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Mathematica Slovaca

DOI: 10.1515/ms-2023-0022
Math. Slovaca **73** (2023), No. 1, 263–273

A NOTE ON STAR PARTIAL ORDER PRESERVERS ON THE SET OF ALL VARIANCE-COVARIANCE MATRICES

Gregor Dolinar* — Bojan Kuzma** — Janko Marovt***, c — Dijana Mosić****

(*Communicated by Marek Balcerzak*)

ABSTRACT. Let $H_n^+(\mathbb{R})$ be the cone of all positive semidefinite $n \times n$ real matrices. We describe the form of all surjective maps on $H_n^+(\mathbb{R})$, $n \geq 3$, that preserve the star partial order in both directions.

1. Introduction

Let $\mathbb{F}$ denote the field of all real numbers $\mathbb{R}$ or the field of all complex numbers $\mathbb{C}$, and let $M_{m,n}(\mathbb{F})$ be the set of all $m \times n$ matrices over $\mathbb{F}$. When $m = n$, we write $M_n(\mathbb{F})$ instead of $M_{n,n}(\mathbb{F})$. By A^* we denote the conjugate transpose of $A \in M_{m,n}(\mathbb{F})$ (if $A \in M_{m,n}(\mathbb{R})$, then $A^* = A^t$, the transpose of A). For $A \in M_{m,n}(\mathbb{F})$, let $\operatorname{Im} A$ and $\operatorname{rank}(A)$ be the image and the rank of A, respectively. Let $H_n(\mathbb{F})$ denote the set of all Hermitian (i.e., symmetric in the real case) matrices in $M_n(\mathbb{F})$. A matrix $A \in H_n(\mathbb{F})$ is said to be positive semidefinite if $x^*Ax \geq 0$ for every $x \in \mathbb{F}^n \equiv M_{n,1}(\mathbb{F})$. We denote by $H_n^+(\mathbb{F})$ the cone of all positive semidefinite matrices in $H_n(\mathbb{F})$, and let $P_n(\mathbb{F})$ be the set of all idempotent matrices in $H_n^+(\mathbb{F})$ (i.e., the set of all orthogonal projector (or self-adjoint idempotent) matrices in $M_n(\mathbb{F})$). The study of positive semidefinite matrices is a flourishing area of mathematical investigation (see, e.g., the monograph [1] and the references therein). Positive semidefinite matrices have become fundamental computational objects in many areas of statistics, engineering, quantum information, and applied mathematics. They appear as variance-covariance matrices (also known as dispersion or covariance matrices) in statistics [5], as elements of the search space in convex and semidefinite programming [17], as kernels in machine learning [22], as density matrices in quantum information [16], and as diffusion tensors in medical imaging [4]. It is known (see, e.g., [5: page 4]) that every variance-covariance matrix is positive semidefinite, and that every real positive semidefinite matrix is a variance-covariance matrix of some multivariate distribution.

2020 Mathematics Subject Classification: 15B48, 47B49, 47L07, 54F05, 62J99.
Keywords: Preserver, star partial order, linear model, variance-covariance matrix.
The first, the second, and the third author acknowledge the financial support from the Slovene Research Agency, ARRS (research programs No. P1-0222, No. P1-0285, and No. P1-0288, and research project No. N1-0210). The fourth author is supported by the Ministry of Education, Science and Technological Development, Republic of Serbia, grant no. 451-03-68/2022-14/200124. The authors also acknowledge the bilateral project between Serbia and Slovenia (Generalized inverses, operator equations and applications) was financially supported by the Ministry of Education, Science and Technological Development, Republic of Serbia (grant no. 337-00-21/2020-09/32) and by the Slovene Research Agency, ARRS (grant BI-RS/20-21-039).
c Corresponding author.

There are many partial orders that may be defined on various sets of matrices and a few of them have applications in statistics especially in the theory of linear models. Let $A, B \in H_n(\mathbb{F})$. Then we say that A is below (or is dominated by) B with respect to *the Löwner partial order* and write

$$A \leq^L B \qquad \text{if} \quad B - A \quad \text{is positive semidefinite.}$$

Let $y = X\beta + \varepsilon$ be the vector form of a linear model. Here y is a real $n \times 1$ random vector of observed quantities, which we try to explain with unknown parameters given by vector β, while the coefficients of the matrix $X \in M_{n,p}(\mathbb{R})$ are known (determined by the model), see, e.g., [5: Chapter 1]. It is assumed that mathematical expectation of ε is 0 and that $\sigma^2 D$ is the variance-covariance matrix of ε. The nonnegative parameter σ^2 and the vector of parameters (real numbers) β are unspecified, and $D \in H_n(\mathbb{R})$ is a known positive semidefinite matrix. We denote this linear model with the triplet $(y, X\beta, \sigma^2 D)$. Statistical analysis often focuses on answering questions about certain linear functions of the form $C\beta$ for a specified real matrix C with p columns. We try to estimate $C\beta$ by a linear function Ay of the response y (here A is a real matrix with n columns). We say that Ay is *a linear unbiased estimator* (LUE) of $C\beta$ if the mathematical expectation of Ay equals $C\beta$ for all possible values of $\beta \in \mathbb{R}^p$. The function $C\beta$ is said to be estimable if it has LUE. *The best linear unbiased estimator* (BLUE) of an estimable $C\beta$ is defined as LUE having the smallest variance-covariance matrix ("smallest" in terms of the Löwner partial order on $H_n^+(\mathbb{R})$).

