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HYPERBOLIC GEOMETRY FOR NON-DIFFERENTIAL TOPOLOGISTS

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ABSTRACT. A *soft* presentation of hyperbolic spaces (as metric spaces), free of differential apparatus, is offered. Fifth Euclid's postulate in such spaces is overthrown and, among other things, it is proved that spheres (equipped with great-circle distances) and hyperbolic and Euclidean spaces are the only locally compact geodesic (i.e., convex) metric spaces that are three-point homogeneous.

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1. Introduction

Hyperbolic geometry of a plane is known as a historically first example of a non-Euclidean geometry; that is, geometry in which all of Euclid's postulates are satisfied, apart from the fifth, called *parallel*, which is false. Discovered in the first half of the 19th century, it is one of the greatest mathematical achievements of those times. Although it merits special attention and mathematicians all over the world hear about it sooner or later, there are plenty of them whose knowledge on hyperbolic geometry is superficial and far from formal details. This sad truth concerns also topologists which do not specialise in differential geometry. One of reasons for this state is concerned with the extent of differential apparatus that one needs to learn in order to get to know and understand hyperbolic spaces. It was one of our two main sakes to propose and prepare an introduction to hyperbolic spaces (and geometry) that will be self-contained, free of differential language and accessible to 'everyone' (e.g., to accomplished mathematicians as well as to students).

The other story about hyperbolic spaces concerns their *free mobility* (or, in other words, *absolute* metric *homogeneity*). A metric space is absolutely homogeneous if all its partial isometries (that is, isometries between its subspaces) extend to global (bijective) isometries. According to a deep result from the 50's of the 20th century, known almost only by differential/Riemannian geometrists, hyperbolic spaces, beside Euclidean spaces and Euclidean spheres, are (in a very strong sense) the only connected locally compact metric spaces that have this property. One may even assume less about a connected locally compact metric space – that only partial isometries between 3-point and 2-point subspaces extend to global isometries. Then such a space is 'equivalent' to one of the aforementioned Riemannian manifolds and therefore is automatically absolutely homogeneous (the equivalence we speak here about does not imply that spaces are isometric, but is much stronger than a statement that they are homeomorphic). So, high level of metric homogeneity makes hyperbolic

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spaces highly exceptional. This is another reason for putting special attention on them. The main result of the present paper (Theorem 7.1 in Section 7 below) gives a full classification, up to isometry, of all connected locally compact metric spaces that are 3-point homogeneous. Although it is a consequence of the Freudenthal's theorem [8] from the 1950's, this classification has never been provided in the literature (according to the best knowledge of the authors). The present paper fills this astonishing lack in the theory of homogeneous metric spaces that have been intensively studied for many years.

The paper is organised as follows. In Section 2, we introduce (one of possible models of) hyperbolic spaces and prove that their metrics satisfy the triangle inequality and are equivalent to Euclidean metrics. In the next third section, we show that hyperbolic spaces are absolutely homogeneous and admit no dilations other than isometries (for dimension greater than 1). The fourth section is devoted to a counterpart of the Lebesgue measure in the hyperbolic realm, whereas the fifth discusses in details straight lines in hyperbolic geometry. We prove there that in a hyperbolic plane there are infinitely many straight lines passing through a given point and disjoint from a fixed straight line that does not pass through the given point. We also show that one-dimensional hyperbolic space is isometric to the ordinary real line. Section 6 deals with Tarski's axioms of the plane geometry in the context of the hyperbolic plane. In particular, we show that all the axioms of this system, except for the one that corresponds to the fifth Euclid's postulate, are satisfied. The last, seventh section is devoted to the classification (up to isometry) of all connected locally compact metric spaces that are 3-point homogeneous. All proofs are included, apart from the proof of a deep and difficult result due to Freudenthal [8] (see Theorem 7.3 below). This result is applied only once in this paper – to classify 3-point homogeneous metric spaces described above.

The reader interested in Riemannian geometry may consult, e.g., [13] or [6]. An axiomatic approach to various kinds of geometries (including Klein's projective model, Poincaré's model of the hyperbolic plane, and others) can be found e.g. in [7] – this material is also free of differential geometry, but totally differs from ours (it has a geometric spirit, whereas ours has a metric one). The reader interested in a presentation of hyperbolic geometry that is closer to ours is referred to [2: Subsection 1.10] by Benz, from where we took the formula (2.1) (see Section 2). Benz skips some non-trivial details, e.g. the proof of the triangle inequality for the hyperbolic metric, which is the most difficult part of the basics of hyperbolic geometry. Besides, he introduces hyperbolic spaces in a geometric fashion – starting from the isometry groups (given *a priori*) of these spaces. Our approach is purely metric space theoretic and we give all details.

Notation and terminology

In this paper, the term *metric* means a function (in two variables) that assigns to a pair of points of a fixed set their distance (so, *metric* does <u>not</u> mean *Riemannian metric*, a common and important notion in differential geometry).

For a pair of vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, we denote by $\langle x, y\rangle$ their standard inner product (that is, $\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$), whereas $\|x\| = \sqrt{\langle x, x\rangle}$ is the Euclidean norm of x. The Euclidean metric (induced by $\|\cdot\|$) will be denoted as d_e. To simplify further arguments and statements, we introduce the following notation:

$$[x] = \sqrt{1 + \|x\|^2}. \tag{1.1}$$

By an *isometric* map between metric spaces we mean any function that preserves the distances, that is, $f\colon (X, d_X) \to (Y, d_Y)$ is isometric if $d_Y(f(p), f(q)) = d_X(p, q)$ for any $p, q \in X$. The term *isometry* is reserved for surjective isometric maps. Additionally, a map f is a *dilation* if there is a constant $c > 0$ such that $d_Y(f(p), f(q)) = c d_X(p, q)$ for any $p, q \in X$ (we do not assume that dilations are surjective). If there exists a bijective dilation between two metric spaces, they are said to be *homothetic*. We will denote by $\mathrm{Iso}(X, d_X)$ the full isometry group of (X, d_X); that is,

$$\mathrm{Iso}(X, d_X) = \{u\colon (X, d_X) \to (X, d_X), \quad u \text{ isometry}\}.$$

A metric space (X, d_X) is said to be *geodesic* (or *convex*) if for any two distinct points a and b of X there exists a dilation $\gamma\colon [0,1] \to (X, d_X)$ such that $\gamma(0) = a$ and $\gamma(1) = b$ (we do not assume the uniqueness of such γ).

