mathrm{i}(\delta_1 - 2)F_{n-1}.$$

Thus

$$(2\Omega - 1)\,F_n + (1 - \Omega)\,F_{n+1} = \operatorname{Re}(G_n) \qquad \text{and} \qquad (\delta_1 - 2)F_{n-1} = \operatorname{Im}(G_n)\,. \tag{6.3}$$

For $\delta_1 = 2$,

$$G_n = (2\Omega - 1)\,F_n + (1 - \Omega)\,F_{n+1},$$

and for $\delta_1 = 2$, $\Omega = 0$,

$$G_n = F_{n-1}.$$

For $\Omega = \delta_1 = 1$, using some combinatorial computations, Proposition 3.1 and (6.3) give for $m \geq 0$,

$$F_{2m+1} = \sum_{k=0}^{2m}\sum_{s=0}^{\lfloor k/2\rfloor}(-1)^{m+k+s}2^{k-2s}\binom{k}{2s}\binom{2m+k}{2k},$$

$$F_{2m+2} = \sum_{k=0}^{2m+1}\sum_{s=0}^{\lfloor (k-1)/2\rfloor}(-1)^{1+m+k+s}2^{k-2s-1}\binom{k}{2s+1}\binom{2m+1+k}{2k}.$$

As a last result in this section, we quote an application of Theorem 5.1. Recall that the Morgan-Voyce polynomials $B_n(x)$ are defined by

$$B_0(x) = 1, \quad B_1(x) = 2 + t, \quad B_n(x) = (2 + x)B_{n-1}(x) - B_{n-2}(x)\,, \qquad n \geq 2.$$

They have been widely studied (see [4, 22, 24, 25]). In [24], the author gives a closed form expression and a generating function

$$B_n(x) = \sum_{k=0}^{n}\binom{n+k+1}{1+2k}x^k, \qquad \sum_{n\geq 0}B_n(x)t^n = \frac{1}{1 - (x+2)t - t^2}.$$

Using (5.4), we are able to express $B_n(x)$ in descending powers of x. Indeed, if we put $A = x+2$, $B = -1$, $r = q = 1$, $a_n = A^n$ for $n \geq 0$, we deduce

$$\sum_{n\geq 0} G_{n+1}^{(0)} t^n = \sum_{n\geq 0} B_n(x) t^n.$$

So, from (3.1) we get for $n \geq 0$,

$$B_n(x) = \sum_{k=0}^{n} \binom{n-k}{k} (-1)^k (x+2)^{n-2k}.$$

We get also, $\sum_{n\geq 0} G_{n+1}^{(0)} t^n = \sum_{n\geq 0} B_n(x) t^n$ by setting $A = 1$, $B = \sqrt{x}$ $(x \geq 0)$, $r = 2$, $q = -1$, $a_n = n+1$. Therefore, for $x \geq 0$ and $n \geq 0$,

$$B_n(x) = \sum_{k=0}^{n} \left\langle {n+k \atop 2k} \right\rangle_{\left(1,\ \sqrt{x},\ n+1,\ \sqrt{x}^n\right)}.$$

7. Some illustrations

We present some examples to illustrate Theorem 4.1 for $p = 0$. The sequence $\left(G_n^{(0)}\right)_n$ is denoted for short by $\left(G_n\right)_n$.

***Example* 2.** Consider $(b_n)_n = (B^n)_n$ and $a_n = A^n \binom{n+k}{k} B^k$ for a fixed k, $k \geq r$. From Corollary 4.1.1, the terms of the sequence $(G_n)_n$ satisfy the recurrence relation

$$\sum_{j=0}^{r} (-A)^j \binom{r}{j} G_{n-j} - \sum_{j=0}^{r} (-A)^j \binom{r}{j} a_{n-j-1} = B^r G_{n-q-r}. \tag{7.1}$$

From Lemma 9.3,

$$\sum_{j=0}^{r} (-A)^j \binom{r}{j} a_{n-j-1} = A^{n-1} \sum_{j=0}^{r} (-1)^j \binom{r}{j} \binom{n+k-1-j}{k} B^k = A^{n-1} \binom{n+k-1-r}{k-r} B^k.$$

Then (7.1) becomes

$$\sum_{j=0}^{r} (-A)^j \binom{r}{j} G_{n-j} - B^r G_{n-q-r} = A^{n-1} \binom{n+k-1-r}{k-r} B^k. \tag{7.2}$$

For $(r,q) = (1,1)$, we obtain

$$G_n - AG_{n-1} - BG_{n-2} = A^{n-1} \binom{n+k-2}{k-1} B^k,$$

which is the nonhomogeneous recurrence relation of $\left(F_n^{(k))}\right)$, the bivariate hyper-Fibonacci polynomials given in [5]. More precisely,

$$G_n = F_n^{(k)}, \qquad \text{where } F_{n+1}^{(k)} = \sum_{j=0}^{\lfloor n/2 \rfloor} A^{n-2j} B^{j+k} \binom{n+k-j}{j+k}.$$

For $(r,q) = (1,q)$, $q > 0$, we obtain

$$G_n - AG_{n-1} - BG_{n-q-1} = A^{n-1} \binom{n+k-2}{k-1} B^k,$$

which is the recurrence relation of $\left(U_n^{(k)}\right)$, the hyper q-Fibonacci polynomials given in [1]:

$$G_n = U_{n+2}^{(k)}, \qquad \text{where } U_{n+1}^{(k)} = \sum_{j=0}^{\lfloor n/(q+1)\rfloor} A^{n-(q+1)j} B^{j+k} \binom{n+k-qj}{j+k}.$$

***Example* 3.** The direction $(r,q)=(1,1)$. It concerns the sequence $W_{n+1} = G_{n+1}^{(1,1,0)}$ given by

$$W_{n+1} = \sum_{k=0}^{\lfloor n/(r+q)\rfloor} \left\langle {n-k \atop k} \right\rangle_{(A,\ B,\ a_n,\ b_n)}.$$

From Theorem 5.1, (W_n) satisfies, for $n \geq 2$,

$$W_n - AW_{n-1} - BW_{n-2} = (a_{n-1} - Aa_{n-2}) + (b_{\frac{n-1}{2}} - Bb_{\frac{n-3}{2}})\,[n \text{ odd}]. \tag{7.3}$$

We give the following examples.

a_n	b_n	W_n
A^nF_{2n-1}	B^nF_{n-1}	$W_n = AW_{n-1} + BW_{n-2} + A^{n-1}F_{2n-4} + B^{\frac{n-1}{2}}F_{\frac{n-7}{2}}\,[n \text{ odd}]$
A^nF_n	B^nF_n	$W_n = AW_{n-1} + BW_{n-2} + A^{n-1}F_{n-3} + B^{\frac{n-1}{2}}F_{\frac{n-5}{2}}\,[n \text{ odd}]$

Remark 7. The recurrence relation associated to (W_n) in the last example of the above table, is given in [7] for $A = B = 1$.

From (7.3) and Theorem 4.2, we are able to write (W_n) as the convolution of two sequences. One has for $n \geq 2$,

$$W_n = \sum_{k=0}^{n-1} X_k Y_{n-k}, \tag{7.4}$$

with $X_0 = 1$, $X_1 = A$, $X_n = AX_{n-1} + BX_{n-2}$, $n \geq 2$ and $Y_0 = 0$, $Y_1 = \Omega$, $Y_n = (a_{n-1} - Aa_{n-2}) + (b_{\frac{n-1}{2}} - Bb_{\frac{n-3}{2}})\,[n \text{ odd}]$, $n \geq 2$.

For $A = B = 1$ and $(a_n)_n$, $(b_n)_n$ two given sequences, (W_n) is the convolution of the Fibonacci sequence and the sequence $(Y_n)_n$.

