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INVERSE TANGENT SERIES INVOLVING PELL AND PELL-LUCAS POLYNOMIALS

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ABSTRACT. By means of the telescoping method, several summation formulae are established for the arctangent function with its argument being Pell and Pell–Lucas polynomials. Numerous infinite series identities involving Fibonacci and Lucas numbers are included as particular cases.

1. Introduction and Motivation

The Fibonacci numbers and Lucas numbers are well known in mathematics, physics and applied sciences (cf. Koshy [5]), which can be defined as follows:

- Recurrence relation:

$$F_n = F_{n-1} + F_{n-2} \quad \text{with} \quad F_0 = 0 \quad \text{and} \quad F_1 = 1,$$
$$L_n = L_{n-1} + L_{n-2} \quad \text{with} \quad L_0 = 2 \quad \text{and} \quad L_1 = 1.$$

- Generating functions:

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2} \quad \text{and} \quad \sum_{n=0}^{\infty} L_n x^n = \frac{2-x}{1-x-x^2}.$$

- For the golden ratio denoted by $\varphi = \frac{1+\sqrt{5}}{2}$, there are the Binet formulae:

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\varphi + \varphi^{-1}} \quad \text{and} \quad L_n = \varphi^n + (-\varphi)^{-n}.$$

There exist numerous formulae about Fibonacci and Lucas numbers in the literature. In 1936, Lehmer discovered the following celebrated result

$$\frac{\pi}{2} = \sum_{n=1}^{\infty} \arctan F_{2n-1}^{-1} \quad \Longleftrightarrow \quad \frac{\pi}{4} = \sum_{n=1}^{\infty} \arctan F_{2n+1}^{-1}. \tag{1.1}$$

A detailed account of it was given by Hoggatt and Ruggles [3], where they found the companion formula below (see also Mahon and Horodam [7])

$$\frac{\arctan 2}{2} = \sum_{n=1}^{\infty} \arctan L_{2n}^{-1}. \tag{1.2}$$

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As a warm up, here we sketch proofs for (1.1) and (1.2). According to the Binet formula, we can rewrite

$$F_{2n-1}^{-1} = \frac{\varphi - \varphi^{-1}}{F_{2n-1}} = \frac{\varphi^2 - \varphi^{-2}}{\varphi^{2n-1} + \varphi^{1-2n}} = \frac{\varphi^{3-2n} - \varphi^{-1-2n}}{1 + \varphi^{2-4n}}$$

which is equivalent to

$$\arctan F_{2n-1}^{-1} = \arctan \varphi^{3-2n} - \arctan \varphi^{-1-2n}.$$

Summing this equation over n from 1 to ∞ by telescoping, we confirm (1.1) as follows:

$$\begin{aligned}\sum_{n=1}^{\infty} \arctan F_{2n-1}^{-1} &= \lim_{m\to\infty} \sum_{n=1}^{m} \arctan F_{2n-1}^{-1} = \arctan \varphi + \arctan \varphi^{-1} \\ &\quad - \lim_{m\to\infty} \big(\arctan \varphi^{1-2m} + \arctan \varphi^{-1-2m}\big) \\ &= \arctan \varphi + \arctan \varphi^{-1} = \frac{\pi}{2}.\end{aligned}$$

Analogously for (1.2), we have

$$L_{2n}^{-1} = \frac{\varphi - \varphi^{-1}}{L_{2n}} = \frac{\varphi - \varphi^{-1}}{\varphi^{2n} + \varphi^{-2n}} = \frac{\varphi^{1-2n} - \varphi^{-1-2n}}{1 + \varphi^{-4n}}$$

which is equivalent to

$$\arctan L_{2n}^{-1} = \arctan \varphi^{1-2n} - \arctan \varphi^{-1-2n}.$$

Then (1.2) follows from the computations below:

$$\begin{aligned}\sum_{n=1}^{\infty} \arctan L_{2n}^{-1} &= \lim_{m\to\infty} \sum_{n=1}^{m} \arctan L_{2n}^{-1} \\ &= \arctan \varphi^{-1} - \lim_{m\to\infty} \arctan \varphi^{-1-2m} \\ &= \arctan \varphi^{-1} = \frac{\arctan 2}{2}.\end{aligned}$$

As a byproduct, we have further

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \arctan \frac{F_{2n}}{2 + F_{2n}^2}. \tag{1.3}$$

In fact, writing $f_n := \arctan F_n$ for $n \in \mathbb{N}$, we have

$$\tan(f_{n+1} - f_{n-1}) = \frac{F_{n+1} - F_{n-1}}{1 + F_{n+1}F_{n-1}} = \frac{F_n}{1 + F_{n+1}F_{n-1}}.$$

It follows that

$$\begin{aligned}\sum_{n=1}^{\infty} \arctan \frac{F_n}{1 + F_{n+1}F_{n-1}} &= \lim_{m\to\infty} \sum_{n=1}^{m} \big(f_{n+1} - f_{n-1}\big) \\ &= \lim_{m\to\infty} \big\{f_m + f_{m+1}\big\} - f_0 - f_1 \\ &= \pi - \frac{\pi}{4} = \frac{3\pi}{4}.\end{aligned}$$

For the last series, consider its bisection according to the Cassini formula

$$F_{n+1}F_{n-1} = (-1)^n + F_n^2.$$

Then the corresponding series with odd indices results in (1.1), while the series with even indices gives exactly (1.3). □

The identities (1.1), (1.2) and (1.3) suggest not only the existence for a larger number of similar formulae concerning Fibonacci and Lucas numbers (cf. [1,3,8]), but also possibly further polynomial extensions, such as Pell and Pell-Lucas polynomials (cf. Koshy [6] and Horadam-Mahon [4, 7]). These polynomials are defined by the recurrence relations

$$\begin{aligned} \mathrm{P}_n(x) &= 2x\mathrm{P}_{n-1}(x) + \mathrm{P}_{n-2}(x), \\ \mathrm{Q}_n(x) &= 2x\mathrm{Q}_{n-1}(x) + \mathrm{Q}_{n-2}(x); \end{aligned}$$

with different initial conditions

$$\begin{aligned} \mathrm{P}_0(x) = 0 \quad &\text{and} \quad \mathrm{P}_1(x) = 1, \\ \mathrm{Q}_0(x) = 2 \quad &\text{and} \quad \mathrm{Q}_1(x) = 2x. \end{aligned}$$