For the reader's convenience, let us recall that the inverse hyperbolic cosine is defined as $\cosh^{-1}(t) = \log(t + \sqrt{t^2 - 1})$ for $t \geq 1$ (throughout this paper 'log' stands for the natural logarithm).

2. Hyperbolic distance

Below we introduce one of many equivalent models of hyperbolic spaces – the one most convenient for us.

Definition 2.1. The *n-dimensional* (real) *hyperbolic space* is a metric space

$$(H^n(\mathbb{R}), d_h)$$

, where $H^n(\mathbb{R}) = \mathbb{R}^n$ and d_h is a metric (called *hyperbolic*) given by

$$d_h(x, y) = \cosh^{-1}([x][y] - \langle x, y\rangle) \qquad (x, y \in \mathbb{R}^n) \tag{2.1}$$

(see (1.1)).

The above formula has its origin in differential/Riemannian geometry (most often it is defined as the length of a geodesic arc – so, to get it one needs to find geodesics and compute integrals related to them), see, e.g., [14] (consult also [1] and [2: Subsection 1.10], where (2.1) appears without derivation). We underline here – at the very beginning of our presentation – that establishing the triangle inequality for d_h using elementary methods is undoubtedly the most difficult part in the whole of this approach.

Remark 2.2. Our definition of $H^n(\mathbb{R})$ is actually a somewhat modified Minkowski model of hyperbolic spaces. Indeed, in the last mentioned model one introduces a bilinear form

$$B\colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \ni ((x_0, \dots, x_n), (y_0, \dots, y_n)) \mapsto x_0 y_0 - \sum_{j=1}^{n} x_j y_j \in \mathbb{R}$$

and deals with an n-dimensional hyperboloid

$$H = \{x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}\colon\ x_0 > 0 \text{ and } B(x, x) = 1\}$$

equipped with a metric $d(x, y) = \cosh^{-1}(B(x, y))$. It is straightforward to check that the function $(H_n(\mathbb{R}), d_h) \ni x \mapsto ([x], x) \in (H, d)$ is a well defined bijective isometry.

Remark 2.3. As we will see in Corollary 5.2, the metric space $(H^1(\mathbb{R}), d_h)$ is isometric to $(\mathbb{R}, d_e)$. This contrasts with all other cases, as for $n > 1$ the metric space $(H^n(\mathbb{R}), d_h)$ admits no dilations other than isometries (see Theorem 3.5 below). In particular, for any integer $n > 1$ and positive real $r \neq 1$ the metric spaces $(H^n(\mathbb{R}), d_h)$ and $(H^n(\mathbb{R}), rd_h)$ are non-isometric but homothetic. Each of the metrics rd_h (with fixed $r > 0$) may serve as a 'standard' hyperbolic metric. Actually, *everything* that will be proved in this paper about the metric spaces $(H^n(\mathbb{R}), d_h)$ remains true when the metric is replaced by rd_h.

Note also that, similarly to Euclidean geometry, for $j < k$ the space $H^j(\mathbb{R})$ can naturally be considered as the subspace $\mathbb{R}^j \times \{0\}^{k-j}$ of $H^k(\mathbb{R})$. Under this identification, the hyperbolic metric of $H^j(\mathbb{R})$ coincides with the metric induced from the hyperbolic one of $H^k(\mathbb{R})$. This is the main reason why we 'forget' the dimension n in the notations 'd_h' and 'd_e.'

The aim of this section is to show that d_h is a metric on $\mathbb{R}^n$ equivalent to d_e.

Lemma 2.4. *For any $x, y \in H^n(\mathbb{R})$, $d_h(x, y)$ is well defined, non-negative and $d_h(x, y) = d_h(y, x)$. Moreover, $d_h(x, y) = 0$ iff $x = y$.*

P r o o f. It follows from the Schwarz inequality that

$$\|x\|^2\|y\|^2 + \|x - y\|^2 \geq \langle x, y\rangle^2 . \tag{2.2}$$

Equivalently, $\|x\|^2\|y\|^2 + \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 \geq \langle x, y\rangle^2$, which gives $1 + \|x\|^2 + \|y\|^2 + \|x\|^2\|y\|^2 \geq 1 + 2\langle x, y\rangle + \langle x, y\rangle^2$ and hence $(1 + \|x\|^2)(1 + \|y\|^2) \geq (1 + \langle x, y\rangle)^2$. Taking square roots from both sides shows that $[x][y] - \langle x, y\rangle \geq 1$ and thus $d_h(x, y)$ is well defined (and, of course, non-negative). Further, we see from (2.2) that $[x][y] - \langle x, y\rangle = 1$ iff $x = y$, which gives the last claim of the lemma. Symmetry is trivial. □

To establish the triangle inequality, first we show its special case, which will be used later in the proof of a general case.

Lemma 2.5. *For any $x, y \in H^n(\mathbb{R})$, $d_h(x, y) \leq d_h(x, 0) + d_h(0, y)$, and equality appears iff either $x = ty$ or $y = tx$ for some $t \leq 0$.*

P r o o f. Observe that

$$\begin{aligned} d_h(x, 0) + d_h(0, y) &= \cosh^{-1}([x]) + \cosh^{-1}([y]) \\ &= \log([x] + \|x\|) + \log([y] + \|y\|) \\ &= \log([x][y] + [x]\,\|y\| + [y]\,\|x\| + \|x\|\,\|y\|). \end{aligned}$$

On the other hand,

$$d_h(x, y) = \cosh^{-1}([x][y] - \langle x, y\rangle) \leq \cosh^{-1}([x][y] + \|x\|\,\|y\|)$$

and equality holds in the above iff $\langle x, y\rangle = -\|x\|\,\|y\|$, or, equivalently, if either $x = ty$ or $y = tx$ for some $t \leq 0$. So, to complete the whole proof, we only need to check that

$$[x][y] + [x]\,\|y\| + [y]\,\|x\| + \|x\|\,\|y\| = \exp(\cosh^{-1}([x][y] + \|x\|\,\|y\|)),$$

which is left to the reader as an elementary exercise. □

To get the triangle inequality for general triples of elements of the hyperbolic space, we will use certain one-dimensional perturbations of the identity map, which turn out to be isometries (with respect to the hyperbolic distance). They are introduced in the following.

