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A NEW FAMILY OF COMPOUND EXPONENTIATED LOGARITHMIC DISTRIBUTIONS WITH APPLICATIONS TO LIFETIME DATA

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ABSTRACT. The logarithmic distribution and a given lifetime distribution are compounded to construct a new family of lifetime distributions. The compounding is performed with respect to maxima. Expressions are derived for lifetime properties like moments and the behavior of extreme values. Estimation procedures for the method of maximum likelihood are also derived and their performance assessed by a simulation study. Three real data (including two lifetime data) applications are described that show superior performance (assessed with respect to Kolmogorov Smirnov statistics, likelihood values, AIC values, BIC values, probability-probability plots and density plots) versus at least five known lifetime models, with each model having the same number of parameters as the model it is compared to.

1. Introduction

There exist numerous distributions for modeling lifetime data, however, some of these distributions lack motivation from a lifetime context. For example, there is no apparent physical motivation for the gamma distribution. It only has a more general mathematical form compared with the exponential distribution with one additional parameter, so it has nicer properties and provides better fits. The same arguments apply to many other distributions.

The aim of this note is to introduce a new family of distributions with sound physical motivation as in [23], [9] and [10]. As explained in subsequent sections, the proposed family encompasses the behavior of and provides better fits (assessed via a comparison of Kolmogorov Smirnov statistics, likelihood values, AIC values, BIC values, probability-probability plots and density plots) compared with several known lifetime distributions having the same number of parameters. We feel that this is a remarkable feature.

The motivation we provide for the new distribution is based on *failures* of a *system*. Here, the words "failures" and "system" should not be interpreted as relating purely to an electronic or a mechanical system, but instead in a more general sense.

Suppose a system has N entities operating independently and identically in parallel, where N is a logarithmic random variable with the PMF:

$$\Pr(N = n) = -\frac{1}{\log p} \frac{(1-p)^n}{n} \qquad \text{for } n = 1, 2, \ldots,$$

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where $0 < p < 1$. A logarithmic distribution for the number entities in a system is not unreasonable. [29], [30] used a logarithmic distribution to characterize the abundance of species of animals or plants. [24] supposed that the number of demands in a batch follows the logarithmic distribution. [4] modeled the number of eggs in a cluster by the logarithmic distribution.

Suppose each entity has α sub-entities operating independently and identically in parallel, where α is an integer. Assume that the lifetimes of the sub-entities are independent of N and have the common PDF $g(\cdot)$ and the common CDF $G(\cdot)$. Then the lifetime of the system, say X, has the CDF

$$F_X(x) = -\frac{1}{\log p} \sum_{n=1}^{\infty} G^{\alpha n}(x) \frac{(1-p)^n}{n} = 1 - \frac{1}{\log p} \log \left\{ p + (1-p) G^{\alpha}(x) \right\} \qquad \text{for } x > 0. \tag{1}$$

The corresponding PDF is:

$$f_X(x) = -\frac{\alpha(1-p) G^{\alpha-1}(x) g(x)}{\log p \left\{ p + (1-p) G^{\alpha}(x) \right\}} \qquad \text{for } x > 0. \tag{2}$$

The corresponding HRF is:

$$h_X(x) = -\frac{\alpha(1-p) G^{\alpha-1}(x) g(x)}{\left\{ p + (1-p) G^{\alpha}(x) \right\} \log \left\{ p + (1-p) G^{\alpha}(x) \right\}} \qquad \text{for } x > 0. \tag{3}$$

The corresponding quantile function is

$$F_X^{-1}(u) = G^{-1} \left[\left(\frac{p}{1-p} \right)^{1/\alpha} \left(p^{-u} - 1 \right)^{1/\alpha} \right] \qquad \text{for } 0 < u < 1,$$

where $G^{-1}(\cdot)$ denotes the inverse function of $G(\cdot)$. Here, α and p are shape parameters. For continuity and simplicity, we shall assume hereafter that α can take any positive real value and not just integers (similar to approximating a discrete distribution by a continuous one).

We shall refer to the distribution given by (1) and (2) as the *Compound Exponentiated Logarithmic* (CEL) distribution. The construction of this distribution is fundamentally different from those of several recently proposed distributions, see [27], [2] and [14]. These papers consider $X = \min(X_1, X_2, \ldots, X_N)$ with the X_i assumed to come from either an exponential or a Weibull distribution. That is, they apply to series systems and the choice for the distribution of X_i is restricted. Here, we impose no such restrictions. [13] consider distributions of $U = \min(X_1, X_2, \ldots, X_N)$ and $V = \max(X_1, X_2, \ldots, X_N)$ with N assumed to be a geometric random variable. However, geometric random variables are "time to event" variables and it makes little sense to model the number of entities in a system by a geometric distribution. In any case, [13] provide no motivation for choosing N to be a geometric random variable. Furthermore, [13] derive properties of U and V only for the particular cases that X_i is an exponential or a Weibull random variable.

The situation giving rise to (1) can encompass a wide range of examples. Here, we discuss just one example. Suppose that N *storms* are observed in one year. The word "storm" could mean a sequence of days experiencing rainfall, a sequence of days experiencing snowfall, etc. Suppose each storm has a fixed duration, α. This is not an unreasonable assumption if storm durations are not too variable (in which case zeros can be added to make the durations equal). Suppose also that N is a logarithmic random variable and let $G(\cdot)$ denote the CDF of daily rainfall, CDF of daily snowfall, etc. Then X will represent the annual maximum rainfall, annual maximum snowfall, etc.

The aim of this note is to study the mathematical properties of the CEL distribution and to illustrate its applicability. The contents are organized as follows. In Section 2, we derive general mathematical properties of the CEL distribution. These include expansions for the PDF and the CDF, shape properties of (2) and (3), moments, moment generating function, characteristic function, and asymptotic distributions of the extreme order statistics. All of these properties are clearly relevant for a random variable representing lifetime. Other properties like mean deviations and entropies can also be derived, but they take complicated forms. The maximum likelihood estimation is considered in Section 3. Section 4 gives applications using three real data sets. Finally, some conclusions are noted in Section 5.

Throughout this note, we shall focus attention on a particular CEL distribution by taking the baseline CDF G to correspond to an exponential distribution with scale parameter β. The rationale for this choice is that the exponential distribution is the first and the most widely used model for failure times. Then (2) becomes

$$f_X(x) = -\frac{\alpha\beta(1-p)}{\log p} \frac{[1-\exp(-\beta x)]^{\alpha-1}\exp(-\beta x)}{p+(1-p)[1-\exp(-\beta x)]^{\alpha}} \tag{4}$$

for $x > 0$, $\alpha > 0$, $\beta > 0$, and $0 < p < 1$. The corresponding HRF is

$$h_X(x) = -\alpha\beta(1-p)\frac{[1-\exp(-\beta x)]^{\alpha-1}\exp(-\beta x)}{\{p+(1-p)[1-\exp(-\beta x)]^{\alpha}\}\log\{p+(1-p)[1-\exp(-\beta x)]^{\alpha}\}} \tag{5}$$

for $x > 0$, $\alpha > 0$, $\beta > 0$, and $0 < p < 1$. We shall refer to (4) as the *Generalized Exponential Logarithmic* (GEL) PDF. The limiting case of (4) for $p \uparrow 1$ is the exponentiated exponential distribution due to [6]. The limiting case of (4) for $p \uparrow 1$ and $\alpha = 1$ is the exponential distribution.