Their Binet forms are

$$\mathrm{P}_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad \mathrm{Q}_n(x) = \alpha^n + \beta^n$$

where

$$\alpha := \alpha(x) = x + \sqrt{x^2+1} \quad \text{and} \quad \beta := \beta(x) = x - \sqrt{x^2+1}.$$

Throughout the paper, we always assume that x is real with $x > 0$, since otherwise, $x < 0$ will result in an exchange between the absolute values of $\alpha(x)$ and $\beta(x)$. In this case, there are the following zero limits

$$\begin{aligned} 0 &= \lim_{n\to\infty} \alpha^{-n}(x) = \lim_{n\to\infty} \arctan \alpha^{-n}(x) \\ &= \lim_{n\to\infty} \beta^{n}(x) = \lim_{n\to\infty} \arctan \beta^{n}(x). \end{aligned}$$

They will be utilized frequently to deduce limiting relations without specific explanations.

These polynomials contain the following well-known numbers as particular cases:

- Fibonacci number $F_n = \mathrm{P}_n(\frac{1}{2})$.
- Lucas number $L_n = \mathrm{Q}_n(\frac{1}{2})$.
- Pell number $P_n = \mathrm{P}_n(1)$.
- Pell–Lucas number $Q_n = \mathrm{Q}_n(1)$.

Similar to (1.1), (1.2) and (1.3), Mahon-Horadam [7] and Melham–Shannon [8] derived several closed formulae for the infinite series containing a single inverse tangent function with its argument involving both Pell and Pell–Lucas polynomials. The objective of the present paper will evaluate the infinite series with their summands being products of two *arctangent* functions.

The crucial ingredients will be the two well-known formulae:

$$\arctan x + \arctan y = \arctan \frac{x+y}{1-xy}, \quad (xy < 1); \tag{1.4}$$

$$\arctan x - \arctan y = \arctan \frac{x-y}{1+xy}, \quad (xy > -1). \tag{1.5}$$

The rest of the paper will be organized as follows. In the next section, we shall prove five summation theorems about products of arctangent function with the Pell polynomials as its arguments. Then in Section 3, five analogous formulae will be shown for products of arctangent function with the Pell-Lucas polynomials as its arguments. Finally in Section 4, six closed formulae will be established for cross products of arctangent function with both Pell and Pell-Lucas polynomials as its arguments. As consequences, several infinite series identities involving Fibonacci and Lucas numbers will be highlighted.

2. Summation Formulae about $\mathrm{P}_k(x)$

According to the Binet form of the Pell polynomial $\mathrm{P}_k(x)$, we have no difficulty to convert (1.4) and (1.5) to the following expressions:

$$\text{[P1]} \quad \arctan \frac{1}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}} = 2\arctan \alpha^{-2k}.$$

$$\text{[P2]} \quad \arctan \frac{x}{\mathrm{P}_{2k-1}(x)\sqrt{x^2+1}} = \arctan \alpha^{2-2k} - \arctan \alpha^{-2k}.$$

$$\text{[P3]} \quad \arctan \frac{1}{\mathrm{P}_{2k}(x)} = \arctan \alpha^{1-2k} + \arctan \alpha^{-1-2k}.$$

$$\text{[P4]} \quad \arctan \frac{2x}{\mathrm{P}_{2k-1}(x)} = \arctan \alpha^{3-2k} - \arctan \alpha^{-1-2k}.$$

$$\text{[P5]} \quad \arctan \frac{2x^2+1}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}} = \arctan \alpha^{2-2k} + \arctan \alpha^{-2-2k}.$$

By making use of the telescoping approach (cf. Chu [2]), we immediately have from [P2]

$$\sum_{k=1}^{n} \arctan \frac{x}{\mathrm{P}_{2k-1}(x)\sqrt{x^2+1}} = \frac{\pi}{4} - \arctan \alpha^{-2n}$$

and its limiting form as $n \to \infty$

$$\sum_{k=1}^{\infty} \arctan \frac{x}{\mathrm{P}_{2k-1}(x)\sqrt{x^2+1}} = \frac{\pi}{4}$$

as well as two particular examples:

$$\boxed{x=\tfrac{1}{2}} \quad \sum_{k=1}^{\infty} \arctan \frac{1}{\sqrt{5}F_{2k-1}} = \frac{\pi}{4},$$

$$\boxed{x=1} \quad \sum_{k=1}^{\infty} \arctan \frac{1}{\sqrt{2}P_{2k-1}} = \frac{\pi}{4}.$$

As done by Mahon and Horadam [7], we have, from [P4], the summation formula

$$\sum_{k=1}^{n} \arctan \frac{2x}{\mathrm{P}_{2k-1}(x)} = \frac{\pi}{2} - \arctan \frac{1}{\mathrm{P}_{2n}(x)}$$

and its limiting form as $n \to \infty$

$$\sum_{k=1}^{\infty} \arctan \frac{2x}{\mathrm{P}_{2k-1}(x)} = \frac{\pi}{2}$$

as well as two known identities:

$$\boxed{x=\tfrac{1}{2}} \quad \sum_{k=1}^{\infty} \arctan \frac{1}{F_{2k-1}} = \frac{\pi}{2},$$

$$\boxed{x=1} \quad \sum_{k=1}^{\infty} \arctan \frac{2}{P_{2k-1}} = \frac{\pi}{2};$$

where the former is just (1.1), while the latter can also be found in Melham-Shannon [8].

Instead, we are now going to analyze products of [P1–P5] and to derive, via the telescoping method, five summation formulae about $\mathrm{P}_k(x)$. As special cases, we deduce also some identities concerning Fibonacci and Pell numbers.

§2.1.

According to [P1], we have

$$\arctan\frac{1}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}}+\arctan\frac{1}{\mathrm{P}_{2k-2}(x)\sqrt{x^2+1}}=2\arctan\alpha^{-2k}+2\arctan\alpha^{2-2k}.$$

Multiplying this with [P2] and then summing the resultant equation over k from 2 to $n+1$ by telescoping, we find the summation formula below.