In the following, some examples are given,

a_n	b_n	W_n	$W_n =$
F_{n+1}	1	$W_n = W_{n-1} + W_{n-2} + F_{n-2}.$	$\sum_{k=0}^{n-1} F_{k+1}F_{n-k-2}$
l_{n+1}	1	$W_n = W_{n-1} + W_{n-2} + l_{n-2}.$	$\sum_{k=0}^{n-1} F_{k+1}l_{n-k-2}$
$-\frac{1}{n}$	1	$W_n = W_{n-1} + W_{n-2} + \frac{1}{n(n+1)}.$	$\sum_{k=0}^{n-1} \frac{F_{k+1}}{(n-k)(n-k+1)}$
$H_n^{(k)}$	1	$W_n = W_{n-1} + W_{n-2} + H_{n-1}^{(k-1)}.$	$\sum_{k=0}^{n-1} F_{k+1}H_{n-k-1}^{(k-1)}$
1	F_{n+1}	$W_n = W_{n-1} + W_{n-2} + F_{\frac{n-3}{2}}\,[n \text{ odd}]$	$\sum_{k=1}^{\lfloor (n-1)/2\rfloor} F_{k-1}F_{n-2k}$
1	l_{n+1}	$W_n = W_{n-1} + W_{n-2} + l_{\frac{n-3}{2}}\,[n \text{ odd}]$	$l_n + \sum_{k=1}^{\lfloor (n-1)/2\rfloor} l_{k-1}l_{n-2k}$
1	$\frac{1}{n+1}$	$W_n = W_{n-1} + W_{n-2} - \frac{4}{n^2-1}\,[n \text{ odd}]$	$F_n - \sum_{k=1}^{\lfloor (n-1)/2\rfloor} \frac{4F_{n-2k}}{(2k+1)^2-1}$

where (F_n) is the Fibonacci sequence, (l_n) is the Lucas sequence and $\left(H_n^{(k)}\right)$ the hyper harmonic sequence defined by $H_n^{(0)} = \frac{1}{n}$ for $k \geq 1$. For more details, see [9, 11].

***Example* 4.** As a consequence of (7.4), we infer an expression for the sum of hyper harmonic numbers in terms of Fibonacci numbers. Indeed, in the last case of the above table, it is easy to see that $\left\langle {n \atop k} \right\rangle_{(1,\,1,\,1,\,\frac{1}{n+1})} = H_{k+1}^{(n-k)}$. So one has the following.

Proposition 7.1. *For any $n \geq 2$, we have*

$$\sum_{k=1}^{\lfloor n/2 \rfloor} H_{k+1}^{(n-2k)} = F_n - \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \frac{4F_{n-2k}}{(2k+1)^2 - 1}.$$

***Example* 5.** The case $A = 0$, $b_n = B^n$, $p = 0$. From Corollary 4.1.1, this case deals with the sequence $\left(V_n^{(r,q)}\right)_n$ satisfying for $n \geq r + q$ the recurrence

$$V_n^{(r,q)} - B^r V_{n-r-q}^{(r,q)} = a_{n-1}. \tag{7.5}$$

The arrays in Tables 1 and 2 were studied by Bruckman [10] and Hoggatt [16], respectively. The column 0 consists of the Fibonacci numbers F_{2n-2} and F_{2n+1}, respectively. Each of the remaining columns j, $j \geq 1$, is obtained by lowering the previous column $j-1$ by one level. Note that these tables are the particular case of the GAT when $A = 0$, $B = 1$ and, respectively, $(a_n)_{n\geq 1} = (F_{2n})_{n\geq 0}$, $(a_n)_{n\geq 0} = (F_{2n+1})_{n\geq 0}$.

Table 5. The $(0, 1, F_{2n}, 1)$ GAT.

1						
1	1					
3	1	1				
8	3	1	1			
21	8	3	1	1		
55	21	8	3	1	1	
⋮	⋮	⋮	⋮	⋮	⋮	⋱

Table 6. The $(0, 1, F_{2n+1}, 1)$ GAT.

1						
2	1					
5	2	1				
13	5	2	1			
34	13	5	2	1		
89	34	13	5	2	1	
⋮	⋮	⋮	⋮	⋮	⋮	⋱

In the Table 5, the n-th row sum is $V_{n+1}^{(1,0)} = F_{2n+1}$ and the rising diagonal sum is $V_{n+1}^{(1,1)} = F_{n+1}^2$. Indeed, from relation (7.5), the sequence $\left(V_n^{(1,0)}\right)_n$ satisfies the recurrence

$$V_n^{(1,0)} - V_{n-1}^{(1,0)} = F_{2n-2},$$

and the sequence $\left(V_n^{(1,1)}\right)_n$ satisfies the recurrence

$$V_n^{(1,1)} - V_{n-2}^{(1,1)} = F_{2n-2}.$$

Also, in the Table 6, $V_{n+1}^{(1,0)} = F_{2n+2}$ and $V_{n+1}^{(1,1)} = F_{n+1}F_n$.

Some examples, with t_n the triangular number and T_n the tetrahedral number, corresponding to this case are given as follows:

a_n	B	r	q	$V_n^{(r,q)}$
$n+1$	1	1	0	t_n
t_{n+1}	1	1	0	T_n
F_{2n+3}	-1	1	0	F_{n+1}^2
l_{n+1}	-1	1	1	F_{n+1}
F_{2n+2}	1	1	1	F_{n+1}^2
l_n	-1	1	1	F_n
$5F_{n+1}$	1	1	3	F_{n+2}
l_{n-1}	1	1	3	F_n

8. The generalized golden ratio

Given two initial terms u_0 and u_1 not both zero, we consider the recurrence relation $u_n = u_{n-1} + u_{n-2}$. It is a well-known property of such sequences that $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \phi$, where ϕ, called the golden ratio, is the greatest root of $t^2 - t - 1 = 0$. Raab [23] extended this result to the sequences defined by the recurrence

$$w_n = xw_{n-1} + yw_{n-m-1},$$

given $m+1$ initial terms. He proves that $\lim\limits_{n\to\infty} \frac{w_{n+1}}{w_n}$ exists and is a root of $t^{m+1} - xt^m - y = 0$.

In this section, we establish the existence of similar limits for more general sequences. For that, let us consider the sequence $(T_n)_n$ defined for $n \geq r+q$ by

$$T_n - A\binom{r}{1}T_{n-1} + \cdots + (-A)^r\binom{r}{r}T_{n-r} = B^r T_{n-q-r}. \tag{8.1}$$

Note that, from Corollary 4.1.2 and Corollary 4.1.1, T_n is the sum of elements lying on the corresponding ray in the GAT generated by the sequences $(a_n)_n = (A^n)_n$ and $(b_n)_n = (B^n)_n$. The characteristic polynomial of $(T_n)_n$ is

$$t^q (t-A)^r - B^r = 0, \tag{8.2}$$

and, from (2.2) and Theorem 1.1, initial conditions are

$$T_{n+1} = \binom{n}{p} A^{n-p} B^p, \; p \leq n < p+r+q.$$

Let us set

$$T_n := \sum_{i=1}^{r+q} c_i t_i^n,$$

where t_i, $i = 1, \ldots, r+q$, are the roots of (8.2) and $g(t) = t^q (t-A)^r - B^r$.

We have $g'(t) = t^{q-1} (t-A)^{r-1} [(r+q)t - Aq]$. or $B \neq 0$, no root of $g'(t)$ is a root of $g(t)$. It follows that $g(t)$ has no multiple root. Moreover, the determinant of the coefficients c_i of the linear system in $c_i, i = 1, \ldots, r+q$,

$$\sum_{i=1}^{r+q} c_i t_i^{n+1} = \binom{n}{p} A^{n-p} B^p, \qquad p \leq n \leq p+r+q-1, \tag{8.3}$$

is not zero. Indeed, by writing (8.3) as

$$\begin{cases} \sum_{i=1}^{r+q} c_i t_i^{p+1} = \binom{p}{p} A^0 B^p \\ \sum_{i=1}^{r+q} c_i t_i^{p+2} = \binom{p+1}{p} A^1 B^p \\ \vdots \\ \sum_{i=1}^{r+q} c_i t_i^{p+r+q} = \binom{p+r+q-1}{p} A^{r+q-1} B^p, \end{cases} \tag{8.4}$$

the determinant which is

$$D = \left| \begin{pmatrix} t_1^{p+1} & t_1^{p+2} & \cdots & t_1^{p+r+q} \\ t_2^{p+1} & t_2^{p+2} & \cdots & t_2^{p+r+q} \\ \vdots & \vdots & \vdots & \vdots \\ t_{r+q}^{p+1} & t_{r+q}^{p+2} & \cdots & t_{r+q}^{p+r+q} \end{pmatrix} \right| = \prod_{k=1}^{r+q} t_k^{p+1} \prod_{1 \le i < j \le r+q}^{r+q} (t_j - t_i).$$

Since all the roots are simple, then $D \neq 0$ and the system (8.4) can be solved uniquely in $c_1, \ldots, c_{r+q}$. Let now t_* be such that for each i, $|t_*| > |t_i|$. We have the following.