We shall see later in Section 4 that the GEL distribution performs well with respect to at least five known models in spite of being the simplest member of the class of CEL distributions. (The GEL distribution corresponds to the failure times following the exponential distribution, the simplest possible model.) Hence, other members of the class of CEL distributions can only be expected to perform better than the GEL distribution. Hence, we feel no need to consider more than one particular case of the class of CEL distributions.

2. Mathematical properties

Section 2.1 shows how $f_X(\cdot)$ and $F_X(\cdot)$ can be expanded in terms of $g(\cdot)$ and $G(\cdot)$. These expansions are useful technical tools for later use. Section 2.2 derives shape characteristics of $f_X(\cdot)$ and $h_X(\cdot)$ in terms of $g(\cdot)$ and $G(\cdot)$. For a given CEL distribution, these characteristics can be used to determine possible shapes of the PDF or the HRF of X. Section 2.3 derives moment properties of X. These can be useful for calculating basic measures like the mean, variance, coefficient of variation, skewness and kurtosis. Moment properties can also be used for estimation. Section 2.4 expresses the extreme value behavior of X in terms of that of a random variable specified by $g(\cdot)$ and $G(\cdot)$. This can be useful in determining the extreme domains of attraction of a given CEL distribution.

2.1. Expansions for PDF and CDF

Some useful expansions for (1) and (2) can be derived using the concept of exponentiated distributions. A random variable is said to have the exponentiated-G distribution with parameter $a > 0$, if its PDF and CDF are

$$h_a(x) = aG^{a-1}(x)g(x) \tag{6}$$

and

$$H_a(x) = G^a(x), \tag{7}$$

respectively. Many authors have studied exponentiated distributions. These include among others [15], [5], [6], [18], [11] and [20].

We now provide expansions for (1) and (2), each in terms of (6) and (7). Expanding the logarithmic term in (1), we can write (1) and (2) as

$$F_X(x) = -\frac{1}{\log p} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1-p)^k}{kp^k} H_{\alpha k}(x) \tag{8}$$

and

$$f_X(x) = -\frac{1}{\log p} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1-p)^k}{kp^k} h_{\alpha k}(x), \tag{9}$$

respectively. So, several properties of the CEL distribution can be obtained by knowing those of exponentiated distributions ([16], [6], [21]).

2.2. Asymptotes and shapes

The asymptotes of (1), (2), and (3) as $x \to 0, \infty$ are given by

$$f_X(x) \sim -\frac{\alpha(1-p)}{p \log p} G^{\alpha-1}(x) g(x) \qquad \text{as } x \to 0, \tag{10}$$

$$f_X(x) \sim -\frac{\alpha(1-p)}{\log p} g(x) \qquad \text{as } x \to \infty, \tag{11}$$

$$F_X(x) \sim -\frac{1-p}{p \log p} G^{\alpha}(x) \qquad \text{as } x \to 0, \tag{12}$$

$$1 - F_X(x) \sim -\frac{\alpha(1-p)}{\log p} \left[1 - G(x)\right] \qquad \text{as } x \to \infty, \tag{13}$$

$$h_X(x) \sim -\frac{\alpha(1-p)}{\log p} g(x) \qquad \text{as } x \to 0, \tag{14}$$

$$h_X(x) \sim \frac{g(x)}{1 - G(x)} \qquad \text{as } x \to \infty. \tag{15}$$

Note that the right hand sides of (10) and (12) are decreasing functions of p. The right hand sides of (11), (13) and (14) are increasing functions of p. It follows from (11) that $f_X(\cdot)$ behaves like $g(\cdot)$ for very large x. Also, $h_X(\cdot)$ behaves like the HRF corresponding to $g(\cdot)$ for very large x.

An analytical description of the shapes of (2) and (3) is possible. The critical points of the PDF are the roots of the equation:

$$(\alpha - 1)\frac{g(x)}{G(x)} + \frac{g^{'}(x)}{g(x)} - \frac{\alpha(1-p)G^{\alpha-1}(x)g(x)}{p + (1-p)G^{\alpha}(x)} = 0. \tag{16}$$

(16) may have one or more roots. If $x = x_0$ is a root of (16), then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether $\lambda(x_0) < 0$, $\lambda(x_0) > 0$ or $\lambda(x_0) = 0$, where

$$\begin{aligned} \lambda(x) = {} & (\alpha - 1)\frac{G(x)g'(x) - g^2(x)}{G^2(x)} + \frac{g(x)g''(x) - (g'(x))^2}{g^2(x)} \\ & - \alpha(1-p)\frac{G^{\alpha-2}(x)\left[(\alpha-1)g(x) + G(x)g'(x)\right]}{p + (1-p)G^{\alpha}(x)} + \alpha^2(1-p)^2\frac{G^{2\alpha-2}(x)g^2(x)}{\left[p + (1-p)G^{\alpha}(x)\right]^2}. \end{aligned}$$

The critical points of the HRF are the roots of:

$$(\alpha - 1)\frac{g(x)}{G(x)} + \frac{g'(x)}{g(x)} - \frac{\alpha(1-p)\left(1/\log p - 1\right)G^{\alpha-1}(x)g(x)}{p + (1-p)G^{\alpha}(x)} = 0. \tag{17}$$

(17) may have one or more roots. If $x = x_0$ is a root of (17), then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether $\lambda(x_0) < 0$, $\lambda(x_0) > 0$ or $\lambda(x_0) = 0$, where

$$\begin{aligned}\lambda(x) = {} & (\alpha-1)\frac{G(x)g'(x)-g^2(x)}{G^2(x)} + \frac{g(x)g''(x)-(g'(x))^2}{g^2(x)} \\ & -\alpha(1-p)\left(1/\log p - 1\right)\frac{G^{\alpha-2}(x)\left[(\alpha-1)g(x)+G(x)g'(x)\right]}{p+(1-p)G^{\alpha}(x)} \\ & +\alpha^2(1-p)^2\left(1/\log p-1\right)\frac{G^{2\alpha-2}(x)g^2(x)}{\left[p+(1-p)G^{\alpha}(x)\right]^2}.\end{aligned}$$

It follows from (16) that $\partial \log f_X(x)/\partial x$ is an increasing function of p with

$$\lim_{p\to 0}\frac{\partial \log f_X(x)}{\partial x} = -\frac{g(x)}{G(x)} + \frac{g'(x)}{g(x)}$$

and

$$\lim_{p\to 1}\frac{\partial \log f_X(x)}{\partial x} = (\alpha-1)\frac{g(x)}{G(x)} + \frac{g'(x)}{g(x)}.$$

It follows from (17) that $\partial \log h_X(x)/\partial x$ is a decreasing function of p with

$$\lim_{p\to 0}\frac{\partial \log h_X(x)}{\partial x} = (2\alpha-1)\frac{g(x)}{G(x)} + \frac{g'(x)}{g(x)}$$

and

$$\lim_{p\to 1}\frac{\partial \log h_X(x)}{\partial x} = (\alpha-1)\frac{g(x)}{G(x)} + \frac{g'(x)}{g(x)}.$$

Calculations using (10)–(15) show that the upper tail of (4) decays exponentially and that the lower tail of (4) decays polynomially. Both the upper and lower tails of (5) approach some constants. In fact, $h_X(0) = -\alpha\lambda(1-p)/\log p$ and $h_X(\infty) = \lambda$.