Another well-known partial order is *the star partial order.* It was introduced in [8] and can be on $M_{m,n}(\mathbb{F})$ defined as follows (see also [21: Chapter 5]). For $A, B \in M_{m,n}(\mathbb{F})$, we write

$$A \leq^* B \qquad \text{if} \quad A^*A = A^*B \text{ and } AA^* = BA^*. \tag{1}$$

Two partial orders that are related to the star partial order are *the left-star* and *the right-star partial orders* [2]. For $A, B \in M_{m,n}(\mathbb{F})$, we say that A is below B with respect to the left-star partial order and write

$$A {*}{\leq}\, B \quad \text{if} \qquad A^*A = A^*B \text{ and } \operatorname{Im} A \subseteq \operatorname{Im} B.$$

Similarly, we define the right-star partial order. For $A, B \in M_{m,n}(\mathbb{F})$, we write

$$A \leq{*} B \qquad \text{if} \quad AA^* = AB^* \text{ and } \operatorname{Im} A^* \subseteq \operatorname{Im} B^*.$$

Note that for $A, B \in M_{m,n}(\mathbb{F})$, $A \leq^* B$ implies $A {*}{\leq}\, B$ and $A \leq{*} B$ (see, e.g., [21: Theorem 6.5.14]), and that the star, the left-star, and the right-star partial orders are the same partial order on $H_n(\mathbb{F})$.

The left-star partial order has applications in the theory of linear models. Let us present two examples of such applications. Linear models $(y, X\beta, \sigma^2 D)$, where $D = I$ is the identity matrix, are called the Gauss-Markov linear models. The following result gives an interpretation of the left-star order in such linear models (see [21: Theorem 15.3.7]).

Proposition 1. *Let $L_1 = (y, X_1\beta, \sigma^2 I)$ and $L_2 = (y, X_2\beta, \sigma^2 I)$ be any two (Gauss-Markov) linear models. Then $X_1 {*}{\leq}\, X_2$ if and only if the following statements hold.*

(i) *The linear models L_1 and $L = (y, (X_2 - X_1)\beta, \sigma^2 I)$ have no common estimable linear function of β;*

(ii) *$X_1\beta$ is estimable under the model L_2;*

(iii) *The BLUE of $X_1\beta$ under the model L_1 is also its BLUE under L_2, and the variance-covariance matrix of the BLUE of $X_1\beta$ under the model L_1 is the same as under the model L_2.*

Let $X \in M_{n,p}(\mathbb{R})$ and let $s\beta$ be an estimable linear parametric function (here s is a $1 \times p$ real vector). A linear function of the response, ly (here l is a $1 \times n$ real vector) is called *a linear zero function* (LZF) if $E(ly) = 0$ for all possible values of β. Here $E(ly)$ denotes the mathematical expectation of the random variable ly. Due to the additivity property of the mathematical expectation, one can easily see that by adding LZFs to LUE, we get other LUEs of $s\beta$. Next we present another application in the theory of comparison of linear models when the model matrices are related via the left-star partial order (see [21: Corollary 15.3.8]).

Proposition 2. *Let $L_1 = (y, X_1\beta, \sigma^2 I)$ and $L_2 = (y, X_2\beta, \sigma^2 I)$ be any two linear models such that $X_1 {*\leq} X_2$. Then the BLUE of every estimable linear function of the parameters under L_1 is a linear zero function under $L = (y, (X_2 - X_1)\beta, \sigma^2 I)$ and vice versa.*

Let $\mathcal{A}$ be some subset of $M_n(\mathbb{F})$ and denote by $\leq$ a partial order on $\mathcal{A}$. We say that a map $\Phi\colon \mathcal{A} \to \mathcal{A}$ is a bi-preserver of the order $\leq$ (or that it preserves the order $\leq$ in both directions) when

$$A \leq B \qquad \text{if and only if} \quad \Phi(A) \leq \Phi(B)$$

for every $A, B \in \mathcal{A}$. Instead of the linear model $M = (y, X\beta, \sigma^2 I)$ one might rather work with the transformed model $\hat{M} = (y, \hat{X}\beta, \sigma^2 I)$, where the new matrix $\hat{X} \in M_n(\mathbb{R})$ is chosen so that its properties are more attractive than $X \in M_n(\mathbb{R})$ (e.g., elements of X that are very close to zero are transformed to zero), and thus it is natural (see [6]) to demand that the transformed model still retains most of the properties of the original model (e.g., has similar relations to other transformed models). Thus, in view of Propositions 1 and 2, it is interesting to know what transformations on $M_n(\mathbb{R})$ or perhaps on some subset of $M_n(\mathbb{R})$ (like $H_n^+(\mathbb{R})$) preserve the left-star partial order. Linear bijective maps on $M_n(\mathbb{F})$ that preserve either the star, or the left-star, or the right-star partial order in one direction (i.e., $A \leq B$ implies $\Phi(A) \leq \Phi(B)$, where $\leq$ is the appropriate order) were investigated in [12] (see also [11]). In [6], the forms of surjective bi-preservers of the left-star partial order and the right-star partial order on $M_n(\mathbb{F})$, $n \geq 3$, were described. Surjective bi-preservers of the star partial order on $M_n(\mathbb{F})$, $n \geq 3$, were characterized in [18]. Since surjective bi-preservers of the star, the left-star, and the right-star partial orders on the full matrix algebra $M_n(\mathbb{F})$, $n \geq 3$, have already been described (see also [13]) and because the set of all positive semidefinite real matrices has an important role in statistics, we focus our attention on the star partial order bi-preservers on $H_n^+(\mathbb{R})$. In [10] (see also [9]), the following result was proved.