Proposition 8.1. *If $A > 0$ and $B > 0$, then*

$$\lim_{n\to\infty} \frac{T_{n+1}}{T_n} = t_*.$$

Proof. Let $g(t) = t^q (t-A)^r - B^r$. We have, $g(A) = -B^r < 0$ and $\lim_{t\to\infty} g(t) = +\infty$. It follows that g admits a real root t_* such that $t_* > A > 0$. Furthermore, we have for $1 \le i \le r+q$, $|t_*| > |t_i|$ with t_i a root of g.

Indeed, suppose that t_i is a root of g such that $|t_i| > t_*$. Thus,

$$|t_i - A| \ge | \, |t_i| - A| \ge |t_i| - A > t_* - A > 0.$$

So,

$$|t_i - A|^r > |t_* - A|^r$$

and it follows that

$$|t_i^q (t_i - A)^r| > |t_*|^q |(t_* - A)|^r = B^r.$$

This is a contradiction.

Consider now the ratio

$$\frac{T_{n+1}}{T_n} = \frac{\sum_{i=1}^{r+q} c_i t_i^{n+1}}{\sum_{i=1}^{r+q} c_i t_i^n} = \frac{t_*^n \left(c_1 t_1 \left(\frac{t_1}{t_*}\right)^n + \cdots + c_* t_* + \cdots + c_{r+q} t_{r+q} \left(\frac{t_{r+q}}{t_*}\right)^n \right)}{t_*^n \left(c_1 \left(\frac{t_1}{t_*}\right)^n + c_* + \cdots + c_{r+q} \left(\frac{t_{r+q}}{t_*}\right)^n \right)}.$$

Since $t_* > 0$ and $t_* > |t_i|$, we get $\lim_{n\to\infty} \frac{T_{n+1}}{T_n} = t_*$. (We suppose without loss of generality that $c_* \neq 0$). □

***Example* 6.** For $(r,q) = (1,1)$, we have $T_n = AT_{n-1} + BT_{n-2}$, $T_0 = 0$, $T_1 = 1$ (see relation (8.1)).

The characteristic polynomial of (T_n) is $t^2 - At - B = 0$, whose roots are $t_1 = \left(A + \sqrt{A^2+4B}\right)/2$ and $t_2 = \left(A - \sqrt{A^2+4B}\right)/2$. Then:

- For $(A,B) = (1,1)$, we obtain $T_n = F_n$, the Fibonacci number and

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \left(1 + \sqrt{5}\right)/2.$$

- For $(A,B)=(2,1)$, we obtain $T_n=P_n$, the Pell number and
$$\lim_{n\to\infty}\frac{P_{n+1}}{P_n}=1+\sqrt{2}.$$
- For $(A,B)=(1,2)$, we obtain $T_n=J_n$, the Jacobsthal number and
$$\lim_{n\to\infty}\frac{J_{n+1}}{J_n}=2.$$
- For $(A,B)=(3,2)$, we obtain $T_n=\phi_n$, the Fermat number and
$$\lim_{n\to\infty}\frac{\phi_{n+1}}{\phi_n}=\left(3+\sqrt{17}\right)/2.$$

Remark 8. For $(r,q)=(1,q)$, the result is given in [23].

9. Lemmata

In this section, we establish some intermediate lemmata which will play a key role in the proof of the recurrence relation associated with the sum of elements along a transversal ray in a GAT.

Lemma 9.1. *Let $(d_n)_{n\in\mathbb{N}}$ be a sequence. Then for $r\geq 1$, $1\leq t\leq r$, $c\geq r-1$, all natural numbers,*
$$\sum_{j=0}^{r}(-1)^j\binom{r}{j}\sum_{i=1}^{c+2-j-t}\binom{c-j-i+1}{t-1}d_i=\sum_{j=0}^{r-t}(-1)^j\binom{r-t}{j}d_{c+2-t-j}.$$

Proof. Let $S_r=\sum\limits_{j=0}^{r}(-1)^j\binom{r}{j}\sum\limits_{i=1}^{c+2-j-r}\binom{c-i-j+1}{r-1}d_i$. Using induction on the nonnegative integer r, we will prove first that
$$S_r=d_{c-r+2}. \tag{9.1}$$
$$S_1=\sum_{j=0}^{1}(-1)^j\binom{1}{j}\sum_{i=1}^{c-j+1}\binom{c-i-j+1}{0}d_i=\sum_{i=1}^{c+1}d_i-\sum_{i=1}^{c}d_i=d_{c+1}.$$
Assume now that $S_r=d_{c-r+2}$ and prove that $S_{r+1}=d_{c-r+1}$.
$$\begin{aligned}S_{r+1}&=\sum_{j=0}^{r+1}(-1)^j\binom{r+1}{j}\sum_{i=1}^{c-j-r+1}\binom{c-i-j+1}{r}d_i\\&=\sum_{j=0}^{r+1}(-1)^j\binom{r}{j}\sum_{i=1}^{c-j-r+1}\binom{c-i-j+1}{r}d_i+\sum_{j=1}^{r+1}(-1)^j\binom{r}{j-1}\sum_{i=1}^{c-j-r+1}\binom{c-i-j+1}{r}d_i\\&=\sum_{j=0}^{r}(-1)^j\binom{r}{j}\sum_{i=1}^{c-j-r+1}\binom{c-i-j+1}{r}d_i-\sum_{j=0}^{r}(-1)^j\binom{r}{j}\sum_{i=1}^{c-j-r}\binom{c-i-j}{r}d_i\\&=\sum_{j=0}^{r}(-1)^j\binom{r}{j}\sum_{i=1}^{c-j-r+1}\binom{c-i-j}{r-1}d_i=d_{c-r+1}.\end{aligned}$$
Let now
$$T_r=\sum_{j=0}^{r}(-1)^j\binom{r}{j}\sum_{i=1}^{c-j-r-t+1}\binom{c-i-j+1}{t-1}d_i.$$

$$T_r = \sum_{j=0}^{r}(-1)^j \sum_{m=0}^{t}\binom{r-t}{j-m}\binom{t}{m}\sum_{i=1}^{c-j-r-t+1}\binom{c-i-j+1}{t-1}d_i,$$

$$T_r = \sum_{m=0}^{t}\sum_{j=0}^{r}(-1)^j \binom{r-t}{j-m}\binom{t}{m}\sum_{i=1}^{c-j-r-t+1}\binom{c-i-j+1}{t-1}d_i.$$

Let $k = j-m$. Since $\binom{n}{k} = 0$ if $k < 0$, we get

$$T_r = \sum_{m=0}^{t}\sum_{k=0}^{r-m}(-1)^{k+m}\binom{r-t}{k}\binom{t}{m}\sum_{i=1}^{c+2-k-m-t}\binom{c-i-k-m+1}{t-1}d_i.$$