Calculations using (16) show that (4) can be either monotonically decreasing or unimodal. Calculations using (17) show that (5) can be either monotonically decreasing, monotonically increasing or upside down bathtub shaped. The exponentiated exponential distribution (the limiting case of the GEL distribution for $p \uparrow 1$) cannot exhibit upside down bathtub shaped hazard rates.

Upside down bathtub hazard rates are common in reliability and survival analysis. For example, such hazard rates can be observed in the course of a disease whose mortality reaches a peak after some finite period and then declines gradually [25]. For other practical examples yielding upside down bathtub hazard rates, see [26].

2.3. Moment properties

Let (1) denote the CDF of a random variable X. Let $h_a(\cdot)$ and $H_a(\cdot)$ denote, respectively, the PDF and CDF of a random variable Z_a. Then, using the expansions, (8) and (9), we have

$$E\left(X^n\right) = -\frac{1}{\log p}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(1-p)^k}{kp^k}E\left[Z_{\alpha k}^n\right] \qquad \text{for } n \geq 1. \tag{18}$$

Similarly, the moment generating function and the characteristic function of X can be expressed as

$$E\left[\exp(tX)\right] = -\frac{1}{\log p}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(1-p)^k}{kp^k}E\left[\exp\left(tZ_{\alpha k}\right)\right] \tag{19}$$

and

$$E\left[\exp\left(\mathrm{i}tX\right)\right] = -\frac{1}{\log p}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(1-p)^k}{kp^k}E\left[\exp\left(\mathrm{i}tZ_{\alpha k}\right)\right], \tag{20}$$

respectively, where $\mathrm{i} = \sqrt{-1}$. The infinite sums on the right hand sides of (18)–(20) can be easily computed by using most computer packages, and even some pocket calculators.

The moments, moment generating function and the characteristic function of the GEL distribution follow immediately from (18)–(20) and the moments, moment generating function and characteristic function of the exponentiated exponential distribution. The latter are given explicitly in [19: Section 2]: if Z is an exponentiated exponential random variable with shape parameter α and scale parameter λ, then

$$E\left(Z^n\right) = \left.\frac{(-1)^n\alpha}{\lambda^n}\frac{\partial^n}{\partial p^n}B\left(\alpha, p+1-\alpha\right)\right|_{p=\alpha},$$

$$E\left[\exp\left(tZ\right)\right] = \alpha B\Big(1-\frac{t}{\lambda}, \alpha\Big),$$

and

$$E\left[\exp\left(\mathrm{i}tZ\right)\right] = \alpha B\Big(1-\frac{\mathrm{i}t}{\lambda}, \alpha\Big),$$

where $B(a,b) = \int\limits_0^1 t^{a-1}(1-t)^{b-1}\mathrm{d}t$ denotes the beta function.

2.4. Extreme values

If $\overline{X} = \left(X_1 + \cdots + X_n\right)/n$ denotes the mean of a random sample from (2), then by the central limit theorem $\sqrt{n}\left[\overline{X} - E(X)\right]/\sqrt{Var(X)}$ approaches the standard normal distribution as $n \to \infty$ under suitable conditions. Sometimes interest lies in the asymptotes of the extreme values $M_n = \max\left(X_1, \ldots, X_n\right)$ and $m_n = \min\left(X_1, \ldots, X_n\right)$.

Suppose firstly that the Gumbel distribution is the max domain of attraction of G. By [12: Chapter 1], there must exist a strictly positive function, say $h(t)$, such that

$$\lim_{t\to\infty}\frac{1-G\left(t+xh(t)\right)}{1-G(t)} = \exp(-x) \qquad \text{for every } x \in (-\infty, \infty).$$

But, using (13), we note that

$$\lim_{t\to\infty}\frac{1-F_X\left(t+xh(t)\right)}{1-F_X(t)} = \lim_{t\to\infty}\frac{1-G\left(t+xh(t)\right)}{1-G(t)} \qquad \text{for every } x \in (-\infty, \infty).$$

So, the Gumbel distribution is also the max domain of attraction of F_X with

$$\lim_{n\to\infty}\Pr\left\{a_n\left(M_n - b_n\right) \le x\right\} = \exp\left\{-\exp(-x)\right\}$$

for some suitable norming constants $a_n > 0$ and b_n.

Suppose secondly that the Fréchet distribution is the max domain of attraction of G. By [12: Chapter 1], there must exist a $\beta < 0$ such that

$$\lim_{t\to\infty}\frac{1-G(tx)}{1-G(t)} = x^{\beta} \qquad \text{for every } x > 0.$$

But, using (13), we note that

$$\lim_{t\to\infty}\frac{1-F_X(tx)}{1-F_X(t)} = \lim_{t\to\infty}\frac{1-G(tx)}{1-G(t)} \qquad \text{for every } x > 0.$$

So, the Fréchet distribution is also the max domain of attraction of F_X with

$$\lim_{n\to\infty} \Pr\left\{a_n\left(M_n - b_n\right) \leq x\right\} = \exp\left(-x^{\beta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.

Suppose thirdly that the Weibull distribution is the max domain of attraction of G. By [**?**:]hapter 1letal1987, there must exist a $c > 0$ such that

$$\lim_{t\to 0} \frac{G(tx)}{G(t)} = x^c$$

for every $x < 0$. But, using (12), we note that

$$\lim_{t\to 0} \frac{F_X(tx)}{F_X(t)} = \lim_{t\to 0} \left[\frac{G(tx)}{G(t)}\right]^{\alpha}.$$

So, the Weibull distribution is also the max domain of attraction of F_X with

$$\lim_{n\to\infty} \Pr\left\{a_n\left(M_n - b_n\right) \leq x\right\} = \exp\left\{-(-x)^{c\alpha}\right\}$$

for some suitable norming constants $a_n > 0$ and b_n.

The same argument applies to min domains of attraction. That is, G and F_X have the same min domain of attraction.

Since the exponential distribution belongs to the max (min) domain of attraction of the Gumbel (Weibull) distribution, we have that the max (min) domain of attraction of the GEL distribution is the Gumbel (Weibull) distribution.

3. Estimation

Section 3.1 estimates the parameters of the CEL distribution by the method of maximum likelihood. It also derives the associated observed information matrix. Section 3.2 assesses the performance of the maximum likelihood estimates with respect to biases and mean squared errors. For this assessment, we consider the GEL distribution, the simplest possible member of the class of CEL distributions.