Theorem 3. *Let $n \geq 3$ be an integer. Then $\Phi\colon H_n^+(\mathbb{R}) \to H_n^+(\mathbb{R})$ is a surjective, additive bi-preserver of the star partial order if and only if there exist an orthogonal matrix $R \in M_n(\mathbb{R})$ and $\lambda > 0$ such that*

$$\Phi(A) = \lambda RAR^t$$

for every $A \in H_n^+(\mathbb{R})$.

In this paper, we study bi-preservers of the star partial order on $H_n^+(\mathbb{R})$ that may be nonadditive. In Section 2, we give some preliminary results. The main result which describes the form of all surjective bi-preservers of the star partial order on $H_n^+(\mathbb{R})$, $n \geq 3$, is stated and proved in Section 3.

2. Preliminaries

The following properties of the star partial order are well-known and can be easily verified (see, e.g., [18: Proposition 2.1, Corollary 3.6]).

Proposition 4. *Let $A, B \in M_n(\mathbb{F})$ and $\alpha \in \mathbb{F}$ with $\alpha \neq 0$. Let $U, V \in M_n(\mathbb{F})$ be unitary matrices (i.e., orthogonal when $\mathbb{F} = \mathbb{R}$). The following statements are then equivalent.*

(i) $A \leq^* B$.

(ii) $\alpha A \leq^* \alpha B$.

(iii) $UAV \leq^* UBV$.

Proposition 5. *Let $A, B \in M_n(\mathbb{F})$. If $A \leq^* B$ and $A \neq B$, then* $\operatorname{rank}(A) < \operatorname{rank}(B)$.

The next lemma states that any Hermitian matrix A is idempotent if and only if the identity matrix I is its upper bound with respect to the star partial order (see [15: Theorem 3.4]).

Lemma 6. *For $A \in H_n(\mathbb{F})$, $A \leq^* I$ if and only if $A^2 = A$.*

Proof. Suppose $A^2 = A$, $A \in H_n(\mathbb{F})$. Then $A^*A = A^2 = A = AI = A^*I$ and similarly $AA^* = IA^*$. So, $A \leq^* I$. Conversely, suppose $A \leq^* I$ for $A \in H_n(\mathbb{F})$. By, e.g., [20: Theorem 8], $A^2 = A$. □

Let $x, y \in \mathbb{F}^n$ be nonzero. We denote by xy^* a rank-one matrix in $M_n(\mathbb{F})$. Note that it represents a rank-one linear operator on $\mathbb{F}^n$ defined with $(xy^*)z = \langle z, y\rangle\, x$ for every $z \in \mathbb{F}^n$ (here $\langle z, y\rangle = y^*z$ is an inner product in $\mathbb{F}^n$). It is known that every rank-one matrix in $M_n(\mathbb{F})$ may be written in this form and that $P \in P_n(\mathbb{F})$ is of rank-one if and only if $P = xx^*$ for some $x \in \mathbb{F}^n$ with $\|x\| = 1$. The following result was proved in [18] (see also [10]).

Proposition 7. *Let $A \in M_n(\mathbb{F})$ and let $x, y \in \mathbb{F}^n$ be nonzero. Then*

$$xy^* \leq^* A \qquad \text{if and only if} \qquad A^*x = \langle x, x\rangle y \quad \text{and} \quad Ay = \langle y, y\rangle x.$$

Corollary 8. *Let $A \in H_n(\mathbb{F})$ and let $x \in \mathbb{F}^n$ be nonzero. Then*

$$xx^* \leq^* A \qquad \text{if and only if} \qquad Ax = \langle x, x\rangle x.$$

Denote by $E_{i,j} \in M_n(\mathbb{F})$ the matrix with all entries equal to zero except the (i, j)-entry which is equal to one, and by $e_1, e_2, \ldots, e_n \in \mathbb{F}^n$ the standard basis vectors.

Lemma 9. *Let xx^*, $yy^* \in H_n^+(\mathbb{F})$ be rank-one matrices with different spectra. Then xx^* and yy^* have a common upper bound with respect to the star partial order if and only if vectors x and y are orthogonal.*

Proof. By using unitary similarity, we may by Proposition 4 assume that $x = \lambda e_1$ for some nonzero $\lambda \in \mathbb{R}$ and $y = \alpha e_1 + \beta e_2$, $\alpha, \beta \in \mathbb{F}$, where $0 < \alpha^2 + \beta^2$. Note that $\lambda^2 \neq \alpha^2 + \beta^2$ since xx^* and yy^* have different spectra.

Suppose first that x and y are orthogonal. Then $\alpha = 0$ and hence $A = xx^* + yy^* = \lambda^2 e_1 e_1^* + \beta^2 e_2 e_2^*$ is a common upper bound of xx^* and yy^*.

Suppose now x and y are not orthogonal, and let us assume that there exists $A = (a_{i,j}) \in H_n^+(\mathbb{F})$ which is a common upper bound of xx^* and yy^*. By Corollary 8, we have $A(\lambda e_1) = \lambda^3 e_1$ and thus $Ae_1 = \lambda^2 e_1$. Also, $A(\alpha e_1 + \beta e_2) = (\alpha^2 + \beta^2)(\alpha e_1 + \beta e_2)$. It follows that

$$\alpha\lambda^2 e_1 + \beta A e_2 = (\alpha^2 + \beta^2)\alpha e_1 + (\alpha^2 + \beta^2)\beta e_2$$

and therefore,

$$\beta A e_2 = (\alpha^2 + \beta^2 - \lambda^2)\alpha e_1 + (\alpha^2 + \beta^2)\beta e_2. \tag{2}$$