Note first, that the coefficient $\binom{r-t}{k}$ in the summation with index k is zero for all $k > r-t$ and $r-m \geq r-t$ since $m \leq t$. So, we can write the summation up to $r-t$ instead of $r-m$ without changing the total sum. Therefore

$$T_r = \sum_{m=0}^{t}\sum_{k=0}^{r-t}(-1)^{k+m}\binom{r-t}{k}\binom{t}{m}\sum_{i=1}^{c+2-k-m-t}\binom{c-i-k-m+1}{t-1}d_i,$$

$$T_r = \sum_{k=0}^{r-t}(-1)^{k}\binom{r-t}{k}\sum_{m=0}^{t}(-1)^m\binom{t}{m}\sum_{i=1}^{c+2-k-m-t}\binom{c-k-m-i+1}{t-1}d_i.$$

But according to (9.1) ,

$$\sum_{m=0}^{t}(-1)^m\binom{t}{m}\sum_{i=1}^{c+2-k-m-t}\binom{c-k-m-i+1}{t-1}d_i = d_{c-k-t+2},$$

then the proof is complete. □

Lemma 9.2. *Let $(d_n)_{n\in\mathbb{N}}$ be a sequence. Then for $t \geq r \geq 1$, $c \geq 0$, all non negative integers*

$$\sum_{j=0}^{r}(-1)^j\binom{r}{j}\sum_{i=1}^{c-j-t+1}\binom{c-j-i+1}{t}d_i = \sum_{i=1}^{c-t+1}\binom{c-i-r+1}{t-r}d_i.$$

P r o o f. We use induction on the non negative integer r. For $r = 1$,

$$\begin{aligned}\sum_{j=0}^{1}(-1)^j\binom{1}{j}\sum_{i=1}^{c-j-t+1}\binom{c-i-j+1}{t}d_i &= \sum_{i=1}^{c-t+1}\binom{c-i+1}{t}d_i - \sum_{i=1}^{c-t}\binom{c-i}{t}d_i \\ &= \binom{t}{t}d_{c-t+1} + \sum_{i=1}^{c-t}\left(\binom{c-i+1}{t} - \binom{c-i}{t}\right)d_i \\ &= \binom{t}{t}d_{c-t+1} + \sum_{i=0}^{c-t}\binom{c-i}{t-1}d_i = \sum_{i=1}^{c-t+1}\binom{c-i}{t-1}d_i.\end{aligned}$$

Let now, $X_r = \sum\limits_{j=0}^{r}(-1)^j\binom{r}{j}\sum\limits_{i=1}^{c-j-t+1}\binom{c-i-j+1}{t}d_i$ and assume that

$$X_r = \sum_{i=1}^{c-t+1}\binom{c-i-r+1}{t-r}d_i.$$

$$\begin{aligned}
X_{r+1} &= \sum_{j=0}^{r}(-1)^j\binom{r}{j}\sum_{i=1}^{c-j-t+1}\binom{c-i-j+1}{t}d_i + \sum_{j=0}^{r+1}(-1)^j\binom{r}{j-1}\sum_{i=1}^{c-j-t+1}\binom{c-i-j+1}{t}d_i \\
&= \sum_{i=1}^{c-t+1}\binom{c-i-r+1}{t-r}d_i - \sum_{j=0}^{r}(-1)^j\binom{r}{j}\sum_{i=1}^{c-j-t}\binom{c-i-j}{t}d_i \\
&= \sum_{i=1}^{c-t+1}\binom{c-i-r+1}{t-r}d_i - \sum_{i=1}^{c-t}\tbinom{c-i-r}{t-r}d_i \\
&= \binom{t-r}{t-r}d_{c-t+1} - \sum_{i=1}^{c-t}\left(\binom{c-i-r+1}{t-r} - \binom{c-i-r}{t-r}\right)d_i \\
&= \binom{t-r}{t-r}d_{c-t+1} - \sum_{i=1}^{c-t}\binom{c-i-r}{t-r-1}d_i \\
&= \sum_{i=1}^{c-t+1}\binom{c-i-r}{t-r-1}d_i. \qquad \square
\end{aligned}$$

Lemma 9.3. *Let a, b and r be non-negative integers. Then*

$$\sum_{j=0}^{r}(-1)^j\binom{r}{j}\binom{a-j}{b} = \begin{cases} \binom{a-r}{b-r} & \text{for } r \le b \le a, \\ 0 & \text{for } r > b. \end{cases}$$

P r o o f. In case where $r \le b \le a$, the proof can be found in [6].
Let now $r > b$ and

$$X_r = \sum_{j=0}^{r}(-1)^j\binom{r}{j}\binom{a-j}{r-1}.$$

By using induction on r, it is easily seen that $X_r = 0$ for $a \ge 2r-1$. Moreover, one has $\binom{r}{j} = \sum_{m=0}^{b+1}\binom{r-b-1}{j-m}\binom{b+1}{m}$, so

$$\begin{aligned}
\sum_{j=0}^{r}(-1)^j\binom{r}{j}\binom{a-j}{b} &= \sum_{j=0}^{r}(-1)^j\sum_{m=0}^{b+1}\binom{r-b-1}{j-m}\binom{b+1}{m}\binom{a-j}{b} \\
&= \sum_{k=0}^{r-b-1}(-1)^k\binom{r-b-1}{k}\sum_{m=0}^{b+1}(-1)^m\binom{b+1}{m}\binom{a-k-m}{b}.
\end{aligned}$$

According to the above, $\sum_{m=0}^{b+1}(-1)^m\binom{b+1}{m}\binom{a-k-m}{b} = 0$, then the proof is complete. $\square$

Lemma 9.4. *Let $r \ge 1$. Then for $0 \le m \le r$, $0 \le s \le r-1$,*

$$\sum_{j=0}^{m}(-1)^j\binom{r}{j}\binom{s+m-j}{s} = (-1)^m\binom{r-s-1}{m}.$$

P r o o f. We again use induction on r. It is easy to verify the formula for $r = 1$. So assume it true for r.

$$\sum_{j=0}^{m}(-1)^j\binom{r+1}{j}\binom{s+m-j}{s} = \sum_{j=0}^{m}(-1)^j\binom{r}{j}\binom{s+m-j}{s} + \sum_{j=1}^{m}(-1)^j\binom{r}{j-1}\binom{s+m-j}{s}$$

$$= \sum_{j=0}^{m}(-1)^j\binom{r}{j}\binom{s+m-j}{s} - \sum_{j=0}^{m-1}(-1)^j\binom{r}{j}\binom{s+m-j-1}{s}$$
$$= (-1)^m\binom{r-s-1}{m} - (-1)^{m-1}\binom{r-s-1}{m-1} = (-1)^m\binom{r-s}{m}. \square$$

Especially for $s=0$, one has for $r \geq 1$ and $0 \leq m \leq r$,

$$\sum_{j=0}^{m}(-1)^j\binom{r}{j} = (-1)^m\binom{r-1}{m}, \tag{9.2}$$

and for $s = r-1$, one has for $r \geq 1$ and $0 \leq t \leq r-1$,

$$\sum_{j=0}^{t}(-1)^j\binom{r}{j}\binom{r+t-j}{r-1} = (-1)^t\binom{r}{t+1}. \tag{9.3}$$

The proof of the principal theorem of this paper is essentially based on the following result.

Lemma 9.5. *Let $r \in \mathbb{N}$, $q \in \mathbb{Z}^+$ and $n \geq r+q+1$ and for $j = 0,\ldots,r$, $\alpha_j = (n-1-j)/(r+q)$. Then only two possibilities hold.*

- *$\alpha_j \notin \mathbb{N}$ for $j = 0,\ldots,r$. In this case, $\lfloor\alpha_j\rfloor = \lfloor\alpha_0\rfloor$, for $j = 0,\ldots,r$.*
- *There is a unique $j_0 \in \{0,\ldots,r\}$ such that $\alpha_{j_0} \in \mathbb{N}$. In such case,*

$$\lfloor\alpha_j\rfloor = \begin{cases} \alpha_{j_0} & \text{for } j \leq j_0, \\ \alpha_{j_0}-1 & \text{for } \geq j_0+1. \end{cases}$$

Proof. Let $n \geq r+q+1$. Then, there exist $k \in \mathbb{N}$ and $\beta < r+q$ such that $n-1 = k(r+q)+\beta$. Let us define, for $j \in \mathbb{N}$, $f(j) = (\beta-j)/(r+q)$. Then for $j_0 = \beta$, $f(j_0) = 0$ and so $\alpha_{j_0} = k$. But $\beta < r+q$ and $q > 0$, so the two following cases arise.