Theorem 2.1.

$$\sum_{k=1}^{n}\arctan\frac{x}{\mathrm{P}_{2k+1}(x)\sqrt{x^2+1}}\left\{\arctan\frac{1}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}}+\arctan\frac{1}{\mathrm{P}_{2k+2}(x)\sqrt{x^2+1}}\right\}$$
$$=2\arctan^2\alpha^{-2}-2\arctan^2\alpha^{-2-2n}.$$

Its limiting form as $n\to\infty$ brings about the infinite series identity.

Corollary 2.2.

$$\sum_{k=1}^{\infty}\arctan\frac{x}{\mathrm{P}_{2k+1}(x)\sqrt{x^2+1}}\left\{\arctan\frac{1}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}}+\arctan\frac{1}{\mathrm{P}_{2k+2}(x)\sqrt{x^2+1}}\right\}$$
$$=2\arctan^2\alpha^{-2}.$$

§2.2.

Analogously by means of [P2], we have

$$\arctan\frac{x}{\mathrm{P}_{2k-1}(x)\sqrt{x^2+1}}+\arctan\frac{x}{\mathrm{P}_{2k+1}(x)\sqrt{x^2+1}}=\arctan\alpha^{2-2k}-\arctan\alpha^{-2-2k}.$$

Multiplying this with [P1] and then summing the resultant equation over k from 1 to n by telescoping, we get another summation formula.

Theorem 2.3.

$$\sum_{k=1}^{n}\arctan\frac{1}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}}\left\{\arctan\frac{x}{\mathrm{P}_{2k-1}(x)\sqrt{x^2+1}}+\arctan\frac{x}{\mathrm{P}_{2k+1}(x)\sqrt{x^2+1}}\right\}$$
$$=\frac{\pi}{2}\arctan\beta^2-2\arctan\alpha^{-2n}\arctan\alpha^{-2-2n}.$$

Its limiting form as $n\to\infty$ gives rise to the infinite series identity.

Corollary 2.4.

$$\sum_{k=1}^{\infty}\arctan\frac{1}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}}\left\{\arctan\frac{x}{\mathrm{P}_{2k-1}(x)\sqrt{x^2+1}}+\arctan\frac{x}{\mathrm{P}_{2k+1}(x)\sqrt{x^2+1}}\right\}$$
$$=\frac{\pi}{2}\arctan\beta^2.$$

Furthermore, it can be verified that both Corollaries 2.2 and 2.4 contain, as special cases, the following same identities about Fibonacci and Pell numbers:

$$\boxed{x=\tfrac{1}{2}}\quad \sum_{k=1}^{\infty}\arctan\frac{2}{\sqrt{5}F_{2k}}\left\{\arctan\frac{1}{\sqrt{5}F_{2k-1}}+\arctan\frac{1}{\sqrt{5}F_{2k+1}}\right\}=\frac{\pi}{4}\arctan\frac{2}{\sqrt{5}},$$

$$\boxed{x=1}\quad \sum_{k=1}^{\infty}\arctan\frac{1}{\sqrt{2}P_k}\arctan\frac{1}{\sqrt{2}P_{k+1}}=\frac{\pi}{4}\arctan\frac{\sqrt{2}}{4};$$

where the last series resolves a problem proposed by Ohtsuka [10], which is justified by the following equivalent bisection series

$$\sum_{k=1}^{\infty}\arctan\frac{1}{\sqrt{2}P_{2k}}\left\{\arctan\frac{1}{\sqrt{2}P_{2k-1}}+\arctan\frac{1}{\sqrt{2}P_{2k+1}}\right\}.$$

§2.3.

In view of [P3], we have

$$\arctan\frac{1}{\mathrm{P}_{2k}(x)}+\arctan\frac{1}{\mathrm{P}_{2k-2}(x)}=\arctan\alpha^{3-2k}+2\arctan\alpha^{1-2k}+\arctan\alpha^{-1-2k}.$$

Now multiplying [P4] with the above equation

$$\begin{aligned}&\arctan\frac{2x}{\mathrm{P}_{2k-1}(x)}\left\{\arctan\frac{1}{\mathrm{P}_{2k}(x)}+\arctan\frac{1}{\mathrm{P}_{2k-2}(x)}\right\}\\&=\big(\arctan\alpha^{3-2k}+\arctan\alpha^{1-2k}\big)^2-\big(\arctan\alpha^{1-2k}+\arctan\alpha^{-1-2k}\big)^2\end{aligned}$$

and then summing over k from 2 to $n+1$ by telescoping, we derive the formula below.

Theorem 2.5.

$$\begin{aligned}&\sum_{k=1}^{n}\arctan\frac{2x}{\mathrm{P}_{2k+1}(x)}\left\{\arctan\frac{1}{\mathrm{P}_{2k}(x)}+\arctan\frac{1}{\mathrm{P}_{2k+2}(x)}\right\}\\&=\big(\arctan\alpha^{-1}+\arctan\alpha^{-3}\big)^2-\big(\arctan\alpha^{-1-2n}+\arctan\alpha^{-3-2n}\big)^2.\end{aligned}$$

Its limiting form as $n\to\infty$ leads to the infinite series identity.

Corollary 2.6.

$$\begin{aligned}&\sum_{k=1}^{\infty}\arctan\frac{2x}{\mathrm{P}_{2k+1}(x)}\left\{\arctan\frac{1}{\mathrm{P}_{2k}(x)}+\arctan\frac{1}{\mathrm{P}_{2k+2}(x)}\right\}\\&=\big(\arctan\alpha^{-1}+\arctan\alpha^{-3}\big)^2.\end{aligned}$$

§2.4.