3.1. Maximum likelihood estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample from (2). Let $\mathbf{\Theta}$ denote a q-dimensional vector containing the parameters in $G(\cdot)$. Then the log-likelihood function, $\log L = \log L\left(p, \alpha, \mathbf{\Theta}\right)$, is

$$\begin{aligned}\log L\left(p, \alpha, \mathbf{\Theta}\right) &= n \log\left[\alpha(1-p)\right] - n\log\left(-\log p\right) + (\alpha - 1)\sum_{i=1}^{n} \log G\left(x_i\right) \\ &\quad + \sum_{i=1}^{n} \log g\left(x_i\right) - \sum_{i=1}^{n} \log\left[p + (1-p)G^{\alpha}\left(x_i\right)\right].\end{aligned} \tag{21}$$

The first derivatives of $\log L$ with respect to p, α and $\mathbf{\Theta}$ are:

$$\frac{\partial \log L}{\partial p} = -\frac{n}{1-p} + \frac{n}{p\log p} - \sum_{i=1}^{n} \frac{1 - G^{\alpha}\left(x_i\right)}{p + (1-p)G^{\alpha}\left(x_i\right)}, \tag{22}$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log G\left(x_i\right) - (1-p)\sum_{i=1}^{n} \frac{G^{\alpha}\left(x_i\right)\log G\left(x_i\right)}{p + (1-p)G^{\alpha}\left(x_i\right)}, \tag{23}$$

$$\frac{\partial \log L}{\partial \mathbf{\Theta}} = (\alpha - 1)\sum_{i=1}^{n} \frac{\partial G\left(x_i\right)/\partial \mathbf{\Theta}}{G\left(x_i\right)} + \sum_{i=1}^{n} \frac{\partial g\left(x_i\right)/\partial \mathbf{\Theta}}{g\left(x_i\right)} - \alpha(1-p)\sum_{i=1}^{n} \frac{G^{\alpha-1}\left(x_i\right)\partial G\left(x_i\right)/\partial \mathbf{\Theta}}{p+(1-p)G^{\alpha}\left(x_i\right)}. \tag{24}$$

The maximum likelihood estimates of $(p, \alpha, \mathbf{\Theta})$, say $\left(\widehat{p}, \widehat{\alpha}, \widehat{\mathbf{\Theta}}\right)$, are the simultaneous solutions of the equations $\partial \log L/\partial p = 0$, $\partial \log L/\partial \alpha = 0$ and $\partial \log L/\partial \mathbf{\Theta} = \mathbf{0}$.

Maximization of (21) can be performed by using well established routines like nlminb or optim in the R statistical package [22]. Our numerical calculations showed that the surface of (21) was smooth for given smooth functions $g(\cdot)$ and $G(\cdot)$. The routines were able to locate the maximum of the likelihood surface for a wide range of smooth functions and for a wide range of starting values. However, to ease the computations it is useful to have reasonable starting values. These can be obtained, for example, by equating the sample and theoretical quantiles. For $r = 1, \ldots, q+2$, let q_r denote the sample quantile corresponding to the probability $r/(q+3)$. Equating these quantiles with the theoretical versions given in Section 1, we have

$$G^{-1}\left[\left(\frac{p}{1-p}\right)^{1/\alpha}\left(1-p^{-r/(q+3)}\right)^{1/\alpha}\right] = q_r \qquad \text{for } r = 1, \ldots, q+2.$$

These equations can be solved simultaneously to obtain the initial estimates.

For interval estimation of $(p, \alpha, \mathbf{\Theta})$ and tests of hypothesis, one requires the Fisher information matrix. We can express the observed Fisher information matrix of $\left(\widehat{p}, \widehat{\alpha}, \widehat{\mathbf{\Theta}}\right)$ as

$$\mathbf{J} = \left(\begin{array}{ccc} J_{11} & J_{12} & \mathbf{J}_{13} \\ J_{12} & J_{22} & \mathbf{J}_{23} \\ \mathbf{J}_{13} & \mathbf{J}_{23} & \mathbf{J}_{33} \end{array}\right),$$

where

$$J_{11} = -\frac{n}{(1-\widehat{p})^2} + \frac{n\left(1+\log \widehat{p}\right)}{\left(\widehat{p}\log \widehat{p}\right)^2} - \sum_{i=1}^{n}\left[\frac{1-G^{\widehat{\alpha}}\left(x_i\right)}{\widehat{p}+(1-\widehat{p})\,G^{\widehat{\alpha}}\left(x_i\right)}\right]^2,$$

$$J_{12} = -\sum_{i=1}^{n} \frac{G^{\widehat{\alpha}}\left(x_i\right)\log G\left(x_i\right)}{\widehat{p}+(1-\widehat{p})\,G^{\widehat{\alpha}}\left(x_i\right)} - (1-\widehat{p})\sum_{i=1}^{n} \frac{G^{\widehat{\alpha}}\left(x_i\right)\log G\left(x_i\right)\left[1-G\left(x_i\right)\right]}{\left[\widehat{p}+(1-\widehat{p})\,G^{\widehat{\alpha}}\left(x_i\right)\right]^2},$$

$$\mathbf{J}_{13} = -\widehat{\alpha}\sum_{i=1}^{n} \frac{G^{\widehat{\alpha}-1}\left(x_i\right)\partial G\left(x_i\right)/\partial \widehat{\mathbf{\Theta}}}{\left[\widehat{p}+(1-\widehat{p})\,G^{\widehat{\alpha}}\left(x_i\right)\right]^2},$$

$$J_{22} = \frac{1}{\widehat{\alpha}^2} + \widehat{p}\,(1-\widehat{p})\sum_{i=1}^{n} \frac{G^{\widehat{\alpha}}\left(x_i\right)\log^2 G\left(x_i\right)}{\left[\widehat{p}+(1-\widehat{p})\,G^{\widehat{\alpha}}\left(x_i\right)\right]^2},$$

$$\begin{aligned}\mathbf{J}_{23} = & -\sum_{i=1}^{n} \frac{\partial G\left(x_i\right)/\partial \widehat{\mathbf{\Theta}}}{G\left(x_i\right)} \\ & + (1-\widehat{p})\sum_{i=1}^{n} \frac{\left[\widehat{p}\widehat{\alpha}\log G\left(x_i\right)+\widehat{p}+(1-\widehat{p})\,G^{\widehat{\alpha}}\left(x_i\right)\right] G^{\widehat{\alpha}-1}\left(x_i\right)\partial G\left(x_i\right)/\partial \widehat{\mathbf{\Theta}}}{\left[\widehat{p}+(1-\widehat{p})\,G^{\widehat{\alpha}}\left(x_i\right)\right]^2},\end{aligned}$$