(1) $\beta \leq r$. In this case, there exist a unique $j_0 \leq r$ such that $\alpha_{j_0} \in \mathbb{N}$.

(2) $r < \beta < r+q$. In this case, $0 < f(j) < 1$ since $0 \leq j \leq r$. It follows that $0 < \alpha_j - k < 1$ and so $\alpha_j \notin \mathbb{N}$ for $0 \leq j \leq r$. $\square$

Lemma 9.6. *Let $r \in \mathbb{N}$, $q \in \mathbb{Z}^+$, $(d_n)_n$ a sequence and let*

$$T_1 := \sum_{j=0}^{r}(-1)^j\binom{r}{j}\sum_{k=2}^{\lfloor\beta_j\rfloor}A^{s+2-(r+q)k}B^{rk}\sum_{i=1}^{s+2-j-(r+q)k}\binom{s-j-qk-i+1}{rk-1}d_i,$$
$$T_2 := \sum_{k=1}^{\lfloor\beta_0-1\rfloor}A^{s+2-(r+q)(k+1)}B^{r(k+1)}\sum_{i=1}^{s+2-(r+q)(k+1)}\binom{s-q(k+1)-i-r+1}{rk-1}d_i,$$

with $\beta_j = \frac{s+2-j}{r+q}$. Then $T_1 = T_2$.

Proof. Let $s \geq r+q-2$. We will prove that $T_1 - T_2 = 0$ by using Lemma 9.5. There are two cases.

<u>*Case 1.*</u> $\beta_j \notin \mathbb{N}$. So, $\lfloor\beta_j\rfloor = \lfloor\beta_0\rfloor$ for $0 \leq j \leq r$ and

$$T_1 - T_2 = \sum_{k=2}^{\lfloor\beta_0\rfloor}A^{s+2-(r+q)k}B^{rk}\sum_{j=0}^{r}(-1)^j\binom{r}{j}\sum_{i=1}^{s+2j-(r+q)k}\binom{s-j-qk-i+1}{rk-1}d_i$$
$$- \sum_{k=1}^{\lfloor\beta_0\rfloor-1}A^{s+2-(r+q)(k+1)}B^{r(k+1)}\sum_{i=1}^{s+2-(r+q)(k+1)}\binom{s-q(k+1)-i-r+1}{rk-1}d_i.$$

By putting $c = s - qk$ and $t = rk - 1$ in Lemma 9.2, we infer

$$T_1 - T_2 = \sum_{k=2}^{\lfloor\beta_0\rfloor} A^{s+2-(r+q)k} B^{rk} \sum_{i=1}^{s+2-(r+q)k} \binom{s-qk-i-r+1}{r(k-1)-1} d_i - \sum_{k=1}^{\lfloor\beta_0\rfloor-1} A^{s+2-(r+q)(k+1)} B^{r(k+1)} \sum_{i=1}^{s+2-(r+q)(k+1)} \binom{s-q(k+1)-i-r+1}{rk-1} d_i.$$

So,

$$T_1 - T_2 = 0. \tag{9.4}$$

<u>*Case 2.*</u> There is a unique j_0, $0 \le j_0 \le r$ such that $\beta_{j_0} \in \mathbb{N}$ and $\beta_j \notin \mathbb{N}$ for $j \neq j_0$ with $\lfloor\beta_j\rfloor = \begin{cases} \beta_{j_0} & \text{for } j \le j_0, \\ \beta_{j_0} - 1 & \text{for } j \ge j_0 + 1. \end{cases}$

Let us write T_1 as three sums by separating the sum indexed by j following $0 \le j \le j_0 - 1$, $j = j_0$ and $j_0 + 1 \le j \le r$.

$$T_1 = \sum_{k=2}^{\beta_{j_0}} A^{s+2-(r+q)k} B^{rk} \sum_{j=0}^{j_0-1} (-1)^j \binom{r}{j} \sum_{i=1}^{s+2-j-(r+q)k} \binom{s-j-qk-i+1}{rk-1} d_i + (-1)^{j_0} \binom{r}{j_0} \sum_{k=2}^{\beta_{j_0}} A^{s+2-(r+q)k} B^{rk} \sum_{i=1}^{s+2-j_0-(r+q)k} \binom{s-j_0-qk-i+1}{rk-1} d_i + \sum_{k=2}^{\beta_{j_0}-1} A^{s+2-(r+q)k} B^{rk} \sum_{j=j_0+1}^{r} (-1)^j \binom{r}{j} \sum_{i=1}^{s+2-j-(r+q)k} \binom{s-j-qk-i+1}{rk-1} d_i.$$

Separating now, by including β_{j_0} value, the sums indexed by k in two parts following $2 \le k \le \beta_{j_0} - 1$ and $k = \beta_{j_0}$. We get by noting that $\binom{s-j_0-q\beta_{j_0}-i+1}{r\beta_{j_0}-1} = 0$ as $s - j_0 - q\beta_{j_0} - i + 1 < r\beta_{j_0} - 1$,

$$T_1 = \sum_{k=2}^{\beta_{j_0}-1} A^{s+2-(r+q)k} B^{rk} \sum_{j=0}^{j_0-1} (-1)^j \binom{r}{j} \sum_{i=1}^{s+2-j-(r+q)k} \binom{s-j-qk-i+1}{rk-1} d_i + A^{j_0} B^{r\beta_{j_0}} \sum_{j=0}^{j_0-1} (-1)^j \binom{r}{j} \sum_{i=1}^{s+2-j-(r+q)\beta_{j_0}} \binom{s-j-q\beta_{j_0}-i+1}{r\beta_{j_0}-1} d_i + (-1)^{j_0} \binom{r}{j_0} \sum_{k=2}^{\beta_{j_0}-1} A^{s+2-(r+q)k} B^{rk} \sum_{i=1}^{s+2-j_0-(r+q)k} \binom{s-j_0-qk-i+1}{rk-1} d_i + \sum_{k=2}^{\beta_{j_0}-1} A^{s+2-(r+q)k} B^{rk} \sum_{j=j_0+1}^{r} (-1)^j \binom{r}{j} \sum_{i=1}^{s+2-j-(r+q)k} \binom{s-j-qk-i+1}{rk-1} d_i.$$

Then,

$$T_1 = A^{j_0} B^{r\beta_{j_0}} \sum_{j=0}^{j_0-1} (-1)^j \binom{r}{j} \sum_{i=1}^{s+2-j-(r+q)\beta_{j_0}} \binom{s-j-q\beta_{j_0}-i+1}{r\beta_{j_0}-1} d_i + \sum_{k=2}^{\beta_{j_0}-1} A^{s+2-(r+q)k} B^{rk} \sum_{j=0}^{r} (-1)^j \binom{r}{j} \sum_{i=1}^{s+2-j-(r+q)k} \binom{s-j-qk-i+1}{rk-1} d_i.$$

If we put in Lemma 9.2, $c = s - qk$ and $t = rk - 1$, we get

$$\begin{aligned} T_1 &= A^{j_0} B^{r\beta_{j_0}} \sum_{j=0}^{j_0-1} (-1)^j \binom{r}{j} \sum_{i=1}^{s+2-j-(r+q)\beta_{j_0}} \binom{s-j-q\beta_{j_0}-i+1}{r\beta_{j_0}-1} d_i \\ &\quad + \sum_{k=2}^{\beta_{j_0}-1} A^{s+2-(r+q)k} B^{rk} \sum_{i=1}^{s+2-(r+q)k} \binom{s-qk-i-r+1}{r(k-1)-1} d_i \\ &= A^{j_0} B^{r\beta_{j_0}} \sum_{j=0}^{j_0-1} (-1)^j \binom{r}{j} \sum_{i=1}^{s+2-j-(r+q)\beta_{j_0}} \binom{s-j-q\beta_{j_0}-i+1}{r\beta_{j_0}-1} d_i \\ &\quad + \sum_{k=1}^{\beta_{j_0}-2} A^{s+2-(r+q)(k+1)} B^{r(k+1)} \sum_{i=1}^{s+2-(r+q)(k+1)} \binom{s-q(k+1)-i-r+1}{rk-1} d_i. \end{aligned}$$