Definition 2.6. For any $y \in H^n(\mathbb{R})$ let a map $T_y\colon H^n(\mathbb{R}) \to H^n(\mathbb{R})$ be defined by

$$T_y(x) = x + \Big([x] + \frac{\langle x, y\rangle}{[y] + 1}\Big)y \qquad (x \in H^n(\mathbb{R})).$$

Lemma 2.7. *For any $y \in H^n(\mathbb{R})$ the map T_y is bijective and fulfills the equation:*

$$d_h(T_y(a), T_y(b)) = d_h(a, b) \qquad (a, b \in H^n(\mathbb{R})). \tag{2.3}$$

Moreover, T_{-y} is the inverse of T_y.

Proof. We start from computing $[T_y(u)]$ for $u \in H^n(\mathbb{R})$:

$$\begin{aligned}[T_y(u)]^2 &= 1 + \left\| u + \left([u] + \frac{\langle u, y\rangle}{[y]+1}\right) y \right\|^2 \\ &= 1 + \|u\|^2 + 2\left([u] + \frac{\langle u, y\rangle}{[y]+1}\right)\langle u, y\rangle + \left([u] + \frac{\langle u, y\rangle}{[y]+1}\right)^2 \|y\|^2 \\ &= [u]^2 + \left(2[u]\langle u, y\rangle + 2\frac{\langle u, y\rangle^2}{[y]+1}\right) + [u]^2\|y\|^2 + \left(2[u]\langle u, y\rangle + \frac{\langle u, y\rangle^2}{[y]+1}\right)\frac{[y]^2-1}{[y]+1} \\ &= [u]^2[y]^2 + 2[u]\langle u, y\rangle [y] + \langle u, y\rangle^2 ,\end{aligned}$$

and thus

$$[T_y(u)] = [u]\,[y] + \langle u, y\rangle . \tag{2.4}$$

For simplicity, put $\alpha(x) = [x] + \frac{\langle x, y\rangle}{[y]+1}$ (then $T_y(x) = x + \alpha(x)y$). Observe that (2.3) is equivalent to

$$[T_y(a)]\,[T_y(b)] - \langle T_y(a), T_y(b)\rangle = [a]\,[b] - \langle a, b\rangle ,$$

which can easily be transformed to an equivalent form:

$$[T_y(a)]\,[T_y(b)] = [a]\,[b] + \alpha(b)\langle a, y\rangle + \alpha(a)\langle b, y\rangle + \alpha(a)\alpha(b)\|y\|^2.$$

The right-hand side expression of the above equation can be transformed as follows:

$$\begin{aligned}&[a]\,[b] + \alpha(b)\langle a, y\rangle + \alpha(a)\langle b, y\rangle + \alpha(a)\alpha(b)\|y\|^2 \\ &= [a][b] + [b]\langle a, y\rangle + \frac{\langle b, y\rangle}{[y]+1}\langle a, y\rangle + [a]\langle b, y\rangle + \frac{\langle a, y\rangle}{[y]+1}\langle b, y\rangle \\ &\quad + [a][b]([y]^2 - 1) + ([b]\langle a, y\rangle + [a]\langle b, y\rangle)\frac{[y]^2-1}{[y]+1} + \frac{\langle a, y\rangle\langle b, y\rangle}{([y]+1)^2}([y]^2-1) \\ &= [a][b][y]^2 + [b]\langle a, y\rangle [y] + [a]\langle b, y\rangle [y] + \langle a, y\rangle\langle b, y\rangle \\ &= ([a][y] + \langle a, y\rangle)([b][y] + \langle b, y\rangle),\end{aligned}$$

which equals $[T_y(a)]\,[T_y(b)]$, by (2.4). So, (2.3) is proved and, combined with Lemma 2.4, implies that T_y is one-to-one. To finish the whole proof, it suffices to check that $T_{-y} \circ T_y$ coincides with the identity map on $H^n(\mathbb{R})$ (because then, by symmetry, also $T_y \circ T_{-y}$ will coincide with the identity map). To this end, we fix $x \in H^n(\mathbb{R})$, and may and do assume that $y \neq 0$. Then there exist a unique vector z and a unique real number β such that $\langle z, y\rangle = 0$ and $x = z + \beta y$. Note that $T_y(x) = z + ([x] + \beta[y])y$ and, consequently,

$$\begin{aligned}T_{-y}(T_y(x)) &= z + ([x] + \beta[y])y - \Big([T_y(x)] - ([x] + \beta[y])([y]-1)\Big)y \\ &= z - [T_y(x)]y + ([x] + \beta[y])[y]y.\end{aligned}$$

Finally, an application of (2.4) enables us continuing the above calculations to obtain:

$$T_{-y}(T_y(x)) = z - ([x]\,[y] + \langle x, y\rangle)y + ([x]\,[y] + \beta[y]^2)y = z - \beta(\|y\|^2 - [y]^2)y = x. \qquad \square$$

We are now able to prove the main result of this section.

Theorem 2.8. *The function d_h is a metric inducing the Euclidean topology.*

Proof. To show the triangle inequality, consider arbitrary three points x, y and z of $H^n(\mathbb{R})$. Since $T_y(0) = y$, it follows from Lemmas 2.7 and 2.5 that

$$d_h(x, z) = d_h(T_{-y}(x), T_{-y}(z)) \le d_h(T_{-y}(x), 0) + d_h(0, T_{-y}(z)) = d_h(x, y) + d_h(y, z).$$

Further, to establish the equivalence of the metrics, observe that

$$d_h(x,0) = \cosh^{-1}([x]) \tag{2.5}$$

and that both T_y and T_{-y} are continuous with respect to the Euclidean metric (and thus they are homeomorphisms in this metric). Thus, for an arbitrary sequence $(x_n)_{n=1}^{\infty}$ of elements of $\mathbb{R}^n$ and any $x \in \mathbb{R}^n$, we have (note that $T_{-x}(x) = 0$):

$$\begin{aligned} x_n \xrightarrow{d_h} x &\iff d_h(x_n, x) \to 0 \iff d_h(T_{-x}(x_n), 0) \to 0 \\ &\iff [T_{-x}(x_n)] \to 1 \iff T_{-x}(x_n) \xrightarrow{d_e} 0 \iff x_n \xrightarrow{d_e} T_x(0) = x. \end{aligned}$$ □

The following is an immediate consequence of Theorem 2.8 (and the fact that the collections of all closed balls around 0 with respect to d_h and d_e, respectively, coincide – only radii change when switching between d_h and d_e). We skip its simple proof.

Corollary 2.9. *For each n, the hyperbolic space $H^n(\mathbb{R})$ is locally compact and connected and the metric d_h is proper (that is, all closed balls are compact) and, in particular, complete.*

3. Absolute (metric) homogeneity

Definition 3.1. A metric space (X, d) is said to be *absolutely* (*metrically*) *homogeneous* if any isometric map $f_0\colon (X_0, d) \to (X, d)$ defined on a non-empty subset X_0 of X extends to an isometry $f\colon (X, d) \to (X, d)$.