$$
\begin{aligned}
\mathbf{J}_{33} = {} & (1-\widehat{\alpha}) \sum_{i=1}^{n} \frac{G\left(x_i\right) \partial^2 G\left(x_i\right) / \partial \widehat{\boldsymbol{\Theta}}^2 - \left[\partial G\left(x_i\right) / \partial \widehat{\boldsymbol{\Theta}}\right]^2}{G^2\left(x_i\right)} \\
& - \sum_{i=1}^{n} \frac{g\left(x_i\right) \partial^2 g\left(x_i\right) / \partial \widehat{\boldsymbol{\Theta}}^2 - \left[\partial g\left(x_i\right) / \partial \widehat{\boldsymbol{\Theta}}\right]^2}{g^2\left(x_i\right)} \\
& + \widehat{\alpha}(1-\widehat{p}) \sum_{i=1}^{n} \frac{\left[\widehat{p}(\widehat{\alpha}-1) - (1-\widehat{p}) G^{\widehat{\alpha}}\left(x_i\right)\right] G^{\widehat{\alpha}-2}\left(x_i\right) \left[\partial G\left(x_i\right) / \partial \widehat{\boldsymbol{\Theta}}\right]^2}{\left[\widehat{p} + (1-\widehat{p}) G^{\widehat{\alpha}}\left(x_i\right)\right]^2} \\
& + \widehat{\alpha}(1-\widehat{p}) \sum_{i=1}^{n} \frac{G^{\widehat{\alpha}-1}\left(x_i\right) \partial^2 G\left(x_i\right) / \partial \widehat{\boldsymbol{\Theta}}^2}{\widehat{p} + (1-\widehat{p}) G^{\widehat{\alpha}}\left(x_i\right)}.
\end{aligned}
$$

For large n, the distribution of $\sqrt{n}\left(\widehat{p}-p, \widehat{\alpha}-\alpha, \widehat{\boldsymbol{\Theta}}-\boldsymbol{\Theta}\right)$ approximates to a $(q+2)$ variate normal distribution with zero means and variance-covariance matrix $\mathbf{J}^{-1}$. The properties of $\left(\widehat{p}, \widehat{\alpha}, \widehat{\boldsymbol{\Theta}}\right)$ can be derived based on this normal approximation.

It is reasonable to ask: how large should n be for the normal approximation to hold? This question is answered in the next section.

3.2. A simulation study

In this section, we assess the performance of the maximum likelihood estimates given by (22)–(24) with respect to the sample size n for the GEL distribution. The assessment of the performance of the maximum likelihood estimates of (α, β, p) is based on a simulation study:

(1) generate ten thousand samples of size n from (4). The inversion method was used to generate samples, i.e., variates of the GEL distribution were generated using

$$
X = -\frac{1}{\beta} \log \left[1 - \left(\frac{p}{1-p}\right)^{1/\alpha} \left(p^{-U} - 1\right)^{1/\alpha}\right],
$$

where $U \sim U(0,1)$ is a uniform variate on the unit interval;

(2) compute the maximum likelihood estimates for the ten thousand samples, say $\widehat{\alpha}_i$, $\widehat{\beta}_i$ and $\widehat{p}_i$ for $i = 1, 2, \ldots, 10000$;

(3) compute the biases and mean squared errors given by

$$
\mathrm{bias}_e(n) = \frac{1}{10000} \sum_{i=1}^{10000} \left(\widehat{e}_i - e\right), \qquad \text{and} \qquad \mathrm{MSE}_e(n) = \frac{1}{10000} \sum_{i=1}^{10000} \left(\widehat{e}_i - e\right)^2
$$

for $e = \alpha, \beta, p$.

We repeated these steps for $n = 10, 20, \ldots, 500$ with $\alpha = 2$, $\beta = 2$ and $p = 0.5$, so computing $\mathrm{bias}_\alpha(n)$, $\mathrm{bias}_\beta(n)$, $\mathrm{bias}_p(n)$, $\mathrm{MSE}_\alpha(n)$, $\mathrm{MSE}_\beta(n)$ and $\mathrm{MSE}_p(n)$ for $n = 10, 20, \ldots, 500$.

Figures 1 and 2 show how the biases and the mean squared errors vary with respect to n. The broken line in Figure 1 corresponds to the biases being zero. The broken line in Figure 2 corresponds to the mean squared errors being zero.

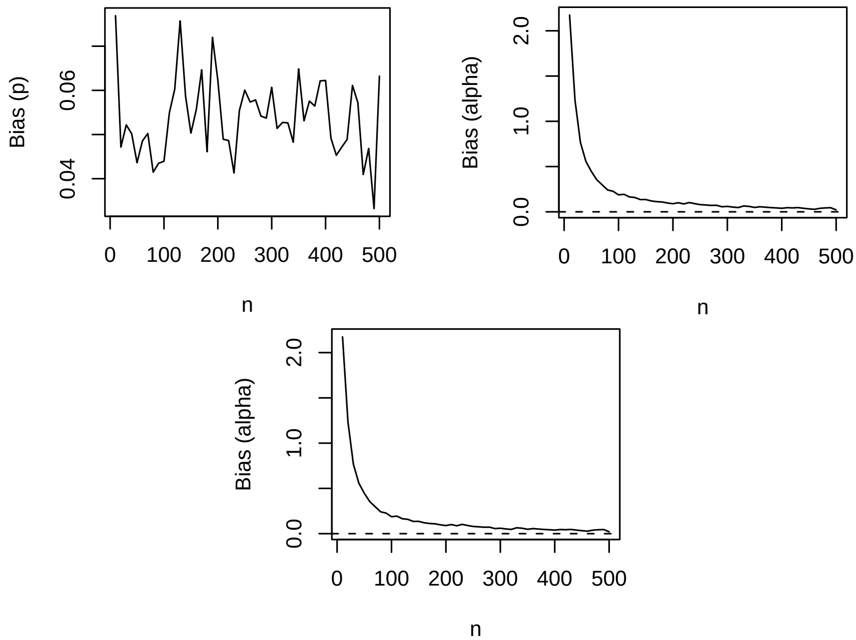

Figure 1. Biases of $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{p}$ versus $n = 10, 20, \ldots, 500$.

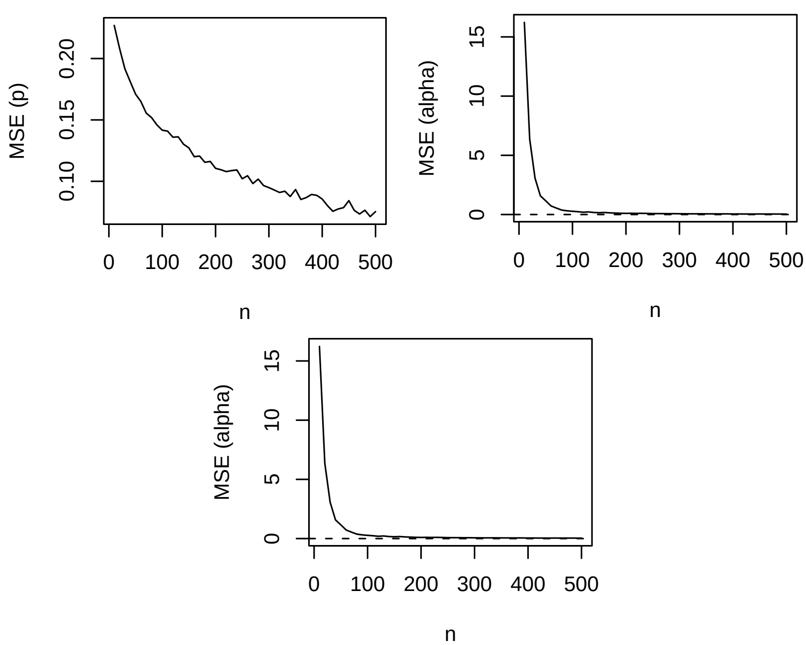

Figure 2. Mean squared errors of $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{p}$ versus $n = 10, 20, \ldots, 500$.