Similarly writing T_2 following $1 \le k \le \beta_{j_0} - 2$ and $k = \beta_{j_0} - 1$. Noting that $\lfloor \beta_0 - 1 \rfloor = \beta_{j_0} - 1$, we obtain

$$\begin{aligned} T_2 = &\sum_{k=1}^{\beta_{j_0}-2} A^{s+2-(r+q)(k+1)} B^{r(k+1)} \sum_{i=1}^{s+2-(r+q)(k+1)} \binom{s-q(k+1)-i-r+1}{rk-1} d_i \\ &+ A^{j_0} B^{r\alpha_{j_0}} \sum_{i=1}^{s+2-(r+q)\beta_{j_0}} \binom{s-q\beta_{j_0}-i-r+1}{r(\beta_{j_0}-1)-1} d_i. \end{aligned}$$

Therefore

$$\begin{aligned} T_1 - T_2 = A^{j_0} B^{r\alpha_{j_0}} &\sum_{j=0}^{j_0-1} (-1)^j \binom{r}{j} \sum_{i=1}^{s+2-j-(r+q)\beta_{j_0}} \binom{s-j-q\beta_{j_0}-i+1}{r\beta_{j_0}-1} d_i \\ &- \sum_{i=1}^{s+2-(r+q)\beta_{j_0}} \binom{s-q\beta_{j_0}-i-r+1}{r(\beta_{j_0}-1)-1} d_i. \end{aligned} \tag{9.5}$$

For $j \ge j_0$, $\binom{s-j-q\beta_{j_0}-i+1}{r\beta_{j_0}-1} = 0$ as $s - j - q\beta_{j_0} - i + 1 < r\beta_{j_0} - 1$ when $1 \le i \le s + 2 - j - (r+q)\beta_{j_0}$. Then

$$\begin{aligned} T_1 - T_2 = A^{j_0} B^{r\alpha_{j_0}} &\sum_{j=0}^{r} (-1)^j \binom{r}{j} \sum_{i=1}^{s+2-j-(r+q)\beta_{j_0}} \binom{s-j-q\beta_{j_0}-i+1}{r\beta_{j_0}-1} d_i \\ &- \sum_{i=1}^{s+2-(r+q)\beta_{j_0}} \binom{s-q\beta_{j_0}-i-r+1}{r(\beta_{j_0}-1)-1} d_i. \end{aligned}$$

Applying Lemma 9.2 for $c = s - q\beta_{j_0}$ and $t = r\beta_{j_0} - 1$, we get $T_1 - T_2 = 0$. □

We set for $r \ge 1$, $q \in \mathbb{Z}^+$, $\beta_j = \frac{s+2-j}{r+q}$ for $0 \le j \le r$. Let then T_3 and T_4 be

$$T_3 := \sum_{j=0}^{r} (-1)^j \binom{r}{j} \sum_{k=2}^{\lfloor \beta_j \rfloor} A^{s+2-(r+q)k} B^{rk} \sum_{i=1}^{rk} \binom{s-j-qk-i+1}{rk-i} d_i,$$

$$T_4 := \sum_{k=1}^{\lfloor \beta_0 - 1 \rfloor} A^{s+2-(r+q)(k+1)} B^{r(k+1)} \sum_{i=1}^{rk} \binom{s-q(k+1)-i-r+1}{rk-i} d_i.$$

Lemma 9.7. *Let $r \ge 1$, $q \in \mathbb{Z}^+$, $(d_n)_n$ a sequence. Then, only one of the two possibilities holds.*

- *For any j, $0 \leq j \leq r$, $\beta_j \notin \mathbb{N}$ and $T_3 = T_4$.*
- *There is a unique $j_0 \in \{0, \ldots, r\}$ such that $\beta_{j_0} \in \mathbb{N}$. In which case,*

$$T_3 - T_4 = A^{j_0} B^{r\alpha_{j_0}} \left[\sum_{j=0}^{j_0-1} (-1)^j \binom{r}{j} \sum_{i=1}^{r\beta_{j_0}} \binom{s-j-q\beta_{j_0}-i+1}{r\beta_{j_0}-i} d_i - \sum_{i=1}^{r\beta_{j_0}-r} \binom{s-q\beta_{j_0}-i-r+1}{r(\beta_{j_0}-1)-i} d_i \right].$$

P r o o f. We follow the same steps as in the proof of Lemma 9.6 by using Lemma 9.3 instead of Lemma 9.2. □

Remark 9. Note that in the second case of Lemma 9.7, as the sum over an empty set is zero and $\binom{n}{k} = 0$ when $n < k$, then

$$T_3 - T_4 = 0 \qquad \text{for } j_0 = 0.$$

10. Proof of Theorem 4.1

Without loss of generality, we will prove Theorem 4.1 for $p = 0$. The sequence $\big(G_n^{(0)}\big)_n$ will be written $\big(G_n\big)_n$ for short.
We set

$$Z := \sum_{j=0}^{r} (-A)^j \binom{r}{j} G_{n-j} - \sum_{j=0}^{r} (-A)^j \binom{r}{j} a_{n-j-1} + \sum_{j=0}^{r-1} M_j \left[\alpha_j \in \mathbb{N}\right],$$

and prove that for $n \geq r+q$, $Z = B^r G_{n-q-r}$. So, replacing G_{n-j} by relation (4.3) and taking account of Remark 1, we obtain

$$Z = \sum_{j=0}^{r} (-1)^j \binom{r}{j} \sum_{k=1}^{\lfloor \alpha_j \rfloor} A^{n-1-(r+q)k} B^{rk} \left(\sum_{i=1}^{n-1-j-(r+q)k} \binom{n-j-2-qk-i}{rk-1} \frac{a_i}{A^i} \right.$$
$$\left. + \sum_{i=1}^{rk} \binom{n-j-2-qk-i}{rk-i} \frac{b_i}{B^i} \right) + \sum_{j=0}^{r} (-A)^j \binom{r}{j} b_{r\alpha_j} \left[\alpha_j \in \mathbb{N}\right] + \sum_{j=0}^{r-1} M_j \left[\alpha_j \in \mathbb{N}\right].$$

Let us set,

$$U = \sum_{j=0}^{r} (-1)^j \binom{r}{j} \sum_{k=1}^{\lfloor \alpha_j \rfloor} A^{n-1-(r+q)k} B^{rk} \sum_{i=1}^{n-1-j-(r+q)k} \binom{n-j-2-qk-i}{rk-1} \frac{a_i}{A^i}$$

and

$$V = \sum_{j=0}^{r} (-1)^j \binom{r}{j} \sum_{k=1}^{\lfloor \alpha_j \rfloor} A^{n-1-(r+q)k} B^{rk} \sum_{i=1}^{rk} \binom{n-j-2-qk-i}{rk-i} \frac{b_i}{B^i}.$$

If we separate the sum in the right-hand side of U isolating the term with $k = 1$, we get

$$U = A^{n-1-(r+q)} B^r \sum_{j=0}^{r} (-1)^j \binom{r}{j} \sum_{i=1}^{n-1-j-(r+q)} \binom{n-j-2-q-i}{r-1} \frac{a_i}{A^i}$$
$$+ \sum_{j=0}^{r} (-1)^j \binom{r}{j} \sum_{k=2}^{\lfloor \alpha_j \rfloor} A^{n-1-(r+q)k} B^{rk} \sum_{i=1}^{n-1-j-(r+q)} \binom{n-j-2-qk-i}{rk-1} \frac{a_i}{A^i}.$$