Alternatively from [P4], we have

$$\begin{aligned}\arctan\frac{2x}{\mathrm{P}_{2k-1}(x)}+\arctan\frac{2x}{\mathrm{P}_{2k+1}(x)}=\;&\arctan\alpha^{3-2k}-\arctan\alpha^{-1-2k}\\&+\arctan\alpha^{1-2k}-\arctan\alpha^{-3-2k}.\end{aligned}$$

Multiplying this with [P3], we can reformulate the result as

$$\begin{aligned}
&\arctan\frac{1}{\mathrm{P}_{2k}(x)}\left\{\arctan\frac{2x}{\mathrm{P}_{2k-1}(x)}+\arctan\frac{2x}{\mathrm{P}_{2k+1}(x)}\right\}\\
&=\left(\arctan\alpha^{1-2k}+\arctan\alpha^{-1-2k}\right)\times\left(\arctan\alpha^{1-2k}+\arctan\alpha^{3-2k}\right)\\
&\quad-\left(\arctan\alpha^{1-2k}+\arctan\alpha^{-1-2k}\right)\times\left(\arctan\alpha^{-1-2k}+\arctan\alpha^{-3-2k}\right).
\end{aligned}$$

Summing the above equation over k from 1 to n by telescoping, we obtain the following companion of the formula stated in Theorem 2.5.

Theorem 2.7.

$$\begin{aligned}
&\sum_{k=1}^{n}\arctan\frac{1}{\mathrm{P}_{2k}(x)}\left\{\arctan\frac{2x}{\mathrm{P}_{2k-1}(x)}+\arctan\frac{2x}{\mathrm{P}_{2k+1}(x)}\right\}\\
&=\left(\arctan\alpha+\arctan\alpha^{-1}\right)\times\left(\arctan\alpha^{-1}+\arctan\alpha^{-3}\right)\\
&\quad-\left(\arctan\alpha^{1-2n}+\arctan\alpha^{-1-2n}\right)\times\left(\arctan\alpha^{-1-2n}+\arctan\alpha^{-3-2n}\right).
\end{aligned}$$

Its limiting form as $n\to\infty$ results in the infinite series identity.

Corollary 2.8.

$$\begin{aligned}
&\sum_{k=1}^{\infty}\arctan\frac{1}{\mathrm{P}_{2k}(x)}\left\{\arctan\frac{2x}{\mathrm{P}_{2k-1}(x)}+\arctan\frac{2x}{\mathrm{P}_{2k+1}(x)}\right\}\\
&=\left(\arctan\alpha+\arctan\alpha^{-1}\right)\left(\arctan\alpha^{-1}+\arctan\alpha^{-3}\right)\\
&=\frac{\pi}{2}\left\{\arctan(\sqrt{x^2+1}-x)+\arctan(\sqrt{x^2+1}-x)^3\right\}.
\end{aligned}$$

Keeping in mind the fact

$$\arctan\left(\frac{\sqrt{5}-1}{2}\right)+\arctan\left(\frac{\sqrt{5}-1}{2}\right)^3=\frac{\pi}{4}$$

we can derive, from both Corollaries 2.6 and 2.8, the following interesting identities:

$$\boxed{x=\tfrac{1}{2}}\qquad\sum_{k=1}^{\infty}\arctan\frac{1}{F_k}\arctan\frac{1}{F_{k+1}}=\frac{\pi^2}{8},$$

$$\boxed{x=1}\qquad\sum_{k=1}^{\infty}\arctan\frac{1}{P_{2k}}\left\{\arctan\frac{2}{P_{2k-1}}+\arctan\frac{2}{P_{2k+1}}\right\}=\frac{\pi^2}{16}+\frac{\pi}{4}\arctan\frac{1}{7};$$

where the former is discovered recently by Fera [9], which follows from the bisection series

$$\sum_{k=1}^{\infty}\arctan\frac{1}{F_{2k}}\left\{\arctan\frac{1}{F_{2k-1}}+\arctan\frac{1}{F_{2k+1}}\right\}.$$

§2.5.

Keeping in mind [P2], we have

$$\arctan\frac{x}{\mathrm{P}_{2k-1}(x)\sqrt{x^2+1}}+\arctan\frac{x}{\mathrm{P}_{2k+1}(x)\sqrt{x^2+1}}=\arctan\alpha^{2-2k}-\arctan\alpha^{-2-2k}.$$

Multiplying this with [P5] and then summing the resultant equation for k from 1 to n by telescoping, we establish the summation formula below.

Theorem 2.9.

$$\sum_{k=1}^{n}\arctan\frac{2x^2+1}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}}\left\{\arctan\frac{x}{\mathrm{P}_{2k-1}(x)\sqrt{x^2+1}}+\arctan\frac{x}{\mathrm{P}_{2k+1}(x)\sqrt{x^2+1}}\right\}$$
$$=\frac{\pi^2}{16}+\arctan^2\alpha^{-2}-\arctan^2\alpha^{-2n}-\arctan^2\alpha^{-2-2n}.$$

Its limiting form as $n\to\infty$ yields the infinite series identity.

Corollary 2.10.

$$\sum_{k=1}^{\infty}\arctan\frac{2x^2+1}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}}\left\{\arctan\frac{x}{\mathrm{P}_{2k-1}(x)\sqrt{x^2+1}}+\arctan\frac{x}{\mathrm{P}_{2k+1}(x)\sqrt{x^2+1}}\right\}$$
$$=\frac{\pi^2}{16}+\arctan^2\alpha^{-2}.$$

Two identities are contained as particular cases.

$$\boxed{x=\tfrac{1}{2}}\quad\sum_{k=1}^{\infty}\arctan\frac{3}{\sqrt{5}F_{2k}}\left\{\arctan\frac{1}{\sqrt{5}F_{2k-1}}+\arctan\frac{1}{\sqrt{5}F_{2k+1}}\right\}=\frac{\pi^2}{16}+\frac{1}{4}\arctan^2\frac{2}{\sqrt{5}},$$

$$\boxed{x=1}\quad\sum_{k=1}^{\infty}\arctan\frac{3}{\sqrt{2}P_{2k}}\left\{\arctan\frac{1}{\sqrt{2}P_{2k-1}}+\arctan\frac{1}{\sqrt{2}P_{2k+1}}\right\}=\frac{\pi^2}{16}+\frac{1}{4}\arctan^2\frac{\sqrt{2}}{4}.$$