In this section, we will show that hyperbolic spaces are absolutely homogeneous. According to Theorem 7.1 (see Section 7 below), this property makes them highly exceptional among all connected locally compact metric spaces.

The following is a reformulation of Lemma 2.7.

Corollary 3.2. *For any $y \in H^n(\mathbb{R})$, the map $T_y\colon (H^n(\mathbb{R}), d_h) \to (H^n(\mathbb{R}), d_h)$ is an isometry.*

Lemma 3.3. *For a map $u\colon A \to H^n(\mathbb{R})$, where A is a subset of $H^n(\mathbb{R})$ containing the zero vector, the following conditions are equivalent:*

(i) *u is isometric (with respect to d_h) and $u(0) = 0$;*

(ii) *$\langle u(x), u(y)\rangle = \langle x, y\rangle$ for all $x, y \in A$.*

P r o o f. Assume (i) holds and fix $x, y \in A$. Then $[u(x)] = \cosh(d_h(u(x), u(0))) = \cosh(d_h(x, 0)) = [x]$. Similarly, $[u(y)] = [y]$ and thus $\langle u(x), u(y)\rangle = [u(x)]\,[u(y)] - \cosh(d_h(u(x), u(y))) = [x]\,[y] - \cosh(d_h(x, y)) = \langle x, y\rangle$. Conversely, if (ii) holds and $x, y \in A$, then $\langle u(x), u(x)\rangle = \langle x, x\rangle$ and hence $[u(x)] = [x]$ (and, similarly, $[u(y)] = [y]$) and $u(0) = 0$. But then easily $d_h(u(x), u(y)) = d_h(x, y)$ and we are done. □

Theorem 3.4. *For any n the hyperbolic space $(H^n(\mathbb{R}), d_h)$ is absolutely homogeneous.*

P r o o f. Fix an isometric map $v\colon A \to H^n(\mathbb{R})$ where A is a non-empty subset of $H^n(\mathbb{R})$. Take any $a \in A$, put $B = T_{-a}(A)$ and $u = T_{-v(a)} \circ v \circ T_a\colon B \to H^n(\mathbb{R})$, and observe that $0 \in B$, $u(0) = 0$ and u is isometric (with respect to d_h). So, the map u satisfies condition (ii) from Lemma 3.3. It is well known (and easy to show) that each such a map extends to a linear map $U\colon \mathbb{R}^n \to \mathbb{R}^n$ that corresponds (in the canonical basis of $\mathbb{R}^n$) to an orthogonal matrix. The last property means precisely that also U fulfills the equation from condition (ii) of Lemma 3.3. Hence, U is isometric with respect to d_h. Then $T_{v(a)} \circ U \circ T_{-a}$ is an isometry that extends v. □

The above proof enables us to describe all isometries of the hyperbolic space $H^n(\mathbb{R})$: all of them are of the form $u = T_a \circ U$ where a is a vector and $U\colon \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal linear map (both a and U are uniquely determined by u). In particular, the isotropy groups $\operatorname{stab}(x) = \{u \in \operatorname{Iso}(H^n(\mathbb{R}), d_h)\colon\ u(x) = x\}$ of elements $x \in H^n(\mathbb{R})$ are pairwise isomorphic and $\operatorname{stab}(0)$ is precisely the group O_n of all orthogonal linear transformations of $\mathbb{R}^n$. The group O_n also coincides with the isotropy group of the zero vector with respect to the isometry group of the n-dimensional Euclidean space. So, hyperbolic spaces are quite similar to Euclidean. However, the last spaces have many (bijective) dilations, which contrasts with hyperbolic geometry, as shown by

Theorem 3.5. *For $n > 1$, every dilation on $(H^n(\mathbb{R}), d_h)$ is an isometry. More generally, if $u\colon H^n(\mathbb{R}) \to H^m(\mathbb{R})$ is a dilation (and $n > 1$), then u is isometric.*

Proof. Assume $u\colon H^n(\mathbb{R}) \to H^m(\mathbb{R})$ satisfies $d_h(u(x), u(y)) = c d_h(x, y)$ for all $x, y \in H^n(\mathbb{R})$ and some constant $c > 0$. Our aim is to show that then $c = 1$. (That is all we need to prove, since any global isometric map on $H^n(\mathbb{R})$ is onto – its inverse map is extendable to an isometry which means that actually it is an isometry.)

Replacing u by $T_{-u(0)} \circ u$, we may and do assume that $u(0) = 0$. Then it follows from (2.5) that for any $x, y \in H^n(\mathbb{R})$:

$$\|u(x)\| = \|u(y)\| \iff \|x\| = \|y\|. \tag{3.1}$$

Further, $d_h(x, 0) = d_h(-x, 0)$ and $d_h(x, -x) = d_h(x, 0) + d_h(0, -x)$. Consequently, $d_h(u(x), 0) = d_h(u(-x), 0)$ and $d_h(u(x), u(-x)) = d_h(u(x), 0) + d_h(0, u(-x))$. So, we infer from Lemma 2.5 (and from (3.1)) that $u(-x) = -u(x)$ for all $x \in H^n(\mathbb{R})$.

Fix arbitrary two vectors $x, y \in H^n(\mathbb{R})$ such that $\langle x, y\rangle = 0$. Then

$$\cosh(d_h(x, y)) = [x]\,[y] = [x]\,[-y] = \cosh(d_h(x, -y)),$$

thus $d_h(u(x), u(y)) = d_h(u(x), u(-y)) = d_h(u(x), -u(y))$. This implies that

$$\langle u(x), u(y)\rangle = 0.$$

Now for arbitrary $t \geq 1$ choose two vectors $x, y \in H^n(\mathbb{R})$ such that $[x] = [y] = t$ and $\langle x, y\rangle = 0$. Then also $\langle u(x), u(y)\rangle = 0$ and (by (3.1)) $[u(y)] = [u(x)] = \cosh(d_h(u(x), 0)) = \cosh(c d_h(x, 0)) = \cosh(c \cosh^{-1}(t))$. It follows from the former property that $c\cosh^{-1}([x]\,[y]) = c d_h(x, y) = d_h(u(x), u(y)) = \cosh^{-1}([u(x)]\,[u(y)])$, which combined with the latter yields

$$\cosh(c\cosh^{-1}(t^2)) = (\cosh(c\cosh^{-1}(t)))^2. \tag{3.2}$$

The above equation is valid for every $t \geq 1$ only if $c = 1$. Although this is a well-known fact, for the reader's convenience we give its brief proof. Observe that

$$\cosh(c\cosh^{-1}(t^2)) = \frac{1}{2}\Big(\big(t^2 + \sqrt{t^4 - 1}\big)^c + \big(t^2 + \sqrt{t^4 - 1}\big)^{-c}\Big)$$

and

$$(\cosh(c\cosh^{-1}(t)))^2 = \frac{1}{4}\Big(\big(t + \sqrt{t^2 - 1}\big)^c + \big(t + \sqrt{t^2 - 1}\big)^{-c}\Big)^2.$$