The following observations can be made:

(1) the biases generally appear positive for each parameter;
(2) the magnitude of the biases generally decreases to zero as $n \to \infty$;
(3) the biases appear smallest for p;
(4) the biases appear largest for α;
(5) the mean squared errors generally decrease to zero as $n \to \infty$;
(6) the mean squared errors appear smallest for p;
(7) the mean squared errors appear largest for α;
(8) the convergence of the biases to zero appears slowest for p with convergence still not reached for n as large as five hundred;
(9) the convergence of the biases to zero appears fastest for α;
(10) the convergence of the mean squared errors to zero appears slowest for p with convergence still not reached for n as large as five hundred;
(11) the convergence of the mean squared errors to zero appears fastest for α.

For brevity, we have presented results only for $\alpha = 2$, $\beta = 2$ and $p = 0.5$, however, the results were similar for other choices for α, β and p.

4. Applications

In this section, we fit the GEL distribution to three real data sets. The first data set consists of waiting times in minutes of one hundred bank customers [3]. The second data set consists of active repair times in hours (see page 156 of [28]). The third data set consists of March precipitation for Minneapolis Saint Paul [8].

We compare the fit of the GEL distribution in (4) with five alternative models each containing three parameters: the generalized exponential-Poisson (GEP) distribution (see [1]) with PDF:

$$f(x) = \frac{\alpha\beta\lambda}{[1-\exp(-\lambda)]^{\alpha}} \left\{1 - \exp\left[-\lambda + \lambda\exp(-\beta x)\right]\right\}^{\alpha-1} \times \exp\left\{-\beta x + \exp\left[-\lambda + \lambda\exp(-\beta x)\right]\right\} \qquad \text{for } \alpha,\ \beta,\ \lambda > 0,$$

the exponential Weibull (EW) distribution (see [16, 17]) with PDF:

$$f(x) = \alpha\beta\lambda^{\beta}x^{\beta-1}\left\{1 - \exp\left[-(\lambda x)^{\beta}\right]\right\}^{a-1} \exp\left[-(\lambda x)^{b}\right] \qquad \text{for } \alpha,\ \beta,\ \lambda > 0,$$

the Weibull Poisson (WP) distribution [7] with PDF

$$f(x) = \frac{\alpha\beta\lambda}{1-\exp(-\lambda)}(\alpha x)^{\beta-1} \exp\left\{-\lambda - (\alpha x)^{\beta} + \lambda\exp\left[-(\alpha x)^{\beta}\right]\right\} \qquad \text{for } \alpha,\ \beta,\ \lambda > 0,$$

the generalized exponential (GE) distribution [6] with PDF:

$$f(x) = \frac{\alpha}{\lambda}\left[1 - \exp\left(-\frac{x-\mu}{\lambda}\right)\right]^{\alpha-1} \exp\left(-\frac{x-\mu}{\lambda}\right) \qquad \text{for } \alpha,\ \beta,\ \lambda > 0,$$

the generalized exponential geometric (GEG) distribution [25] with PDF:

$$f(x) = \frac{\alpha\beta(1-p)\exp(-\beta x)\left[1-\exp(-\beta x)\right]^{\alpha-1}}{\left\{1 - p\left[1-\exp(-\beta x)\right]\right\}^{\alpha+1}} \qquad \text{for } \alpha,\ \beta > 0 \text{ and } 0 < p < 1.$$

The maximum likelihood estimates, corresponding standard errors, the log-likelihood values, the Kolmogorov Smirnov statistics, associated p-values, the AIC values and the BIC values are shown in Tables 1 to 3. The standard errors were computed by inverting the observed information matrices, see Section 3.1. We can see that the largest log-likelihood value, the largest p-value, the smallest AIC value and the smallest BIC value are obtained for the GEL distribution. The results show that the GEL distribution yields the best fits.

Table 1. Parameter estimates and associated values for the first data set.

Distribution	Estimates (standard errors)	Log-likelihood	K-S	p-value	AIC	BIC
GEL	(2.3266, 0.1478, 0.5671)	−317.0107	0.0761	0.7141	640.0214	647.8369
	(2.1942, 0.1069, 0.4053)					
WP	(0.0596, 1.7228, 2.9720)	−318.3816	0.0821	0.4847	642.7631	650.5787
	(0.0158, 1.6526, 0.8982)					
EW	(2.5729, 0.9060, 0.1904)	−317.1054	0.0858	0.5690	640.2108	648.0263
	(1.3821, 0.5958, 0.1671)					
GEP	(2.7193, 0.1593, 0.7999)	−317.5271	0.1145	0.2271	641.0542	648.8697
	(0.9171, 0.1387, 0.0325)					
GE	(1.8929, 7.6563, 0.3460)	−318.8549	0.1077	0.1828	643.7097	651.5252
	(1.3703, 1.6305, 0.0237)					
GEG	(3.2316, 0.1397, 0.4231)	−318.0609	0.0776	0.5572	642.1218	649.9373
	(1.4594, 0.1258, 0.0531)					

Table 2. Parameter estimates and associated values for the second data set.

Distribution	Estimates (standard errors)	Log-likelihood	K-S	p-value	AIC	BIC
GEL	(1.5085, 0.2040, 0.0700)	−102.7108	0.1195	0.7582	211.4217	216.9721
	(1.0042, 0.1718, 0.0544)					
WP	(0.1168, 1.0968, 3.5363)	−103.7616	0.1497	0.4878	213.5232	219.0736
	(0.1036, 0.9061, 2.3973)					
EW	(2.9085, 0.5779, 1.2000)	−103.2490	0.1883	0.2251	212.4979	218.0484
	(1.6546, 0.3251, 0.2387)					
GEP	(1.1434, 0.2119, 0.1000)	−109.5163	0.1439	0.5383	225.0326	230.5830
	(1.0036, 0.0798, 0.0828)					
GE	(0.2636, 47.0700, 0.2000)	−110.7692	0.2828	0.0152	227.5383	233.0887
	(0.1793, 10.0844, 0.0135)					
GEG	(9.7999, 0.0950, 0.9810)	−103.5101	0.1205	0.7493	213.0202	218.5706
	(0.1225, 0.0768, 0.7290)					

It is pleasing that the standard errors are less than the parameter estimates for each fitted distribution. It is also pleasing that the p-values based on Kolmogorov Smirnov statistics suggest that each fitted distribution adequately describes the data.

The probability-probability plots for the six fitted models and for each data set are shown in Figures 3 to 5. We can see that the GEL distribution has the points closest to the diagonal line for each data set.

A density plot compares the fitted PDFs of the models with the empirical histogram of the observed data. The density plots for the three data sets are shown in Figures 6 to 8. Again the fitted PDFs for the GEL distribution appear to capture the general pattern of the empirical histograms best.