3. Summation Formulae about $\mathrm{Q}_k(x)$

By combining the Binet form of the Pell-Lucas polynomial $\mathrm{Q}_k(x)$ with (1.4) and (1.5), it is not hard to show the following expressions:

$$\text{[Q1]}\quad \arctan\frac{2}{\mathrm{Q}_{2k-1}(x)}=2\arctan\alpha^{1-2k}.$$

$$\text{[Q2]}\quad \arctan\frac{2x}{\mathrm{Q}_{2k}(x)}=\arctan\alpha^{1-2k}-\arctan\alpha^{-1-2k}.$$

$$\text{[Q3]}\quad \arctan\frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)}=\arctan\alpha^{2-2k}+\arctan\alpha^{-2k}.$$

$$\text{[Q4]}\quad \arctan\frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)}=\arctan\alpha^{2-2k}-\arctan\alpha^{-2-2k}.$$

$$\text{[Q5]}\quad \arctan\frac{2(2x^2+1)}{\mathrm{Q}_{2k-1}(x)}=\arctan\alpha^{3-2k}+\arctan\alpha^{-1-2k}.$$

The relation in [Q2] can be used promptly via telescoping to construct the summation formula

$$\sum_{k=1}^{n}\arctan\frac{2x}{\mathrm{Q}_{2k}(x)}=\arctan\alpha^{-1}-\arctan\alpha^{-1-2n}$$

and its limiting version as $n \to \infty$

$$\sum_{k=1}^{\infty} \arctan \frac{2x}{Q_{2k}(x)} = \arctan \alpha^{-1}$$

as well as two particular cases

$$\boxed{x = \tfrac{1}{2}} \quad \sum_{k=1}^{\infty} \arctan \frac{1}{L_{2k}} = \frac{\arctan 2}{2},$$

$$\boxed{x = 1} \quad \sum_{k=1}^{\infty} \arctan \frac{2}{Q_{2k}} = \frac{\pi}{8}.$$

Among these two formulae just displayed, the first one is anticipated in (1.2). Further identities of similar form can be found in Mahon and Horodam [7].

Alternatively, we have from [Q4]

$$\sum_{k=1}^{n} \arctan \frac{4x\sqrt{x^2+1}}{Q_{2k}(x)} = \frac{\pi}{4} + \arctan \alpha^{-2} - \arctan \frac{2\sqrt{x^2+1}}{Q_{2n+1}(x)}$$

and its limiting version as $n \to \infty$

$$\sum_{k=1}^{\infty} \arctan \frac{4x\sqrt{x^2+1}}{Q_{2k}(x)} = \frac{\pi}{4} + \arctan \alpha^{-2}$$

as well as two particular cases

$$\boxed{x = \tfrac{1}{2}} \quad \sum_{k=1}^{\infty} \arctan \frac{\sqrt{5}}{L_{2k}} = \frac{\pi}{4} + \frac{1}{2} \arctan \frac{2}{\sqrt{5}},$$

$$\boxed{x = 1} \quad \sum_{k=1}^{\infty} \arctan \frac{4\sqrt{2}}{Q_{2k}} = \frac{\pi}{4} + \frac{1}{2} \arctan \frac{\sqrt{2}}{4}.$$

By employing the telescoping method, we are now going to examine products of [Q1–Q5] and to establish five summation formulae for $Q_k(x)$, which contain, as special cases, some identities about Lucas and Pell-Lucas numbers.

§3.1.

According to [Q1], we have

$$\arctan \frac{2}{Q_{2k-1}(x)} + \arctan \frac{2}{Q_{2k+1}(x)} = 2 \arctan \alpha^{1-2k} + 2 \arctan \alpha^{-1-2k}.$$

Multiplying this with [Q2] and then summing over k from 1 to n by telescoping, we get the summation formula below.

Theorem 3.1.

$$\sum_{k=1}^{n} \arctan \frac{2x}{Q_{2k}(x)} \left\{ \arctan \frac{2}{Q_{2k-1}(x)} + \arctan \frac{2}{Q_{2k+1}(x)} \right\}$$
$$= 2 \arctan^2 \alpha^{-1} - 2 \arctan^2 \alpha^{-1-2n}.$$

Its limiting form as $n \to \infty$ brings about the infinite series identity.

Corollary 3.2.

$$\sum_{k=1}^{\infty}\arctan\frac{2x}{Q_{2k}(x)}\left\{\arctan\frac{2}{Q_{2k-1}(x)}+\arctan\frac{2}{Q_{2k+1}(x)}\right\}$$
$$=2\arctan^2\left(\sqrt{x^2+1}-x\right).$$

§3.2.

Analogously by means of [Q2], we have

$$\arctan\frac{2x}{Q_{2k-2}(x)}+\arctan\frac{2x}{Q_{2k}(x)}=\arctan\alpha^{3-2k}-\arctan\alpha^{-1-2k}.$$

Multiplying this with [Q1] and then summing over k from 1 to n by telescoping, we deduce another summation formula.

Theorem 3.3.

$$\sum_{k=1}^{n}\arctan\frac{2}{Q_{2k-1}(x)}\left\{\arctan\frac{2x}{Q_{2k-2}(x)}+\arctan\frac{2x}{Q_{2k}(x)}\right\}$$
$$=2\arctan\alpha^{-1}\arctan\alpha-2\arctan\alpha^{-1-2n}\arctan\alpha^{1-2n}.$$

Its limiting form as $n\to\infty$ gives rise to the infinite series identity.

Corollary 3.4.

$$\sum_{k=1}^{n}\arctan\frac{2}{Q_{2k-1}(x)}\left\{\arctan\frac{2x}{Q_{2k-2}(x)}+\arctan\frac{2x}{Q_{2k}(x)}\right\}$$
$$=2\arctan(\sqrt{x^2+1}-x)\arctan(\sqrt{x^2+1}+x).$$

From both Corollaries 3.2 and 3.4, we can show two further interesting formulae:

$$\boxed{x=\tfrac{1}{2}}\quad\sum_{k=1}^{\infty}\arctan\frac{2}{L_{2k-1}}\left\{\arctan\frac{1}{L_{2k-2}}+\arctan\frac{1}{L_{2k}}\right\}=\frac{1}{2}\arctan 2(\pi-\arctan 2),$$
$$\boxed{x=1}\quad\sum_{k=1}^{\infty}\arctan\frac{2}{Q_k}\arctan\frac{2}{Q_{k+1}}=\frac{\pi^2}{32};$$

where the last identity was a *Monthly problem* proposed by Ohtsuka and resolved by Vowe [9], which is simplified from the bisection series

$$\sum_{k=1}^{\infty}\arctan\frac{2}{Q_{2k}}\left\{\arctan\frac{2}{Q_{2k-1}}+\arctan\frac{2}{Q_{2k+1}}\right\}.$$

§3.3.