If $\alpha = 0$, then $y = \beta e_2$ and thus x and y are orthogonal, a contradiction. If $\beta = 0$, then we get a contradiction by (2) since $\alpha \neq 0$ and $\alpha^2 + \beta^2 - \lambda^2 \neq 0$. By $Ae_1 = \lambda^2 e_1$, we obtain $a_{2,1} = 0$, and since $\beta \neq 0$, we get (see (2)) that $a_{1,2} \neq 0$. So, $A \notin H_n(\mathbb{F})$, a contradiction. □

Let $A \in M_n(\mathbb{F})$ be nonzero. As a direct corollary of the singular value decomposition there is a unique decomposition

$$A = \sum_{j=1}^{k} t_j V_j,$$

where $t_1 > t_2 > \cdots > t_k > 0$ and $V_1, V_2, \dots, V_k$ are pairwise orthogonal nonzero partial isometries. This decomposition is known as the Penrose decomposition. Here rank $A = \sum_{j=1}^{k} \text{rank } (V_j)$. In [14: page 371] (see also [18: Theorem 3.3]), the following result was proved.

Proposition 10. *Let A, $B \in M_n(\mathbb{F})$ have the Penrose decompositions*

$$A = \sum_{j=1}^{k} t_j V_j, \qquad B = \sum_{i=1}^{l} u_i W_i.$$

Then $A \leq^* B$ if and only if there exists a function $\varphi\colon \{1, 2, \dots, k\} \to \{1, 2, \dots, l\}$ such that $t_j = u_{\varphi(j)}$ and $V_j \leq^* W_{\varphi(j)}$ for all $j \in \{1, 2, \dots, k\}$.

Remark 11. Note that the function φ from Proposition 10 is injective. Indeed, if $\varphi(i) = \varphi(j)$, then $t_i = u_{\varphi(i)} = u_{\varphi(j)} = t_j$ which implies by the definition of the Penrose decomposition that $i = j$.

Let now $A \in H_n^+(\mathbb{F})$ be nonzero. By the spectral theorem there exists a unique (eigenvalue) decomposition

$$A = \sum_{j=1}^{k} \lambda_j P_j,$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$ are (nonzero) eigenvalues of A and $P_1, P_2, \dots, P_k$ are pairwise orthogonal nonzero projectors. Observe that this is in fact also the Penrose decomposition of A. We thus have the following direct corollary of Proposition 10.

Corollary 12. *Let $A, B \in H_n^+(\mathbb{F})$ have the eigenvalue decompositions*

$$A = \sum_{j=1}^{k} \lambda_j P_j, \qquad B = \sum_{i=1}^{l} \mu_i Q_i.$$

Then $A \leq^* B$ if and only if there exists a function $\varphi\colon \{1, 2, \dots, k\} \to \{1, 2, \dots, l\}$ such that $\lambda_j = \mu_{\varphi(j)}$ and $P_j \leq^* Q_{\varphi(j)}$ for all $j \in \{1, 2, \dots, k\}$.

Let $\mathbb{R}^+ = [0, \infty)$ denote the set of all nonnegative real numbers. For $A \in H_n^+(\mathbb{F})$, let $\text{Sp}(A)$ denote the spectrum of A, i.e., the set of all eigenvalues of A, and let $\#\,\text{Sp}(A)$ denote the number of (pairwise distinct) eigenvalues of A.

Definition 1. Let $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a function. For $A \in H_n^+(\mathbb{F})$, let $r = \#\,\text{Sp}(A)$ and let p_A be the polynomial of the degree $r - 1$ such that the restriction of p_A to $\text{Sp}(A)$ equals the restriction of f to $\text{Sp}(A)$. For this A, we define

$$f(A) := p_A(A).$$

Remark 13. Note that for every fixed $A \in H_n^+(\mathbb{F})$ the polynomial p_A, introduced in Definition 1, exists and is unique. Moreover, we could take any polynomial q_A such that its restriction to $\text{Sp}(A)$ equals the restriction of f to $\text{Sp}(A)$. Namely, for any $\lambda \in \text{Sp}(A)$, we have $(q_A - p_A)(\lambda) = f(\lambda) - f(\lambda) = 0$ and thus $(q_A - p_A)(A) = 0$.

Remark 14. Note that when f is analytic or even measurable, $f(A)$ as defined in Definition 1 matches $f(A)$ obtained via the functional calculus.

Remark 15. Let $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a function with f(0)=0, and let $\Phi\colon H_n^+(\mathbb{F}) \to H_n^+(\mathbb{F})$ be defined as $\Phi(A) = f(A)$. Let $A = \sum_{j=1}^{k} \lambda_j P_j$ be the eigenvalue decomposition of $A \in H_n^+(\mathbb{F})$. Observe, by the orthogonality of projectors involved, that then $\Phi(A) = \sum_{j=1}^{k} f(\lambda_j) P_j$ and thus $\Phi(H_n^+(\mathbb{F})) \subseteq H_n^+(\mathbb{F})$.

Lemma 16. *Let $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a bijective function with $f(0) = 0$ and let $\Phi\colon H_n^+(\mathbb{F}) \to H_n^+(\mathbb{F})$ be defined as*

$$\Phi(A) = f(A).$$

Then Φ is a bijective star partial order bi-preserver.