As a consequence, $\lim\limits_{t\to\infty} \frac{\cosh(c\cosh^{-1}(t^2))}{t^{2c}} = 2^c$, whereas

$$\lim_{t\to\infty} \frac{(\cosh(c\cosh^{-1}(t)))^2}{t^{2c}} = 2^{2c-1}.$$

So, if (3.2) holds for all $t \geq 1$, then $2^c = 2^{2c-1}$ and $c = 1$. $\square$

We underline that the claim of Theorem 3.5 is false for $n = 1$ (see Corollary 5.2 below).

As an immediate consequence of the above result, we obtain:

Corollary 3.6. *For any $n > 1$, the metric spaces $(H^n(\mathbb{R}), d_h)$ and $(\mathbb{R}^n, d_e)$ are non-isometric.*

4. "Hyperbolic" measure

For the purposes of this section, let us call a positive Borel measure on a metric space (M, d_M) an *iso-measure* if it is finite on compact sets and invariant under each isometry of (M, d_M). (Recall that a Borel measure μ on a topological space X is invariant under a map $u\colon X \to X$ if $\mu(u^{-1}(A)) = \mu(A)$ for any Borel set $A \subset X$.)

The Lebesgue measure on $\mathbb{R}^n$ can be characterised in many different ways. One of them is related to metric homogeneity, namely: the Lebesgue measure is a unique (up to a scalar multiple) iso-measure on $(\mathbb{R}^n, d_e)$. Actually, this property is a very special case of an old result due to Loomis [11, 12], applicable also to hyperbolic spaces. It asserts that every (non-empty) metrically homogenenous metric space in which all closed balls are compact admits a unique (up to a multiple constant) iso-measure. The aim of this short section is to find an iso-measure on a hyperbolic space. It can be considered as a "hyperbolic" counterpart of the Lebesgue measure. Being unique, it is a natural "ingredient" of the hyperbolic "world."

Theorem 4.1. *The iso-measure m_h on the hyperbolic space $H^n(\mathbb{R})$ and the n-dimensional Lebesgue measure are mutually absolutely continuous. More precisely, for any Borel set $A \subset \mathbb{R}^n$:*

$$m_h(A) = \int\limits_A \frac{\mathrm{d}x}{[x]}, \tag{4.1}$$

where the right-hand side integral is computed with respect to the Lebesgue measure on $\mathbb{R}^n$.

Proof. We only need to check that the measure m_h defined by (4.1) is invariant under any isometry u of $H^n(\mathbb{R})$. To this end, write $u = T_a \circ U$ where $U \in O_n$ and $a \in H^n(\mathbb{R})$. We will show separately the invariance under U and T_a, applying the change-of-variables theorem for the Lebesgue integral.

For the orthogonal matrix U this is quite easy: substituting $y = U(x)$, we get:

$$m_h(U^{-1}(A)) = \int\limits_{U^{-1}(A)} \frac{\mathrm{d}x}{[x]} = \int\limits_A \frac{\mathrm{d}y}{[U^{-1}(y)]} = m_h(A),$$

since $[U^{-1}(y)] = [y]$. For T_a, we set $y = T_a(x)$ and use the following two properties:

- $T_a^{-1} = T_{-a}$;
- the Jacobian of T_{-a} at y equals $1 - \left\langle \frac{y}{[y]} - \frac{a}{1+[a]}, a \right\rangle$ (and is positive).

The latter property easily follows from the fact that the derivative of T_{-a} at y is a linear map of the form

$$z \mapsto z - \left\langle z, \frac{y}{[y]} - \frac{a}{1+[a]} \right\rangle a.$$

So, by (2.4):

$$m_h(T_a^{-1}(A)) = \int\limits_A \frac{1 - \left\langle \frac{y}{[y]} - \frac{a}{1+[a]}, a \right\rangle}{[T_{-a}(y)]} \mathrm{d}y = \int\limits_A \frac{\frac{1+[a]+\|a\|^2}{1+[a]} - \frac{\langle a,y \rangle}{[y]}}{[a][y] - \langle a, y \rangle} \mathrm{d}y$$

$$= \int\limits_A \frac{[a] - \frac{\langle a,y \rangle}{[y]}}{[a][y] - \langle a, y \rangle} \mathrm{d}y = \int\limits_A \frac{\mathrm{d}y}{[y]} = m_h(A).$$

$\square$

Denoting by $\bar{B}_\varrho(a,r)$ the closed ball around a of radius r with respect to a metric ϱ, we have (of course) $m(\bar{B}_{d_e}(a,r)) = \alpha \int_0^r t^{n-1}\,\mathrm{d}t$ where m stands for the Lebesgue measure on $\mathbb{R}^n$ and α is a constant that depends only on the dimension n of the space $\mathbb{R}^n$. A counterpart of this formula for hyperbolic spaces is formulated in the next result and reveals another connection with hyperbolic functions (first such a connection is exposed in the formula for d_h).

Proposition 4.2. *For any $n > 0$, $a \in H^n(\mathbb{R})$ and $R > 0$:*

$$m_h(\bar{B}_{d_h}(a,R)) = \gamma \int_0^R \sinh^{n-1}(t)\,\mathrm{d}t$$

where γ is a constant that depends only on n.