TABLE 3. Parameter estimates and associated values for the third data set.

Distribution	Estimates (standard errors)	Log-likelihood	K-S	p-value	AIC	BIC
GEL	(3.6258, 1.0771, 0.5333)	−38.1570	0.0846	0.9744	82.3139	86.5175
	(2.6047, 0.0934, 0.0535)					
WP	(0.3990, 2.5829, 2.9895)	−39.8804	0.1231	0.7257	85.7609	89.9645
	(0.1012, 1.8698, 1.4216)					
EW	(2.8421, 1.0931, 1.2956)	−40.4429	0.2060	0.1479	88.3301	92.5337
	(1.1871, 1.0023, 0.2747)					
GEP	(3.7942, 1.3426, 0.3050)	−39.2024	0.1721	0.3193	84.4048	88.6084
	(1.4932, 1.0194, 0.1686)					
GE	(2.4402, 0.9992,−0.2501)	−39.1574	0.1243	0.7152	84.3149	88.5184
	(1.9948, 0.9868, 0.0932)					
GEG	(4.0001, 1.1323, 0.4223)	−39.0193	0.1624	0.3874	84.0386	88.2422
	(3.1755, 0.4400, 0.3087)					

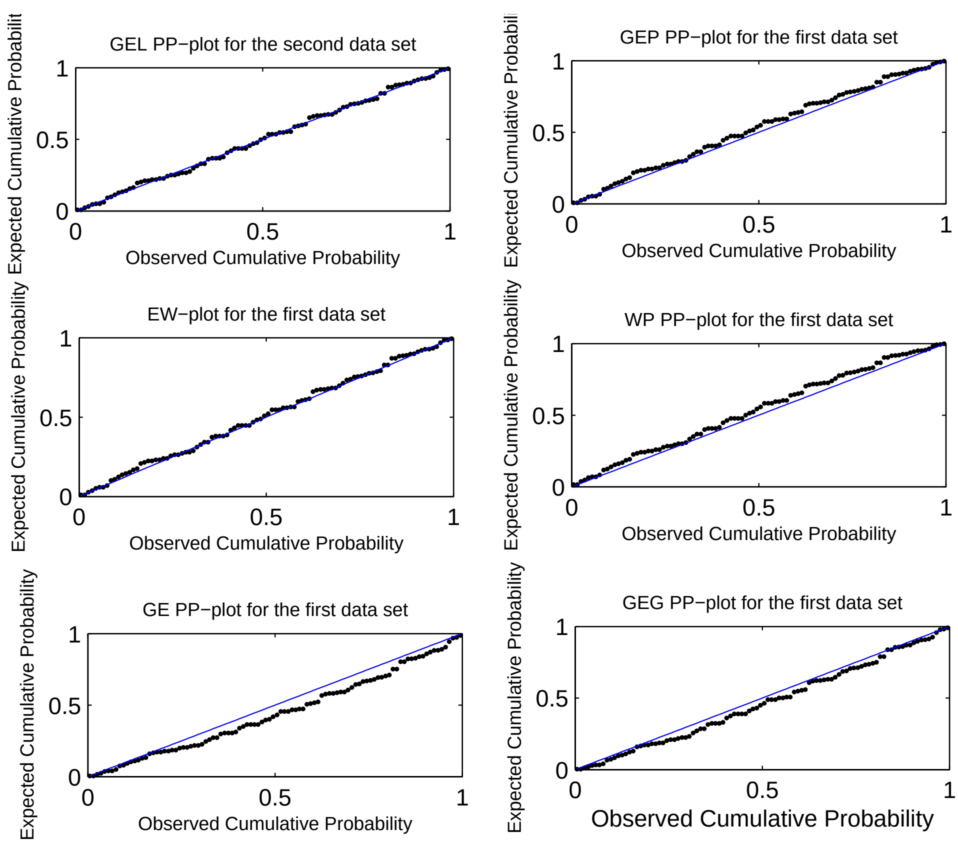

FIGURE 3. Probability-probability plots for the fitted models for the first data set.

5. Conclusions

We have proposed a class of distributions by compounding the logarithmic distribution with any lifetime distribution. We have derived some mathematical properties of the class including shape properties, moment properties and the asymptotic distributions of the extreme order statistics. We have also discussed maximum likelihood estimation for the class of distributions and performed a simulation study to assess the performance of the maximum likelihood estimates.

The flexibility of the class is illustrated by fitting the generalized exponential logarithmic distribution, a particular member of the class, to three real data sets. We have compared the fit of the generalized exponential logarithmic distribution with five other distributions each having the same number of parameters. Evidence based on Kolmogorov Smirnov statistics, likelihood values, AIC values, BIC values, probability-probability plots and density plots shows that the generalized exponential logarithmic distribution outperforms all of the other distributions.

The Compound Exponentiated Logarithmic distribution in (1) and (2) is defined in terms of a random variable of the form $\max(Y_1, Y_2, \ldots, Y_N)$. In classical insurance mathematics, the distribution of $\max(Y_1, Y_2, \ldots, Y_N)$ is compared with the distribution of $Y_1 + Y_2 + \cdots + Y_N$ and this leads to the notion of subexponentiality. A future work is to see if subexponentiality has connections to the Compound Exponentiated Logarithmic distribution.

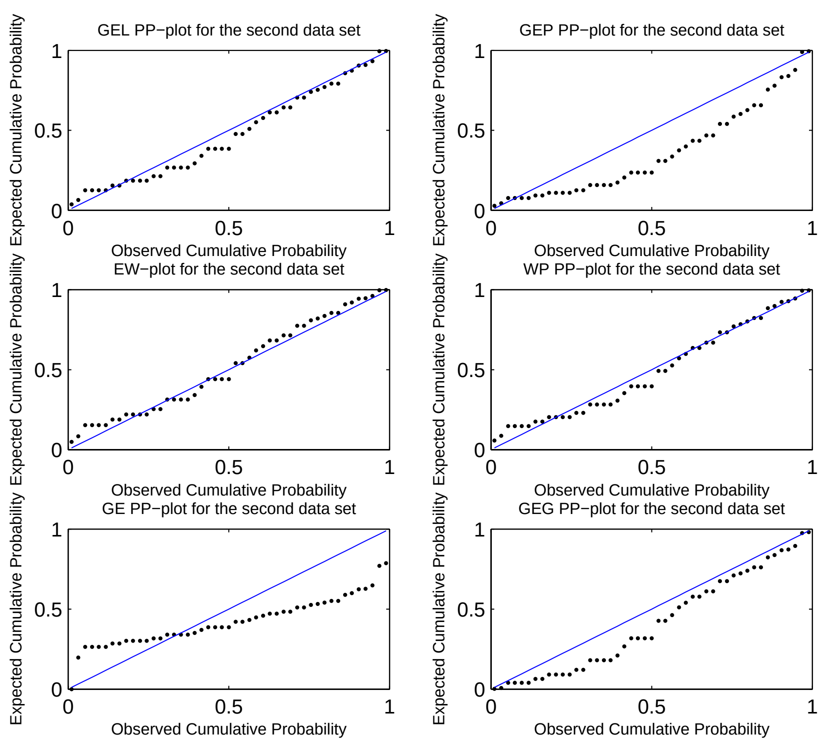

FIGURE 4. Probability-probability plots for the fitted models for the second data set.