Proof. Let $B \in H_n^+(\mathbb{F})$ be nonzero and let $B = \sum_{i=1}^{l} \mu_i Q_i$ be the eigenvalue decomposition of B. Let $A = \sum_{i=1}^{l} f^{-1}(\mu_i) Q_i$. This is up to a permutation of indices the eigenvalue decomposition of A. Thus

$$\Phi(A) = \sum_{i=1}^{l} f(f^{-1}(\mu_i)) Q_i = \sum_{i=1}^{l} \mu_i Q_i = B.$$

Since $f(0) = 0$, we have $\Phi(0) = 0$ and thus Φ is surjective. Let now $\Phi(A) = \Phi(B) \neq 0$ and let $\sum_{j=1}^{k} \lambda_j P_j$ be the eigenvalue decomposition of $\Phi(A)$. Then $A = \sum_{j=1}^{k} f^{-1}(\lambda_j) P_j = B$. If $\Phi(A) = \Phi(B) = 0$, then $A = B = 0$ since $f(0) = 0$ and f is injective. We may conclude that Φ is injective and thus bijective.

Suppose now $A \leq^* B$ for some nonzero $A, B \in H_n^+(\mathbb{F})$. Let $A = \sum_{j=1}^{k} \lambda_j P_j$, $B = \sum_{i=1}^{l} \mu_i Q_i$ be the eigenvalue decompositions of A and B, respectively. By Corollary 12, there exists a function $\varphi\colon \{1, 2, \ldots, k\} \to \{1, 2, \ldots, l\}$ such that $\lambda_j = \mu_{\varphi(j)}$ and $P_j \leq^* Q_{\varphi(j)}$ for all $j \in \{1, 2, \ldots, k\}$. Also,

$$\Phi(A) = \sum_{j=1}^{k} f(\lambda_j) P_j, \qquad \Phi(B) = \sum_{i=1}^{l} f(\mu_i) Q_i$$

and therefore, since $\lambda_j = \mu_{\varphi(j)}$ implies $f(\lambda_j) = f(\mu_{\varphi(j)})$, we may again by Corollary 12 conclude that $\Phi(A) \leq^* \Phi(B)$. If $A = 0$, then $\Phi(A) = f(0) = 0$ and so (see (1)) $A \leq^* B$ and $\Phi(A) \leq^* \Phi(B)$ for every $B \in H_n^+(\mathbb{F})$. We proved that $A \leq^* B$ implies $\Phi(A) \leq^* \Phi(B)$ for every $A, B \in H_n^+(\mathbb{F})$. Since f is bijective, we may similarly show that $\Phi(A) \leq^* \Phi(B)$ implies $A \leq^* B$. □

3. Statement and proof of the main result

We use the notation $A <^* B$ if $A \leq^* B$ and $A \neq B$, and denote by $\operatorname{diag}(d_i) \in M_n(\mathbb{R})$ a diagonal matrix with real diagonal elements $d_i, i = 1, 2, \ldots, n$. For a subspace V of $\mathbb{R}^n$, we denote by $P_V \in P_n(\mathbb{R})$ the orthogonal projector matrix with $\operatorname{Im} P_V = V$. Observe first (see [3]) that then for any pair of matrices $P, Q \in P_n(\mathbb{R})$, we have

$$P \leq^* Q \quad \text{if and only if} \quad P = QP = PQ \quad \text{if and only if} \quad \operatorname{Im} P \subseteq \operatorname{Im} Q. \tag{3}$$

Our main result follows.

Theorem 17. *Let $n \geq 3$ be an integer. Then $\Phi\colon H_n^+(\mathbb{R}) \to H_n^+(\mathbb{R})$ is a surjective bi-preserver of the star partial order if and only if there exist an orthogonal matrix $Q \in M_n(\mathbb{R})$, a positive $\lambda \in \mathbb{R}$, and a bijective function $f\colon \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $f(0) = 0$ such that*

$$\Phi(A) = \lambda Q f(A) Q^t$$

for every $A \in H_n^+(\mathbb{R})$.

Proof. Let $\Phi\colon H_n^+(\mathbb{R}) \to H_n^+(\mathbb{R})$ be of the form $\Phi(A) = \lambda Q f(A) Q^t$, $A \in H_n^+(\mathbb{R})$, where Q, λ and f are as above. From Proposition 4 and Lemma 16, it follows that Φ is a bijective bi-preserver of the star partial order.

Conversely, let $\Phi\colon H_n^+(\mathbb{R}) \to H_n^+(\mathbb{R})$ be a surjective bi-preserver of the star partial order. We will split the proof into several steps. The first four steps are standard and we include them for the sake of completeness.

Step 1. Φ is bijective. Let $\Phi(A) = \Phi(B)$ for $A, B \in H_n^+(\mathbb{R})$. Thus $\Phi(A) \leq^* \Phi(B)$ and $\Phi(B) \leq^* \Phi(A)$ and since Φ is a bi-preserver of the star partial order, we have $A \leq^* B$ and $B \leq^* A$. So $A = B$, i.e., Φ is injective and hence bijective.

Step 2. $\Phi(0) = 0$. By (1), $0 \leq^* A$ for every $A \in H_n^+(\mathbb{R})$. So on the one hand $0 \leq^* \Phi(0)$ and on the other hand, since Φ^{-1} has the same properties as Φ, $0 \leq^* \Phi^{-1}(0)$. Thus $\Phi(0) \leq^* 0$ and therefore $\Phi(0) = 0$.