Proof. For simplicity, set $F(R) = m_h(\bar{B}_{d_h}(a,R))$. By metric homogeneity of $H^n(\mathbb{R})$ and invariance of m_h under any isometry, we have $F(R) = m_h(\bar{B}_{d_h}(0,R))$. Observe that $\bar{B}_{d_h}(0,R) = \bar{B}_{d_e}(0,\sinh(R))$. So, $F(R) = \int_{\bar{B}_{d_e}(0,\sinh(R))} \frac{\mathrm{d}x}{[x]}$. When $n > 1$, a usage of the hyperspherical variables transforms this integral to

$$\int_{[0,\pi]^{n-2}\times[0,2\pi]} \left(\int_0^{\sinh(R)} \frac{r^{n-1}\sin^{n-2}(\varphi_1)\cdot\ldots\cdot\sin^0(\varphi_{n-1})}{\sqrt{1+r^2}}\,\mathrm{d}r \right) \mathrm{d}\varphi_1 \ldots \mathrm{d}\varphi_{n-1},$$

and hence $F(R) = \gamma \int_0^{\sinh(R)} \frac{r^{n-1}}{\sqrt{1+r^2}}\,\mathrm{d}r$ (this formula is correct also for $n = 1$, with $\gamma = 2$). Now to finish the proof it remains to substitute $r = \sinh(t)$ where $0 \leq t \leq R$ and notice that then $\mathrm{d}r = \cosh(t)\,\mathrm{d}t$ and $\cosh(t) = \sqrt{1+\sinh^2(t)}$. □

The proof presented above shows that for $n = 1$, $m_h(\bar{B}_{d_h}(a,R)) = 2R$ and hence the one-dimensional measure m_h coincides with the one-dimensional Lebesgue measure after an isometric identification of $(H^1(\mathbb{R}), d_h)$ and $(\mathbb{R}, d_e)$ (cf. Corollary 5.2 below).

5. Straight lines in hyperbolic spaces

In this section, we study in detail counterparts of straight lines in hyperbolic spaces. Our final result here (Theorem 5.4 below) is a negation of a possible interpretation of Euclid's fifth postulate. The property formulated in that result made hyperbolic geometry iconic. We discuss all the axioms of a contemporary geometry in the context of the hyperbolic plane in the next section.

Recall that a *straight line* in a metric space is an isometric image of the real line, and a *straight line segment* is an isometric image of the compact interval in $\mathbb{R}$. Additionally, for simplicity, we call three points a, b, c in a metric space (X, d) *metrically collinear* if for some x, y, z with $\{x, y, z\} = \{a, b, c\}$ we have $d(x,z) = d(x,y) + d(y,z)$.

To avoid confusions, straight lines in $(\mathbb{R}^n, d_e)$ (that is, one-dimensional affine subspaces) will be called *Euclidean* lines, whereas straight lines in $(H^n(\mathbb{R}), d_h)$ will be called *hyperbolic* lines. We will use analogous names for other geometric notions.

Theorem 5.1. *Let a and b be two distinct points of $H^n(\mathbb{R})$.*

(A) *The metric segment* $I(a,b) = \{x \in H^n(\mathbb{R})\colon\ d_h(a,b) = d_h(a,x) + d_h(x,b)\}$ *is a unique straight line segment in* $H^n(\mathbb{R})$ *that joins* a *and* b.

(B) *The set* $L(a,b)$ *of all* $x \in H^n(\mathbb{R})$ *such that* x, a, b *are metrically collinear is a unique hyperbolic line passing through* a *and* b.

(C) *Every isometric map* $\gamma\colon \mathbb{R} \to H^n(\mathbb{R})$ *that sends* 0 *to* a *is of the form* $\gamma(t) = T_a(\sinh(t)z)$ *where* $z \in H^n(\mathbb{R})$ *is such that* $\|z\| = 1$.

Proof. First assume $a = 0$. All the assertions of the theorem in this case will be shown in a few steps.

For any $z \in H^n(\mathbb{R})$ with $\|z\| = 1$ denote by $\gamma_z\colon \mathbb{R} \to H^n(\mathbb{R})$ a map given by $\gamma_z(t) = \sinh(t)z$. This map is isometric, which can be shown by a straightforward calculation:

$$\begin{aligned} d_h(\gamma(s),\gamma(t)) &= \cosh^{-1}([\sinh(s)z]\,[\sinh(t)z] - \sinh(s)\sinh(t)) \\ &= \cosh^{-1}\big(\sqrt{(\sinh(s)^2+1)(\sinh(t)^2+1)} - \sinh(s)\sinh(t)\big) \\ &= \cosh^{-1}(\cosh(s)\cosh(t) - \sinh(s)\sinh(t)) \\ &= \cosh^{-1}(\cosh(s-t)) = |s-t|. \end{aligned}$$

Observe that the image of γ_z coincides with the linear span of the vector z. Thus, every Euclidean line passing through the zero vector is also a hyperbolic line.

Now let $x, y, z \in H^n(\mathbb{R})$ satisfy $d_h(x,z) = d_h(x,y) + d_h(y,z)$ and let $0 \in \{x,y,z\}$. We claim that x, y, z lie on a Euclidean line. Indeed, if $y = 0$, it suffices to apply Lemma 2.5; and otherwise, we may assume, without loss of generality, that $x = 0$. In that case, we proceed as follows. The linear span L of y is a hyperbolic line (by the previous paragraph). So, there exists an isometry $q\colon L \to L$ that sends y to 0. By absolute homogeneity established in Theorem 3.4, there exists an isometry $Q\colon H^n(\mathbb{R}) \to H^n(\mathbb{R})$ that extends q. So, $Q(L) = L$, $Q(y) = 0$ and $d_h(Q(x),Q(z)) = d_h(Q(x),Q(y)) + d_h(Q(y),Q(z))$. Again, Lemma 2.5 implies that $Q(z) = tQ(x)$ for some $t \leq 0$ (as $Q(x) \neq 0 = Q(y)$). But $Q(x) = Q(0) \in Q(L) = L$ and therefore also $Q(z) \in L$. So, $x, y, z \in L$ and we are done.

Now we prove items (A)–(C) in the case $a = 0$. Let $z \in H^n(\mathbb{R})$ be a vector such that $\|z\| = 1$ and $b \in \gamma_z(\mathbb{R})$. It follows from the last paragraph that each element $x \in H^n(\mathbb{R})$ such that $x, 0, b$ are metrically collinear belongs to $\gamma_z(\mathbb{R})$. We also know that $\gamma_z(\mathbb{R})$ is a hyperbolic line. This shows that $L(0,b) = \gamma_z(\mathbb{R})$ and proves (B), from which (A) easily follows. Finally, if $\gamma\colon \mathbb{R} \to H^n(\mathbb{R})$ is an isometric map such that $\gamma(0) = 0$, then for any $t \in \mathbb{R}$, the points 0, $\gamma(1)$ and $\gamma(t)$ are metrically collinear, so $\gamma(t) \in L(0,\gamma(1))$. It follows from the above argument that $L(0,\gamma(1)) = \gamma_z(\mathbb{R})$ for some unit vector $z \in H^n(\mathbb{R})$. Then $v = \gamma_z^{-1} \circ \gamma\colon \mathbb{R} \to \mathbb{R}$ is a Euclidean isometry sending 0 to 0. Thus $v(t) = t$ or $v(t) = -t$. In the former case, we get $\gamma = \gamma_z$, whereas in the latter we have $\gamma = \gamma_{-z}$, which finishes the proof of (C).