In view of [Q3], we have

$$\arctan\frac{2\sqrt{x^2+1}}{Q_{2k-1}(x)}+\arctan\frac{2\sqrt{x^2+1}}{Q_{2k+1}(x)}=\arctan\alpha^{2-2k}+\arctan\alpha^{-2-2k}+2\arctan\alpha^{-2k}.$$

Multiplying this with [Q4] and then summing over k from 1 to n by telescoping, we arrive at the summation formula below.

Theorem 3.5.

$$\sum_{k=1}^{n} \arctan \frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)} \left\{ \arctan \frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)} + \arctan \frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k+1}(x)} \right\}$$
$$= \left(\frac{\pi}{4} + \arctan \beta^2\right)^2 - \left(\arctan \alpha^{-2n} + \arctan \alpha^{-2-2n}\right)^2.$$

Its limiting form as $n \to \infty$ leads to the infinite series identity.

Corollary 3.6.

$$\sum_{k=1}^{\infty} \arctan \frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)} \left\{ \arctan \frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)} + \arctan \frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k+1}(x)} \right\} = \left(\frac{\pi}{4} + \arctan \beta^2\right)^2.$$

§3.4.

Alternatively from [Q4], we have

$$\arctan \frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)} + \arctan \frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k-2}(x)}$$
$$= \arctan \alpha^{2-2k} - \arctan \alpha^{-2k} + \arctan \alpha^{4-2k} - \arctan \alpha^{-2-2k}.$$

Multiplying this with [Q3] and then summing over k from 1 to n by telescoping, we can show the following companion formula of that in Theorem 3.5.

Theorem 3.7.

$$\sum_{k=1}^{n} \arctan \frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)} \left\{ \arctan \frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)} + \arctan \frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k-2}(x)} \right\}$$
$$= \left(\arctan \alpha + \arctan \alpha^2\right) \times \left(\arctan \alpha + \arctan \alpha^{-2}\right)$$
$$- \left(\arctan \alpha^{-2n} + \arctan \alpha^{2-2n}\right) \times \left(\arctan \alpha^{-2n} + \arctan \alpha^{-2-2n}\right).$$

Its limiting form as $n \to \infty$ results in the infinite series identity.

Corollary 3.8.

$$\sum_{k=1}^{\infty} \arctan \frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)} \left\{ \arctan \frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)} + \arctan \frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k-2}(x)} \right\}$$
$$= \frac{3\pi^2}{16} + \arctan \alpha^2 \arctan \alpha^{-2}.$$

Both Corollaries 3.6 and 3.8 contain the following two particular cases.

$$\boxed{x = \tfrac{1}{2}} \quad \sum_{k=1}^{\infty} \arctan \frac{\sqrt{5}}{L_k} \arctan \frac{\sqrt{5}}{L_{k+1}} = \left(\frac{\pi}{4} + \frac{1}{2} \arctan \frac{2}{\sqrt{5}}\right)^2,$$

$$\boxed{x = 1} \quad \sum_{k=1}^{\infty} \arctan \frac{2\sqrt{2}}{Q_{2k-1}} \left\{ \arctan \frac{4\sqrt{2}}{Q_{2k}} + \arctan \frac{4\sqrt{2}}{Q_{2k-2}} \right\} = \frac{3\pi^2}{16} + \frac{1}{2} \arctan \frac{\sqrt{2}}{4} \arctan(3+2\sqrt{2}).$$

§3.5.

Keeping in mind [Q2], we have

$$\arctan\frac{2x}{\mathrm{Q}_{2k}(x)}+\arctan\frac{2x}{\mathrm{Q}_{2k-2}(x)}=\arctan\alpha^{3-2k}-\arctan\alpha^{-1-2k}.$$

Multiplying this with [Q5] and then summing over k from 1 to n by telescoping, we establish the summation formula below.

Theorem 3.9.

$$\sum_{k=1}^{n}\arctan\frac{2(2x^2+1)}{\mathrm{Q}_{2k-1}(x)}\left\{\arctan\frac{2x}{\mathrm{Q}_{2k}(x)}+\arctan\frac{2x}{\mathrm{Q}_{2k-2}(x)}\right\}$$
$$=\arctan^2\alpha+\arctan^2\alpha^{-1}-\arctan^2\alpha^{1-2n}-\arctan^2\alpha^{-1-2n}.$$

Its limiting form as $n\to\infty$ yields the infinite series identity.

Corollary 3.10.

$$\sum_{k=1}^{\infty}\arctan\frac{2(2x^2+1)}{\mathrm{Q}_{2k-1}(x)}\left\{\arctan\frac{2x}{\mathrm{Q}_{2k}(x)}+\arctan\frac{2x}{\mathrm{Q}_{2k-2}(x)}\right\}$$
$$=\arctan^2\alpha+\arctan^2\alpha^{-1}.$$

From this corollary, we have two further identities as consequences.

$$\boxed{x=\tfrac{1}{2}}\quad\sum_{k=1}^{\infty}\arctan\frac{3}{L_{2k-1}}\left\{\arctan\frac{1}{L_{2k}}+\arctan\frac{1}{L_{2k-2}}\right\}=\arctan^2\varphi+\operatorname{arccot}^2\varphi,$$

$$\boxed{x=1}\quad\sum_{k=1}^{\infty}\arctan\frac{6}{Q_{2k-1}}\left\{\arctan\frac{2}{Q_{2k}}+\arctan\frac{2}{Q_{2k-2}}\right\}=\frac{5\pi^2}{32}.$$

4. Series Involving both $\mathrm{P}_k(x)$ and $\mathrm{Q}_k(x)$

By examining cross products between [P1–P5] and [Q1–Q5], we shall establish further summation formulae containing both Pell and Pell-Lucas polynomials.