Step 3. Φ preserves rank. Let $A \in H_n^+(\mathbb{R})$ with $\operatorname{rank}(A) = k$. There exists an orthogonal matrix $U \in M_n(\mathbb{R})$ such that

$$A = UDU^t,$$

where $D = \operatorname{diag}(d_i)$ with $d_i \geq 0$ for every $i = 1, 2, \ldots, n$. Without loss of generality we may assume that $d_1 \geq d_2 \geq \cdots \geq d_k > 0 = d_{k+1} = d_{k+2} = \cdots = d_n$ and write $D = d_1E_{1,1} + d_2E_{2,2} + \cdots + d_kE_{k,k}$. Clearly,

$$\begin{aligned} 0 &<^* d_1E_{1,1} <^* d_1E_{1,1} + d_2E_{2,2} <^* \cdots <^* d_1E_{1,1} + d_2E_{2,2} + \cdots + d_kE_{k,k} \\ &= D <^* d_1E_{1,1} + d_2E_{2,2} + \cdots + d_kE_{k,k} + E_{k+1,k+1} <^* \cdots \\ &<^* d_1E_{1,1} + d_2E_{2,2} + \cdots + d_kE_{k,k} + E_{k+1,k+1} + \cdots + E_{n,n}. \end{aligned}$$

By Proposition 4 and since congruence preserves rank, we have

$$\begin{aligned} 0 &<^* U(d_1E_{1,1})U^t <^* U(d_1E_{1,1} + d_2E_{2,2})U^t <^* \cdots \\ &<^* A <^* U(d_1E_{1,1} + d_2E_{2,2} + \cdots + d_kE_{k,k} + E_{k+1,k+1})U^t <^* \cdots \\ &<^* U(d_1E_{1,1} + d_2E_{2,2} + \cdots + d_kE_{k,k} + E_{k+1,k+1} + \cdots + E_{n,n})U^t. \end{aligned}$$

Since $\Phi(0) = 0$ and since Φ preserves the order $\leq^*$ and is injective, we obtain

$$\begin{aligned} 0 &<^* \Phi(U(d_1E_{1,1})U^t) <^* \Phi(U(d_1E_{1,1} + d_2E_{2,2})U^t) <^* \cdots \\ &<^* \Phi(A) <^* \Phi(U(d_1E_{1,1} + d_2E_{2,2} + \cdots + d_kE_{k,k} + E_{k+1,k+1})U^t) <^* \cdots \\ &<^* \Phi(U(d_1E_{1,1} + d_2E_{2,2} + \cdots + d_kE_{k,k} + E_{k+1,k+1} + \cdots + E_{n,n})U^t). \end{aligned}$$

By Proposition 5, every successor in the above chain of matrices is of rank strictly greater than its predecessor. Since rank of any matrix in $H_n^+(\mathbb{R})$ can not be greater than n, it follows that $\operatorname{rank}\Phi(A) = k$.

Step 4. We may without loss of generality assume that $\Phi(I) = I$. By the previous step, $\Phi(I)$ is of rank n. It follows that there exist an orthogonal matrix $U \in M_n(\mathbb{R})$ and $\lambda_i > 0$, $i = 1, 2, \ldots, n$, with $\Phi(I) = U \operatorname{diag}(\lambda_i) U^t$. We may thus without loss of generality assume that $\Phi(I) = \operatorname{diag}(\lambda_i)$. Suppose there are λ_i, λ_j with $\lambda_i \neq \lambda_j$, $i, j \in \{1, 2, \ldots, n\}$. By Corollary 12, there are exactly two rank-one matrices that are below $\lambda_iE_{i,i} + \lambda_jE_{j,j}$ with respect to the star partial order, namely $\lambda_i E_{i,i}, \lambda_j E_{j,j} \in H_n^+(\mathbb{R})$. Since Φ preserves the order in both directions, we have $\Phi^{-1}(\lambda_i E_{i,i} + \lambda_j E_{j,j}) \leq^* I$. By Lemma 6 and the previous step, $\Phi^{-1}(\lambda_i E_{i,i} + \lambda_j E_{j,j})$ is a rank-two self-adjoint idempotent matrix. This is a contradiction, since on the one hand $\Phi^{-1}(\lambda_i E_{i,i})$ and $\Phi^{-1}(\lambda_j E_{j,j})$ are the only two rank-one matrices that are below $\Phi^{-1}(\lambda_i E_{i,i} + \lambda_j E_{j,j})$ but on the other hand, since $\Phi^{-1}(\lambda_i E_{i,i} + \lambda_j E_{j,j})$ is a self-adjoint idempotent, for every rank-one self-adjoint idempotent $P \in M_n(\mathbb{R})$ with $\operatorname{Im} P \subseteq \operatorname{Im} \Phi^{-1}(\lambda_i E_{i,i} + \lambda_j E_{j,j})$, we have $P \leq^* \Phi^{-1}(\lambda_i E_{i,i} + \lambda_j E_{j,j})$, and there are infinitely many such matrices P (see (3)). It follows that $\Phi(I) = \lambda I$ for some $\lambda > 0$. Let now $\Psi \colon H_n^+(\mathbb{R}) \to H_n^+(\mathbb{R})$ be the map defined with

$$\Psi(A) = \frac{1}{\lambda}\Phi(A), \qquad A \in H_n^+(\mathbb{R}).$$

Then Ψ is a surjective map that by Proposition 4 preserves the order $\leq^*$ in both directions. Moreover, $\Psi(I) = I$.

We will thus from now on assume that

$$\Phi(I) = I.$$

Step 5. There exists a bijective function $f \colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(0) = 0$ and $\Phi(\lambda xx^*) = f(\lambda)\Phi(xx^*)$ for every $x \in \mathbb{R}^n$ with $\|x\| = 1$ and every $\lambda > 0$.