Now we consider a general case. When a is arbitrary, $b \neq a$ and $\gamma\colon \mathbb{R} \to H^n(\mathbb{R})$ is isometric and sends 0 to a, it is easy to verify that $T_{-a}(I(a,b)) = I(0, T_{-a}(b))$, $T_{-a}(L(a,b)) = L(0, T_{-a}(b))$ and $T_{-a} \circ \gamma$ is an isometric map from $\mathbb{R}$ into $H^n(\mathbb{R})$ that sends 0 to 0. So, the first part of the proof implies that $I(a,b)$ and $L(a,b)$ are a unique straight line segment joining a and b, and – respectively – a unique hyperbolic line passing through a and b. Similarly, $T_{-a} \circ \gamma = \gamma_z$ for some unit vector z. Then $\gamma = T_a \circ \gamma_z$ and we are done. □

It follows from the above result that two hyperbolic lines either are disjoint or have a single common point, or coincide.

As a consequence of Theorem 5.1, we obtain the following result, which is at least surprising when one compares the formulas for d_h and d_e.

Corollary 5.2. *The map* $(\mathbb{R}, d_e) \ni t \mapsto \sinh(t) \in (H^1(\mathbb{R}), d_h)$ *is an isometry.*

The above result implies, in particular, that $H^1(\mathbb{R})$ admits many non-isometric dilations. Its assertion is the reason for considering (in almost whole existing literature) only hyperbolic spaces of dimension greater than one.

Another immediate consequence of Theorem 5.1 reads as follows.

Corollary 5.3. *Hyperbolic spaces are geodesic.*

The next result may be named *Failure of the Classical Euclid's Fifth Postulate.*

Theorem 5.4. *Let* $n > 1$, L *be a hyperbolic line in* $H^n(\mathbb{R})$ *and* $a \notin L$. *There are infinitely many hyperbolic lines that pass through* a *and are disjoint from* L.

Proof. Thanks to the metric homogeneity of $H^n(\mathbb{R})$, we may and do assume that $a = 0$. So, L is a hyperbolic line that does not pass through 0. We claim that then there exist two linearly independent vectors a and b such that

$$L = \{\sinh(t)a + \cosh(t)b\colon\ t \in \mathbb{R}\}. \tag{5.1}$$

Indeed, we know that $L = \gamma(\mathbb{R})$ where $\gamma = T_y \circ \gamma_z$, $y \neq 0$, $\|z\| = 1$ and γ_z is given by $\gamma_z(t) = \sinh(t)z$. Then $\gamma(t) = \sinh(t)z + ([\sinh(t)z] + \frac{\sinh(t)\langle z,y\rangle}{[y]+1})y = \sinh(t)(z + \frac{\langle z,y\rangle}{[y]+1}y) + \cosh(t)y$. So, substituting $a = z + \frac{\langle z,y\rangle}{[y]+1}y$ and $b = y$, it remains to check that a and b are linearly independent to get (5.1). If a and b were not such, then L would be a subset of the linear span Y of y. But Y is a hyperbolic line and therefore we would obtain that $L = Y$ and hence $0 \in L$. This proves (5.1).

Now let $\mu \in \mathbb{R}$ be such that $|\mu| > 1$. Put $c = \mu a + b$ and let L_μ be the linear span of c. We claim that L_μ is disjoint from L. Indeed, let s and t be real and assume, on the contrary, that $sc = \sinh(t)a + \cosh(t)b$ (cf. (5.1)). It follows from the linear independence of a and b that $\sinh(t) = s\mu$ and $\cosh(t) = s$. Consequently, $|\sinh(t)| = |\cosh(t)\mu| > |\cosh(t)|$, which is impossible.

So, for any real μ with $|\mu| > 1$ the set L_μ is a hyperbolic line disjoint from L and passing through 0. Since a and b are linearly independent, these sets L_μ are all different. □

The following theorem shows one more non-Euclidean property of the discussed space.

Theorem 5.5. *For any two hyperbolic lines in* $H^2(\mathbb{R})$ *there exists a hyperbolic line disjoint from both of them.*

Proof. Observe that for two orthogonal vectors a and b, the hyperbolic lines $T_a(\mathbb{R}b)$ and $T_{-a}(\mathbb{R}b)$ are contained in different components of $\mathbb{R}^2 \setminus \mathbb{R}b$. This, by metric homogeneity, proves that for an arbitrary hyperbolic line, each component of its complement contains a hyperbolic line.

If the given two lines K and L are disjoint (or equal), we can pick a line from the component of $\mathbb{R} \setminus K$ different than the one containing L.

Now proceed to the remaining case of the two intersecting lines. We may and do assume that they intersect at 0. Let them be the linear spans of independent unit vectors, a and b, respectively. We claim that there exist $\beta > 0$ such that $T_{-\beta(a+b)}(\mathbb{R}(a-b))$ is a hyperbolic line disjoint from $\mathbb{R}a$ and $\mathbb{R}b$.

For convenience, let us transform the whole setting by $T_{\beta(a+b)}$, obtaining hyperbolic lines $\{T_{\beta(a+b)}(ta)\colon\ t \in \mathbb{R}\}$, $\{T_{\beta(a+b)}(tb)\colon\ t \in \mathbb{R}\}$ and $\{s(a-b)\colon\ s \in \mathbb{R}\}$. We want to prove that for sufficiently large β neither $T_{\beta(a+b)}(ta) = s(a-b)$ nor $T_{\beta(a+b)}(tb) = s(a-b)$, regardless of the values of real parameters s and t. Here we focus only on the former equation (the latter can be treated analogously). Using the explicit formula for the isometry, we obtain (in the former equation):

$$ta + \left([ta] + \frac{\langle ta, \beta(a+b)\rangle}{[\beta(a+b)]+1}\right)\beta(a+b) = s(a-b).$$

Using linear independence of a and b, we obtain a system of two equations:

$$\begin{cases} t + \beta\sqrt{t^2+1} + t\dfrac{\beta^2\langle a, a+b\rangle}{[\beta(a+b)]+1} & = s \\ \beta\sqrt{t^2+1} + t\dfrac{\beta^2\langle a, a+b\rangle}{[\beta(a+b)]+1} & = -s \end{cases}.$$

Adding the equations, we obtain

$$t + 2\beta\sqrt{t^2+1} + t\frac{2\beta^2\langle a, a+b\rangle}{[\beta(a+b)]+1} = 0. \tag{5.2}$$

Since a and b are unit vectors, $2\langle a, a+b\rangle = \|a+b\|^2$. Expanding $[\beta(a+b)]$ and simplifying, we arrive at

$$2\beta\sqrt{t^2+1} = -t[\beta(a+b)]. \tag{5.3}$$

Observe that the above equation is fully symmetric with respect to a and b.