Lemma 9.1 gives

$$U = B^r a_{n-1-q-r} + \sum_{j=0}^{r}(-1)^j \binom{r}{j} \sum_{k=2}^{\lfloor \alpha_j \rfloor} A^{n-1-(r+q)k} B^{rk} \sum_{i=1}^{n-1-j-(r+q)} \binom{n-j-2-qk-i}{rk-1} \frac{a_i}{A^i}.$$

We do the same for V,

$$\begin{aligned} V &= A^{n-1-(r+q)} B^r \sum_{j=0}^{r}(-1)^j \binom{r}{j} \sum_{i=1}^{r} \binom{n-j-2-q-i}{r-i} \frac{b_i}{B^i} \\ &+ \sum_{j=0}^{r}(-1)^j \binom{r}{j} \sum_{k=2}^{\lfloor \alpha_j \rfloor} A^{n-1-(r+q)k} B^{rk} \sum_{i=1}^{rk} \binom{n-j-2-qk-i}{rk-i} \frac{b_i}{B^i}. \end{aligned}$$

Lemma 9.3 gives

$$V = \sum_{j=0}^{r}(-1)^j \binom{r}{j} \sum_{k=2}^{\lfloor \alpha_j \rfloor} A^{n-1-(r+q)k} B^{rk} \sum_{i=1}^{rk} \binom{n-j-2-qk-i}{rk-i} \frac{b_i}{B^i}.$$

Moreover, replacing G_{n-q-r} by relation (4.3), we get

$$\begin{aligned} B^r G_{n-q-r} = &\sum_{k=1}^{\lfloor \alpha_0 - 1 \rfloor} A^{n-1-(r+q)(k+1)} B^{r(k+1)} \Bigg(\sum_{i=1}^{n-1-(r+q)(k+1)} \binom{n-2-q(k+1)-i-r}{rk-1} \frac{a_i}{A^i} \\ &+ \sum_{i=1}^{rk} \binom{n-2-q(k+1)-i-r}{rk-i} \frac{b_i}{B^i} \Bigg) + B^r a_{n-q-r-1} + B^r b_{r(\alpha_0-1)} \left[\alpha_0 - 1 \in \mathbb{N}\right]. \end{aligned}$$

Let us set now,

$$\begin{aligned} U^* &= \sum_{k=1}^{\lfloor \alpha_0 - 1 \rfloor} A^{n-1-(r+q)(k+1)} B^{r(k+1)} \sum_{i=1}^{n-1-(r+q)(k+1)} \binom{n-2-q(k+1)-i-r}{rk-1} \frac{a_i}{A^i} + B^r a_{n-q-r-1}, \\ V^* &= \sum_{k=1}^{\lfloor \alpha_0 - 1 \rfloor} A^{n-1-(r+q)(k+1)} B^{r(k+1)} \sum_{i=1}^{rk} \binom{n-2-q(k+1)-i-r}{rk-i} \frac{b_i}{B^i}. \end{aligned}$$

Then,

$$Z - B^r G_{n-q-r} = (U - U^*) + (V - V^*) + W, \tag{10.1}$$

with

$$\begin{aligned} U - U^* = &\sum_{j=0}^{r}(-1)^j \binom{r}{j} \sum_{k=2}^{\lfloor \alpha_j \rfloor} A^{n-1-(r+q)k} B^{rk} \sum_{i=1}^{n-1-j-(r+q)k} \binom{n-j-2-qk-i}{rk-1} \frac{a_i}{A^i} \\ &- \sum_{k=1}^{\lfloor \alpha_0 - 1 \rfloor} A^{n-1-(r+q)(k+1)} B^{r(k+1)} \sum_{i=1}^{n-1-(r+q)(k+1)} \binom{n-2-q(k+1)-i-r}{rk-1} \frac{a_i}{A^i}, \\ V - V^* = &\sum_{j=0}^{r}(-1)^j \binom{r}{j} \sum_{k=2}^{\lfloor \alpha_j \rfloor} A^{n-1-(r+q)k} B^{rk} \sum_{i=1}^{rk} \binom{n-j-2-qk-i}{rk-i} \frac{b_i}{B^i} \\ &- \sum_{k=1}^{\lfloor \alpha_0 - 1 \rfloor} A^{n-1-(r+q)(k+1)} B^{r(k+1)} \sum_{i=1}^{rk} \binom{n-2-q(k+1)-i-r}{rk-i} \frac{b_i}{B^i}, \end{aligned}$$

and

$$W=\sum_{j=0}^{r}(-A)^j\binom{r}{j}b_{r\alpha_j}\left[\alpha_j\in\mathbb{N}\right]+\sum_{j=0}^{r-1}M_j\left[\alpha_j\in\mathbb{N}\right]-B^r b_{r(\alpha_0-1)}\left[\alpha_0-1\in\mathbb{N}\right].$$

We aim to prove that $Z-B^rG_{n-q-r}$, given by formula (10.1), is zero by Lemma 9.5. So, we will consider the two cases for each of the terms $U-U^*$, $V-V^*$ and W.

Case 1. If $\alpha_j\notin\mathbb{N}$, $\lfloor\alpha_j\rfloor=\lfloor\alpha_0\rfloor$ for $0\le j\le r$, then $W=0$ and by replacing s by $n-3$ in Lemmata 9.6 and 9.7, we get

$$U-U^*=T_1-T_2=0,\qquad V-V^*=T_3-T_4=0.$$

Therefore $Z-B^rG_{n-q-r}=0$.

Case 2. There exists j_0, $0\le j_0\le r$ such that $\alpha_{j_0}\in\mathbb{N}$ and for $j\neq j_0$, $\alpha_j\notin\mathbb{N}$. So, we are dealing with two alternatives: $j_0=0$ and $j_0\neq 0$.

1. Suppose that $j_0=0$. So from Lemma 9.5, $\alpha_0\in\mathbb{N}$ and $\alpha_j\notin\mathbb{N}$ with $\lfloor\alpha_j\rfloor=\alpha_0-1$ for $j\neq 0$. From relation (4.1), $W=0$ and by replacing s by $n-3$ in Lemmata 9.6 and 9.7, we get

$$U-U^*=T_1-T_2=0,\qquad V-V^*=T_3-T_4.$$

From Remark 9, $V-V^*=0$. Therefore $Z-B^rG_{n-q-r}=0$.

2. Suppose now that $j_0\neq 0$. From Lemma 9.5, $\alpha_{j_0}\in\mathbb{N}$ and for $j\neq j_0$, $\alpha_j\notin\mathbb{N}$ with $\lfloor\alpha_j\rfloor=\begin{cases}\alpha_{j_0} & \text{for } j\le j_0,\\ \alpha_{j_0}-1 & \text{for } j\ge j_0+1.\end{cases}$ In this case, by relation (4.2),

$$W=(-A)^{j_0}\binom{r}{j_0}b_{r\alpha_{j_0}}+M_{j_0}.$$

Replacing s by $n-3$ in Lemmata 9.6 and 9.7, we get, $U-U^*=T_1-T_2=0$ and

$$\begin{aligned}V-V^*&=T_3-T_4\\&=A^{j_0}B^{r\alpha_{j_0}}\left[\sum_{j=0}^{j_0-1}(-1)^j\binom{r}{j}\sum_{i=1}^{r\alpha_{j_0}}\binom{n-2-j-q\alpha_{j_0}-i}{r\alpha_{j_0}-i}\frac{b_i}{B^i}-\sum_{i=1}^{r\alpha_{j_0}-r}\binom{n-2-q\alpha_{j_0}-i-r}{r(\alpha_{j_0}-1)-i}\frac{b_i}{B^i}\right].\end{aligned}$$

Introducing the value of α_{j_0}, we rewrite $V-V^*$ as

$$V-V^*=A^{j_0}B^{r\alpha_{j_0}}\left(\sum_{j=0}^{j_0-1}(-1)^j\binom{r}{j}\sum_{i=1}^{r\alpha_{j_0}}\binom{r\alpha_{j_0}-1+j_0-i-j}{j_0-j-1}\frac{b_i}{B^i}-\sum_{i=1}^{r\alpha_{j_0}-r}\binom{r\alpha_{j_0}-1+j_0-i-r}{j_0-1}\frac{b_i}{B^i}\right).$$