§4.1.

For the two equalities [P1] and [Q4], by adding their product with respect to k from 1 to n and then simplifying the sum by telescoping, we get the summation formula below.

Theorem 4.1.

$$\sum_{k=1}^{n}\arctan\frac{1}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}}\arctan\frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)}$$
$$=\frac{\pi}{2}\arctan\alpha^{-2}-2\arctan\alpha^{-2n}\arctan\alpha^{-2-2n}.$$

Its limiting form as $n\to\infty$ brings about the infinite series identity.

Corollary 4.2.

$$\sum_{k=1}^{\infty}\arctan\frac{1}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}}\arctan\frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)}=\frac{\pi}{2}\arctan\alpha^{-2}.$$

Two special cases are highlighted as follows:

$$\boxed{x=\tfrac{1}{2}} \quad \sum_{k=1}^{\infty} \arctan\frac{1}{\sqrt{2}P_{2k}} \arctan\frac{4\sqrt{2}}{Q_{2k}} = \frac{\pi}{4}\arctan\frac{\sqrt{2}}{4},$$

$$\boxed{x=1} \quad \sum_{k=1}^{\infty} \arctan\frac{2}{\sqrt{5}F_{2k}} \arctan\frac{\sqrt{5}}{L_{2k}} = \frac{\pi}{4}\arctan\frac{2}{\sqrt{5}}.$$

§4.2.

Analogously by considering the product of [P4] and [Q1], we have the next formula.

Theorem 4.3.

$$\sum_{k=1}^{n} \arctan\frac{2x}{\mathrm{P}_{2k-1}(x)} \arctan\frac{2}{\mathrm{Q}_{2k-1}(x)} = 2\arctan(\sqrt{x^2+1}+x)\arctan(\sqrt{x^2+1}-x) - 2\arctan\alpha^{1-2n}\arctan\alpha^{-1-2n}.$$

Its limiting form as $n\to\infty$ gives rise to the infinite series identity.

Corollary 4.4.

$$\sum_{k=1}^{\infty} \arctan\frac{2x}{\mathrm{P}_{2k-1}(x)} \arctan\frac{2}{\mathrm{Q}_{2k-1}(x)} = 2\arctan(\sqrt{x^2+1}+x)\arctan(\sqrt{x^2+1}-x).$$

For the above series, we record two examples of it.

$$\boxed{x=\tfrac{1}{2}} \quad \sum_{k=1}^{\infty} \arctan\frac{1}{F_{2k-1}} \arctan\frac{2}{L_{2k-1}} = \frac{\pi}{2}\arctan 2 - \frac{1}{2}\arctan^2 2,$$

$$\boxed{x=1} \quad \sum_{k=1}^{\infty} \arctan\frac{2}{P_{2k-1}} \arctan\frac{2}{Q_{2k-1}} = \frac{3\pi^2}{32}.$$

§4.3.

Further, by examining the product of [P2] and [Q3], we derive the formula below.

Theorem 4.5.

$$\sum_{k=1}^{n} \arctan\frac{x}{\mathrm{P}_{2k-1}(x)\sqrt{x^2+1}} \arctan\frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)} = \frac{\pi^2}{16} - \arctan^2\beta^{2n}.$$

Letting $n\to\infty$ in this theorem, we find the following infinite series identity.

Corollary 4.6.

$$\sum_{k=1}^{\infty} \arctan\frac{x}{\mathrm{P}_{2k-1}(x)\sqrt{x^2+1}} \arctan\frac{2\sqrt{x^2+1}}{\mathrm{Q}_{2k-1}(x)} = \frac{\pi^2}{16}.$$

The last identity is remarkable in the sense that resulting value of the series is incredibly independent of the variable $x>0$. Two immediate consequences read as follows:

$$\boxed{x=\tfrac{1}{2}} \quad \sum_{k=1}^{\infty} \arctan\frac{\sqrt{5}}{5F_{2k-1}} \arctan\frac{\sqrt{5}}{L_{2k-1}} = \frac{\pi^2}{16},$$

$$\boxed{x=1} \quad \sum_{k=1}^{\infty} \arctan\frac{\sqrt{2}}{2P_{2k-1}} \arctan\frac{2\sqrt{2}}{Q_{2k-1}} = \frac{\pi^2}{16}.$$

Six elegant expressions for π^2 are highlighted further.

$$\boxed{x=\tfrac{\sqrt{5}}{2}} \qquad \sum_{k=1}^{\infty}\arctan\frac{3\sqrt{5}}{5F_{4k-2}}\arctan\frac{\sqrt{5}}{L_{4k-2}}=\frac{\pi^2}{16},$$

$$\boxed{x=2\sqrt{2}} \qquad \sum_{k=1}^{\infty}\arctan\frac{3\sqrt{2}}{2P_{4k-2}}\arctan\frac{4\sqrt{2}}{Q_{4k-2}}=\frac{\pi^2}{16};$$

$$\boxed{x=2} \qquad \sum_{k=1}^{\infty}\arctan\frac{4\sqrt{5}}{5F_{6k-3}}\arctan\frac{2\sqrt{5}}{L_{6k-3}}=\frac{\pi^2}{16},$$

$$\boxed{x=7} \qquad \sum_{k=1}^{\infty}\arctan\frac{7\sqrt{2}}{2P_{6k-3}}\arctan\frac{10\sqrt{2}}{Q_{6k-3}}=\frac{\pi^2}{16};$$

$$\boxed{x=\tfrac{3\sqrt{5}}{2}} \qquad \sum_{k=1}^{\infty}\arctan\frac{7\sqrt{5}}{5F_{8k-4}}\arctan\frac{3\sqrt{5}}{L_{8k-4}}=\frac{\pi^2}{16},$$

$$\boxed{x=12\sqrt{2}} \qquad \sum_{k=1}^{\infty}\arctan\frac{17\sqrt{2}}{2P_{8k-4}}\arctan\frac{24\sqrt{2}}{Q_{8k-4}}=\frac{\pi^2}{16}.$$

§4.4.

Analogously, the product of [P3] and [Q2] leads to the following formula.