To see this, let $x \in \mathbb{R}^n$ with $\|x\| = 1$ and let $\lambda > 0$. By unitary similarity we may without loss of generality assume that $x = e_1$, i.e., $xx^* = E_{1,1}$. The matrices $\lambda E_{1,1}$, $(\lambda+2)\,E_{2,2}$, $(\lambda+3)\,E_{3,3}$, $\dots$, $(\lambda+n)\,E_{n,n}$ are pairwise orthogonal and have pairwise different spectra. Similarly, the matrices $E_{1,1}$, $(\lambda+2)\,E_{2,2}$, $(\lambda+3)\,E_{3,3}$, $\dots$, $(\lambda+n)\,E_{n,n}$ are pairwise orthogonal and have pairwise different spectra. By Lemma 9 and since Φ preserves the star partial order, it follows that matrices

$$\Phi(\lambda E_{1,1}), \Phi((\lambda+2)\,E_{2,2}), \Phi((\lambda+3)\,E_{3,3}), \dots, \Phi((\lambda+n)\,E_{n,n})$$

are pairwise orthogonal and similarly

$$\Phi(E_{1,1}), \Phi((\lambda+2)\,E_{2,2}), \Phi((\lambda+3)\,E_{3,3}), \dots, \Phi((\lambda+n)\,E_{n,n})$$

are also pairwise orthogonal. Since $\Phi(\lambda E_{1,1})$ and $\Phi(E_{1,1})$ are pairwise orthogonal to the same set of $n-1$ pairwise orthogonal rank-one matrices, they must be linearly dependent, i.e., $\Phi(\lambda E_{1,1}) = \lambda_{e_1}\Phi(E_{1,1})$, $\lambda_{e_1} > 0$. So, $\Phi(\lambda xx^*) = \lambda_x \Phi(xx^*)$, $\lambda_x > 0$, for every $x \in \mathbb{R}^n$ with $\|x\| = 1$ and $\lambda > 0$.

Suppose now $x, y \in \mathbb{R}^n$ with $\|x\| = \|y\| = 1$ are linearly independent and let $\lambda > 0$. We have $\Phi(\lambda xx^*) = \lambda_x \Phi(xx^*)$ and $\Phi(\lambda yy^*) = \lambda_y \Phi(yy^*)$. To show that $\lambda_x = \lambda_y$, let $P \in P_n(\mathbb{R})$ be of rank-two with $x, y \in \operatorname{Im} P$. By Corollary 12, $\lambda xx^*, \lambda yy^* \leq^* \lambda P$. Note that $\Phi(\lambda P)$ is of rank-two and suppose $\Phi(\lambda P) = \mu_1 z_1 z_1^* + \mu_2 z_2 z_2^*$ is the eigenvalue decomposition of $\Phi(\lambda P)$ with $\mu_1 \neq \mu_2$. By Corollary 12, $\mu_1 z_1 z_1^*$, $\mu_2 z_2 z_2^*$ are the only two rank-one matrices below $\Phi(\lambda P)$ with respect to the star partial order. This is a contradiction, since Φ preserves the order in both directions and there are infinitely many rank-one matrices below λP. So, $\Phi(\lambda P) = \mu Q$, where $\mu > 0$ and $Q \in P_n(\mathbb{R})$ is of rank-two. It follows that $\Phi(\lambda xx^*), \Phi(\lambda yy^*) \leq^* \mu Q$ and thus by Corollary 12, $\lambda_x = \mu = \lambda_y$.

Recall that $\Phi(0) = 0$. We may conclude that there exists a function $f \colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(0) = 0$ and $\Phi(\lambda xx^*) = f(\lambda)\Phi(xx^*)$ for every $x \in \mathbb{R}^n$ with $\|x\| = 1$ and every $\lambda > 0$. Function f is bijective since Φ is bijective.

Step 6. Let $xx^*, yy^* \in P_n(\mathbb{R})$ be pairwise orthogonal. Then $\Phi(xx^*), \Phi(yy^*)$ are pairwise orthogonal.

By unitary similarity, we may without loss of generality assume that $xx^* = E_{1,1}$ and $yy^* = E_{2,2}$. Lemma 9 implies $\Phi(E_{1,1})$ and $\Phi(2E_{2,2})$ are pairwise orthogonal, and *Step 5* yields $\Phi(2E_{2,2}) = f(2)\Phi(E_{22})$. Thus, $\Phi(E_{2,2})$ is orthogonal to $\Phi(E_{1,1})$.

Step 7. We may without loss of generality assume that $\Phi(P) = P$ for every $P \in P_n(\mathbb{R})$. Let $P \in P_n(\mathbb{R})$. It follows that $P \leq^* I$ by Lemma 6. Therefore $\Phi(P) \leq^* \Phi(I) = I$. Again, by Lemma 6,

$\Phi(P) \in P_n(\mathbb{R})$. Since Φ^{-1} has the same properties as Φ, we may conclude that

$$P \in P_n(\mathbb{R}) \qquad \text{if and only if} \qquad \Phi(P) \in P_n(\mathbb{R}),$$

i.e., Φ preserves the set of all orthogonal projector matrices. Recall that we may identify subspaces of $\mathbb{R}^n$ with elements of $P_n(\mathbb{R})$. Let $\mathcal{C}(\mathbb{R}^n)$ be the lattice of all subspaces of $\mathbb{R}^n$. It follows that the map Φ induces a lattice automorphism, i.e., a bijective map $\tau \colon \mathcal{C}(\mathbb{R}^n) \to \mathcal{C}(\mathbb{R}^n)$ such that