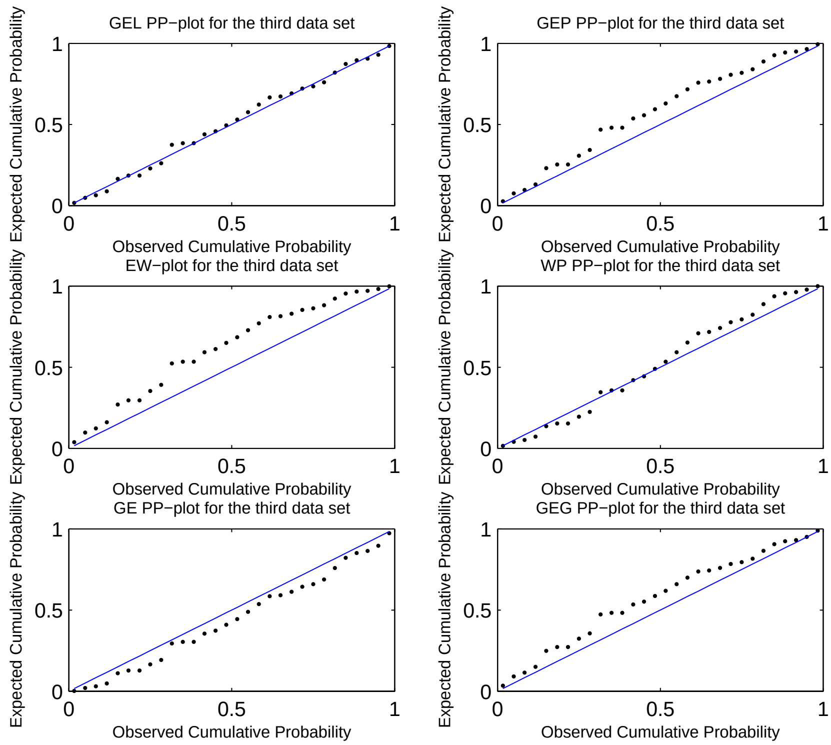

FIGURE 5. Probability-probability plots for the fitted models for the third data set.

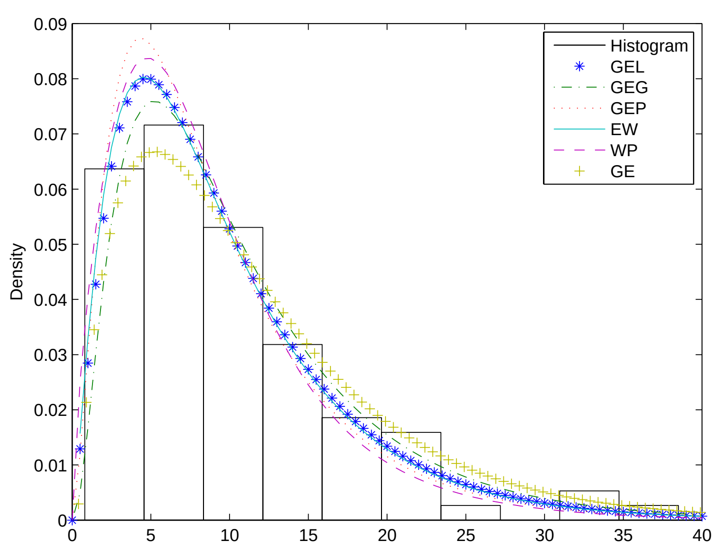

FIGURE 6. Fitted PDFs and the observed histogram for the first data set.

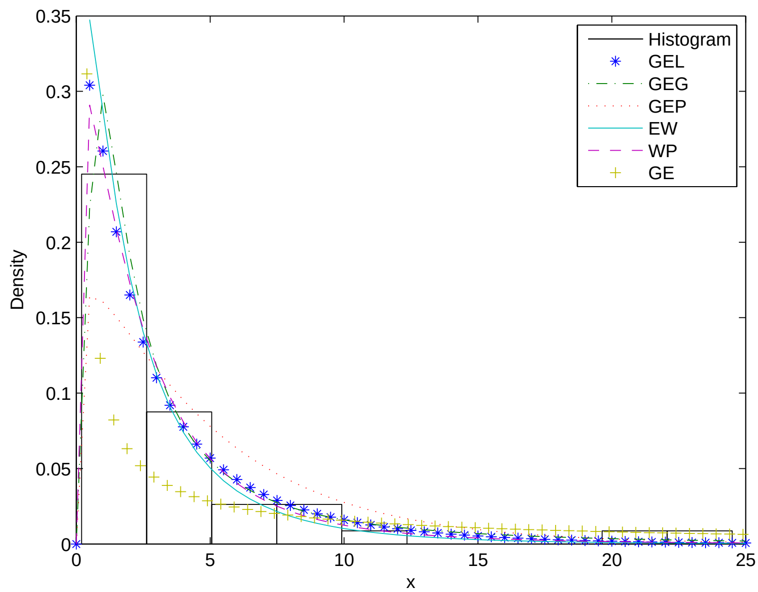

FIGURE 7. Fitted PDFs and the observed histogram for the second data set.

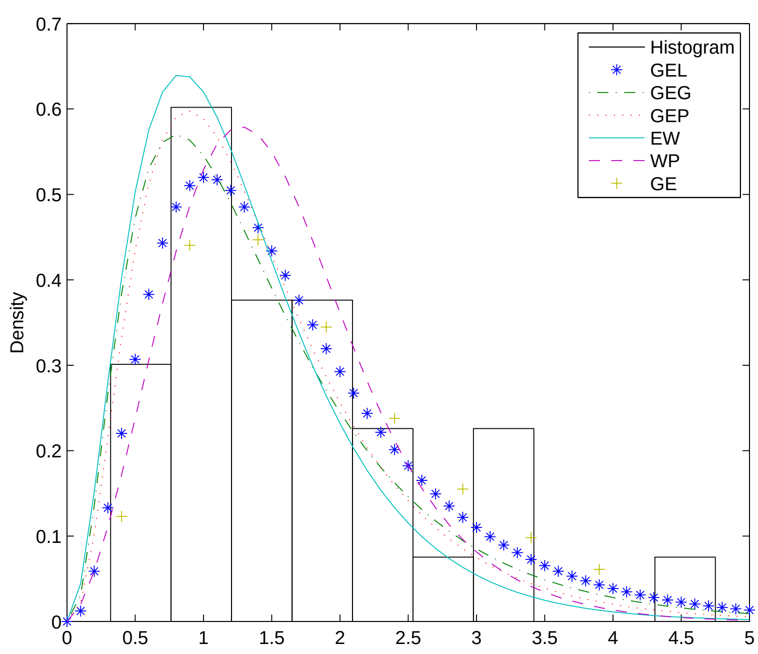

FIGURE 8. Fitted PDFs and the observed histogram for the third data set.

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