By replacing $U-U^*$, $V-V^*$, W in (10.1), we get

$$\begin{aligned}Z-B^rG_{n-q-r}&=A^{j_0}B^{r\alpha_{j_0}}\sum_{j=0}^{j_0-1}(-1)^j\binom{r}{j}\sum_{i=1}^{r\alpha_{j_0}}\binom{r\alpha_{j_0}-1+j_0-i-j}{j_0-j-1}\frac{b_i}{B^i}\\&\quad-A^{j_0}B^{r\alpha_{j_0}}\sum_{i=1}^{r\alpha_{j_0}-r}\binom{r\alpha_{j_0}-1+j_0-i-r}{j_0-1}\frac{b_i}{B^i}+(-A)^{j_0}\binom{r}{j_0}b_{r\alpha_{j_0}}+M_{j_0}.\end{aligned}$$

Now, we write the first sum in the right-hand side of $Z-B^rG_{n-q-r}$ by isolating $i=r\alpha_{j_0}$ to obtain

$$\begin{aligned}Z-B^rG_{n-q-r}&=A^{j_0}B^{r\alpha_{j_0}}\sum_{j=0}^{j_0-1}(-1)^j\binom{r}{j}\sum_{i=1}^{r\alpha_{j_0}-1}\binom{r\alpha_{j_0}-1+j_0-i-j}{j_0-j-1}\frac{b_i}{B^i}\\&\quad+A^{j_0}\sum_{j=0}^{j_0-1}(-1)^j\binom{r}{j}\binom{j_0-j-1}{j_0-j-1}b_{r\alpha_{j_0}}\end{aligned}$$

$$- A^{j_0}B^{r\alpha_{j_0}}\sum_{i=1}^{r\alpha_{j_0}-r}\binom{r\alpha_{j_0}-1+j_0-i-r}{j_0-1}\frac{b_i}{B^i}+(-A)^{j_0}\binom{r}{j_0}b_{r\alpha_{j_0}}+M_{j_0}.$$

From relation (4.2) and (9.2)

$$M_{j_0}=-A^{j_0}\left(\sum_{s=0}^{j_0}(-1)^s\binom{r}{s}b_{r\alpha_{j_0}}+\sum_{t=1}^{r-j_0}B^t\lambda_t^{(j_0)}b_{r\alpha_{j_0}-t}\right).$$

Then

$$Z-B^rG_{n-q-r}=A^{j_0}B^{r\alpha_{j_0}}\left(\sum_{j=0}^{j_0-1}(-1)^j\binom{r}{j}\sum_{i=1}^{r\alpha_{j_0}-1}\binom{r\alpha_{j_0}-1+j_0-i-j}{j_0-j-1}\frac{b_i}{B^i}\right.$$
$$\left.-\sum_{i=1}^{r\alpha_{j_0}-r}\binom{r\alpha_{j_0}-1+j_0-i-r}{j_0-1}\frac{b_i}{B^i}\right)-A^{j_0}\sum_{t=1}^{r-j_0}B^t\lambda_t^{(j_0)}b_{r\alpha_{j_0}-t}.$$

Now, we separate the first sum in $Z-B^rG^0_{n-q-r}$ following the index i in three parts; $1\le i\le r\alpha_{j_0}-r$, $r\alpha_{j_0}-r+1\le i\le r\alpha_{j_0}-r-1+j_0$ and $r\alpha_{j_0}-r+j_0\le i\le r\alpha_{j_0}-1$ to obtain

$$Z-B^rG_{n-q-r}=A^{j_0}B^{r\alpha_{j_0}}\left(Z_1+Z_2+Z_3-Z_4-A^{j_0}\sum_{t=1}^{r-j_0}B^t\lambda_t^{(j_0)}b_{r\alpha_{j_0}-t}\right),$$

with

$$Z_1=\sum_{j=0}^{j_0-1}(-1)^j\binom{r}{j}\sum_{i=1}^{r\alpha_{j_0}-r}\binom{r\alpha_{j_0}-1+j_0-i-j}{j_0-j-1}\frac{b_i}{B^i},$$
$$Z_2=\sum_{j=0}^{j_0-1}(-1)^j\binom{r}{j}\sum_{i=r\alpha_{j_0}-r+1}^{r\alpha_{j_0}-r-1+j_0}\binom{r\alpha_{j_0}-1+j_0-i-j}{j_0-j-1}\frac{b_i}{B^i},$$
$$Z_3=\sum_{j=0}^{j_0-1}(-1)^j\binom{r}{j}\sum_{i=r\alpha_{j_0}-r+j_0}^{r\alpha_{j_0}-1}\binom{r\alpha_{j_0}-1+j_0-i-j}{j_0-j-1}\frac{b_i}{B^i}$$
$$Z_4=\sum_{i=1}^{r\alpha_{j_0}-r}\binom{r\alpha_{j_0}-1+j_0-i-r}{j_0-1}\frac{b_i}{B^i}.$$

Since $\binom{r\alpha_{j_0}-1+j_0-i-j}{j_0-j-1}=0$ for $j\ge j_0$, Z_1 becomes

$$Z_1=\sum_{j=0}^{r}(-1)^j\binom{r}{j}\sum_{i=1}^{r\alpha_{j_0}-r}\binom{r\alpha_{j_0}-1+j_0-i-j}{j_0-j-1}\frac{b_i}{B^i}$$
$$=\sum_{j=0}^{r}(-1)^j\binom{r}{j}\sum_{i=1}^{r\alpha_{j_0}-r}\binom{r\alpha_{j_0}-1+j_0-i-j}{r\alpha_{j_0}-i}\frac{b_i}{B^i}\quad\text{as}\quad\binom{n}{k}=\binom{n}{n-k}.$$

Then Lemma 9.3 gives

$$Z_1=\sum_{i=1}^{r\alpha_{j_0}-r}\binom{r\alpha_{j_0}-1+j_0-i-r}{j_0-1}\frac{b_i}{B^i}.$$

So,

$$Z_1-Z_4=0.$$

Otherwise, by setting $k = i - r\alpha_{j_0} + r$, it is easy to see that

$$Z_2 = \sum_{j=0}^{r}(-1)^j\binom{r}{j}\sum_{k=1}^{j_0-1}\binom{r-1+j_0-k-j}{r-k}\frac{b_{k+r\alpha_{j_0}-r}}{B^{k+r\alpha_{j_0}-r}}$$
$$= \sum_{k=1}^{j_0-1}\frac{b_{k+r\alpha_{j_0}-r}}{B^{k+r\alpha_{j_0}-r}}\sum_{j=0}^{r}(-1)^j\binom{r}{j}\binom{r-1+j_0-k-j}{r-k}.$$

Then according to Lemma 9.3, $Z_2 = 0$. So,

$$Z - B^r G_{n-q-r} = A^{j_0}B^{r\alpha_{j_0}}Z_3 - A^{j_0}\sum_{t=1}^{r-j_0}B^t\lambda_t^{(j_0)}b_{r\alpha_{j_0}-t},$$

with

$$Z_3 = \sum_{j=0}^{j_0-1}\sum_{s=1}^{r-j_0}(-1)^j\binom{r}{j}\binom{s+j_0-j-1}{s}\frac{b_{r\alpha_{j_0}-s}}{B^{r\alpha_{j_0}-s}}$$

by setting $s = r\alpha_{j_0} - i$, and from Lemma 9.4,

$$Z_3 = \sum_{s=1}^{r-j_0}(-1)^{j_0-1}\binom{r-s-1}{j_0-1}\frac{b_{r\alpha_{j_0}-s}}{B^{r\alpha_{j_0}-s}} = \sum_{s=1}^{r-j_0}\lambda_s^{(j_0)}\frac{b_{r\alpha_{j_0}-s}}{B^{r\alpha_{j_0}-s}}.$$

Therefore, $Z - B^r G_{n-q-r} = 0$ and the proof is complete.

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