$$M \subseteq N \qquad \text{if and only if} \qquad \tau(M) \subseteq \tau(N)$$

for all $M, N \in \mathcal{C}(\mathbb{R}^n)$. It was proved in [19: page 246] (see also [7: pages 820 and 823]) that for $n \geq 3$ every such a map is induced by an invertible linear operator, i.e., there exists an invertible linear operator $T \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\tau(M) = T(M)$ for every $M \in \mathcal{C}(\mathbb{R}^n)$. For the map Φ, we thus have

$$\Phi(P) = P_{T(\operatorname{Im} P)} \tag{4}$$

for every $P = P_{\operatorname{Im} P} \in P_n(\mathbb{R})$. Let $xx^*, yy^* \in P_n(\mathbb{R})$ be pairwise orthogonal. By *Step 6*, $\Phi(xx^*), \Phi(yy^*)$ are pairwise orthogonal and hence by (4),

$$0 = \langle Tx, Ty \rangle$$

and therefore $\langle T^tTx, y \rangle = 0$. This equation holds for every $y \in \mathbb{R}^n$ with $\|y\| = 1$ and $\langle x, y \rangle = 0$. Since $\langle T^tTx, y \rangle = \|x\| \, \|y\| \left\langle T^tT \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle$, we may conclude that for any fixed $x \in \mathbb{R}^n$, we have $\langle T^tTx, y \rangle = 0$ for every $y \in \mathbb{R}^n$ with $\langle x, y \rangle = 0$. So T^tTx is a scalar multiple of x, i.e., T^tT and I are locally linearly dependent. It is known that for linear operators of rank at least 2 local linear dependence implies (global) linear dependence (see, e.g., [23: page 1869]). Note that $T^tT \in H_n^+(\mathbb{R})$. Therefore,

$$T^tT = \alpha I$$

for some $\alpha > 0$. Let now $Q = \frac{1}{\sqrt{\alpha}} T$. It follows that $Q^tQ = \frac{1}{\alpha} T^tT = I$. So, Q is a linear isometry and since it is also invertible (and thus surjective), it is also a coisometry ($QQ^t = I$). For any $P \in P_n(\mathbb{R})$, we thus have $\Phi(P) = P_{Q(\operatorname{Im} P)}$, where Q is an orthogonal operator, i.e., it may be represented with an orthogonal matrix Q, where $QQ^t = Q^tQ = I$. Therefore for every $P \in P_n(\mathbb{R})$

$$\operatorname{Im} \Phi(P) = Q(\operatorname{Im} P) = QP(\mathbb{R}^n) = QPQ^t(\mathbb{R}^n) = \operatorname{Im} QPQ^t.$$

Since clearly $QPQ^t \in P_n(\mathbb{R})$, we conclude that

$$\Phi(P) = QPQ^t$$

for every $P \in P_n(\mathbb{R})$.

Without loss of generality, we assume from now on that

$$\Phi(P) = P$$

for every $P \in P_n(\mathbb{R})$.

Step 8. Conclusion of the proof. Let $A \in H_n^+(\mathbb{R})$. Then we may write

$$A = \sum_{i=1}^{n} \lambda_i x_i x_i^*,$$

where $\lambda_i \geq 0$, $i = 1, 2, \ldots, n$, are the eigenvalues of A and $\{x_1, x_2, \ldots, x_n\}$ is an orthonormal basis of $\mathbb{R}^n$. Then $\lambda_i x_i x_i^* \leq^* A$ for every $i = 1, 2, \ldots, n$ and therefore

$$f(\lambda_i) x_i x_i^* = f(\lambda_i) \Phi(x_i x_i^*) = \Phi(\lambda_i x_i x_i^*) \leq^* \Phi(A),$$

where $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is the function obtained in *Step 5*. By Corollary 12 and since Φ preserves rank, it follows that

$$\Phi(A) = \sum_{i=1}^{n} f(\lambda_i)\Phi(x_i x_i^*) = \sum_{i=1}^{n} f(\lambda_i) x_i x_i^*.$$

Recall Definition 1 to conclude that $\Phi(A) = f(A)$ for every $A \in H_n^+(\mathbb{R})$. Taking into account the assumptions from *Steps 4* and *7*, we may conclude that if $\Phi\colon H_n^+(\mathbb{R}) \to H_n^+(\mathbb{R})$ is a surjective bi-preserver of the star partial order, there exist an orthogonal matrix $Q \in M_n(\mathbb{R})$, a positive λ, and a bijective function $f\colon \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $f(0) = 0$ such that

$$\Phi(A) = \lambda Q f(A) Q^t$$

for every $A \in H_n^+(\mathbb{R})$. □

Acknowledgement. The authors wish to thank the anonymous reviewers for helpful and constructive comments that improved the presentation of this paper.

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Received 20. 10. 2021
Accepted 1. 2. 2022

* *University of Ljubljana*
Faculty of Electrical Engineering
Tržaška cesta 25
SI-1000 Ljubljana

IMFM
Jadranska 19
SI-1000 Ljubljana
SLOVENIA
E-mail: gregor.dolinar@fe.uni-lj.si

** *University of Primorska*
Glagoljaška 8
SI-6000 Koper

IMFM
Jadranska 19
SI-1000 Ljubljana
SLOVENIA
E-mail: bojan.kuzma@upr.si

*** *University of Maribor*
Faculty of Economics and Business
Razlagova 14
SI-2000 Maribor

IMFM
Jadranska 19
SI-1000 Ljubljana
SLOVENIA
E-mail: janko.marovt@um.si

**** *Faculty of Sciences and Mathematics*
University of Niš
P.O. Box 224, 18000 Niš
SERBIA
E-mail: dijana@pmf.ni.ac.rs