Theorem 4.7.

$$\sum_{k=1}^{n}\arctan\frac{1}{\mathrm{P}_{2k}(x)}\arctan\frac{2x}{\mathrm{Q}_{2k}(x)}=\arctan^2\beta-\arctan^2\beta^{2n+1}.$$

Letting $n\to\infty$, we come to the infinite series identity.

Corollary 4.8.

$$\sum_{k=1}^{\infty}\arctan\frac{1}{\mathrm{P}_{2k}(x)}\arctan\frac{2x}{\mathrm{Q}_{2k}(x)}=\arctan^2\beta.$$

Two particular cases are given below for this corollary.

$$\boxed{x=\tfrac{1}{2}} \qquad \sum_{k=1}^{\infty}\arctan\frac{1}{F_{2k}}\arctan\frac{1}{L_{2k}}=\frac{\arctan^2 2}{4},$$

$$\boxed{x=1} \qquad \sum_{k=1}^{\infty}\arctan\frac{1}{P_{2k}}\arctan\frac{2}{Q_{2k}}=\frac{\pi^2}{64}.$$

§4.5.

Now, the product of [P4] and [Q5] gives rise to the following formula.

Theorem 4.9.

$$\sum_{k=1}^{n}\arctan\frac{2x}{\mathrm{P}_{2k-1}(x)}\arctan\frac{2(2x^2+1)}{\mathrm{Q}_{2k-1}(x)}$$
$$=\arctan^2\alpha+\arctan^2\beta-\arctan^2\alpha^{1-2n}-\arctan^2\alpha^{-1-2n}.$$

Its limiting form as $n\to\infty$ results in the infinite series identity.

Corollary 4.10.

$$\sum_{k=1}^{\infty} \arctan \frac{2x}{\mathrm{P}_{2k-1}(x)} \arctan \frac{2(2x^2+1)}{\mathrm{Q}_{2k-1}(x)} = \arctan^2 \alpha + \arctan^2 \beta.$$

This identity contains the two formulae below as particular cases.

$$\boxed{x = \tfrac{1}{2}} \quad \sum_{k=1}^{\infty} \arctan \frac{1}{F_{2k-1}} \arctan \frac{3}{L_{2k-1}} = \arctan^2 \varphi + \operatorname{arccot}^2 \varphi,$$

$$\boxed{x = 1} \quad \sum_{k=1}^{\infty} \arctan \frac{2}{P_{2k-1}} \arctan \frac{6}{Q_{2k-1}} = \frac{5\pi^2}{32}.$$

§4.6.

Alternatively, the product of [P5] and [Q4] yields the summation formula below.

Theorem 4.11.

$$\sum_{k=1}^{n} \arctan \frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)} \arctan \frac{2x^2+1}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}}$$
$$= \frac{\pi^2}{16} + \arctan^2 \frac{1}{\alpha^2} - \arctan^2 \frac{1}{\alpha^{2n}} - \arctan^2 \frac{1}{\alpha^{2n+2}}.$$

Its limiting form as $n \to \infty$ becomes the infinite series identity.

Corollary 4.12.

$$\sum_{k=1}^{\infty} \arctan \frac{4x\sqrt{x^2+1}}{\mathrm{Q}_{2k}(x)} \arctan \frac{2x^2+1}{\mathrm{P}_{2k}(x)\sqrt{x^2+1}} = \frac{\pi^2}{16} + \arctan^2 \beta^2.$$

This identity implies two further summation formulae.

$$\boxed{x = \tfrac{1}{2}} \quad \sum_{k=1}^{\infty} \arctan \frac{\sqrt{5}}{L_{2k}} \arctan \frac{3}{\sqrt{5}F_{2k}} = \frac{\pi^2}{16} + \frac{1}{4} \arctan^2 \frac{2}{\sqrt{5}},$$

$$\boxed{x = 1} \quad \sum_{k=1}^{\infty} \arctan \frac{4\sqrt{2}}{Q_{2k}} \arctan \frac{3}{\sqrt{2}P_{2k}} = \frac{\pi^2}{16} + \frac{1}{4} \arctan^2 \frac{\sqrt{2}}{4}.$$

5. Concluding Comments

The telescoping approach has been shown efficient to deal with the sums of the arctangent function with its argument being Pell and Pell-Lucas polynomials. Apart from the 16 formulae highlighted in previous sections, it is possible to find more identities. For instance, from the following relations

$$\arctan \frac{4x^2+1}{\mathrm{P}_{2k}(x)} = \arctan \alpha^{3-2k} + \arctan \alpha^{-3-2k},$$

$$\arctan \frac{2x(4x^2+3)}{\mathrm{Q}_{2k}(x)} = \arctan \alpha^{3-2k} - \arctan \alpha^{-3-2k};$$

we find the summation formula

$$\sum_{k=1}^{n} \arctan\frac{4x^2+1}{\mathrm{P}_{2k}(x)} \arctan\frac{2x(4x^2+3)}{\mathrm{Q}_{2k}(x)} = \arctan^2\alpha + \arctan^2\alpha^{-1} + \arctan^2\alpha^{-3} - \arctan^2\alpha^{1-2n} - \arctan^2 -\alpha^{-1-2n} - \arctan^2\alpha^{3-2n},$$

its limiting form

$$\sum_{k=1}^{\infty} \arctan\frac{4x^2+1}{\mathrm{P}_{2k}(x)} \arctan\frac{2x(4x^2+3)}{\mathrm{Q}_{2k}(x)} = \arctan^2\alpha + \arctan^2\alpha^{-1} + \arctan^2\alpha^{-3},$$

as well as two particular cases:

$$\boxed{x=\tfrac{1}{2}} \quad \sum_{k=1}^{\infty} \arctan\frac{2}{F_{2k}} \arctan\frac{4}{L_{2k}} = \arctan^2\varphi + \operatorname{arccot}^2\varphi + \operatorname{arccot}^2\varphi^3,$$

$$\boxed{x=1} \quad \sum_{k=1}^{\infty} \arctan\frac{5}{P_{2k}} \arctan\frac{14}{Q_{2k}} = \frac{5\pi^2}{32} + \frac{1}{4}\arctan^2\frac{1}{7}.$$

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