Consider the asymptotic behaviour of the $[\,\cdot\,]$ function:

$$\lim_{\beta\to\infty}\frac{[\beta(a+b)]}{\beta\|a+b\|} = 1.$$

Since $\|a+b\| < 2$, for sufficiently large parameter β, $2\beta > [\beta(a+b)]$. Then $|-t[\beta(a+b)]| < 2\beta|t| < 2\beta\sqrt{t^2+1}$, hence the equation (5.3) holds for no t. $\square$

The reader interested in establishing further geometric properties by means of the metric is referred to [4].

6. Tarski's axioms of the plane geometry

While the Euclidean geometry for centuries used to be a canonical example of deductive approach, the foundations given by Euclid came out to be not fully sufficient from the viewpoint of mathematical logic. The first successful system of axioms for the geometry was completed by David Hilbert in 1899. We will discuss a different famous system, invented by Alfred Tarski, and prove how its statements are satisfied in our model. Since hyperbolic geometry is non-Euclidean not all the axioms would hold.

Tarski's axioms of geometry base on a universe of points and two relations (there is also the relation "=" of identity, but it is more a part of the logic framework than an element of the constructed geometry): a quaternary relation of *equidistance* denoted by $ab \equiv cd$ and a ternary relation of *betweenness* which is usually denoted as $B(abc)$. The narrow set of primitive notions is one of the most significant advantages of Tarski's approach.

The initial list of axioms presented in 1926 consisted of twenty statements and one axiom schema. Over the years, several axioms were shown to be derivable. Below we present the list of eleven axioms (the axiom 11 is not a first-order sentence; it can be replaced by an axiom schema while it is out of our interest in this paper) sufficient for Tarski's system of Euclidean geometry. The order coincides with presented in [15] and [16].

Ax. 1.: (Reflexivity of equidistance) For any points a, b,
$$ab \equiv ba.$$
Ax. 2.: (Transitivity of equidistance) For any points a, b, c, d, e, f,
$$(ab \equiv cd \wedge ab \equiv ef) \implies cd \equiv ef.$$
Ax. 3.: (Identity for equidistance) For any points a, b, c,
$$ab \equiv cc \implies a = b.$$
Ax. 4.: (Segment Construction) For any points a, b, c, q,
$$\exists x \ (B(qax) \wedge ax \equiv bc).$$
Ax. 5.: (Five-Segment Axiom) For any points $a, b, c, d, a', b', c', d'$,
$$\Big(a \neq b \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ac \equiv a'c' \\ \wedge bd \equiv b'd' \wedge B(abd) \wedge B(a'b'd')\Big) \implies cd \equiv c'd'.$$
Ax. 6.: (Identity for betweenness) For any points a, b,
$$B(aba) \implies a = b.$$
Ax. 7.: (Pasch Axiom) For any points a, b, c, p, q,
$$B(apb) \wedge B(aqc) \implies \exists x \ B(cxp) \wedge B(bxq).$$
Ax. 8.: (Lower bound for dimension) There exist three non-collinear points.

Ax. 9.: (Upper bound for dimension) For any points a, b, p, q, r,
$$(a \neq b \wedge ap \equiv pb \wedge aq \equiv qb \wedge ar \equiv rb) \implies (B(pqr) \vee B(qrp) \vee B(rpq)).$$
Ax. 10.: (Euclid's Axiom) For any points a, b, c, d, t,
$$(a \neq d \wedge B(adt) \wedge B(bdc)) \implies \exists x, y \ (B(abx) \wedge B(acy) \wedge B(xty)).$$
Ax. 11.: (Axiom of Continuity) For any sets X and Y of points,
$$(\exists a \ \forall x \in X \ \forall y \in Y \ B(axy)) \implies (\exists b \ \forall x \in X \ \forall y \in Y \ B(xby)).$$

Axioms 8 and 9 are specific for two-dimensional geometry. To determine higher dimension different statements are required. Examples of such axioms can be found e.g. in [10: Remark 4] and [15: p. 23]. Note that the form of higher dimension axioms presented in [16] (and cited in several places) is incorrect. The four-dimensional Euclidean space satisfies 'Ax. $8^{(n)}$' (formulated therein) for every $n \geq 2$.

For further remarks on the system, its history and importance the reader is referred to [16]. A systematic development of Euclidean geometry based on Tarski's axioms can be found in the book [15].

Definition 6.1. In the hyperbolic space $(H^n(\mathbb{R}), d_h)$, we define relations of equidistance and betweenness in the following way $(a, b, c, d \in H^n(\mathbb{R}))$:
$$\begin{aligned} ab \equiv cd \quad &\iff \quad d_h(a, b) = d_h(c, d), \\ B(abc) \quad &\iff \quad d_h(a, b) + d_h(b, c) = d_h(a, c) \quad \iff \quad b \in I(a, c). \end{aligned}$$

Observe that all the relations are strictly preserved by isometries, that means for any formula $\phi(p_1, \ldots, p_k)$ of k free variables and fixed points $x_1, \ldots, x_k$, the following are equivalent

(i) $\phi(x_1, \ldots, x_k)$,

(ii) $\exists_{\Phi \in \mathrm{Iso}(H^n(\mathbb{R}), d_e)} \ \phi(\Phi(x_1), \ldots, \Phi(x_k))$,

(iii) $\forall_{\Phi \in \mathrm{Iso}(H^n(\mathbb{R}), d_e)}\ \phi(\Phi(x_1), \ldots, \Phi(x_k))$.

We will show that with elementary relations introduced as above, our model of hyperbolic geometry satisfies all of the Tarski's axioms except for the Euclid's Axiom, what shall be expected. We focus on the two dimensional case (we have listed Tarski's axioms particularly for dimension 2), while most of the proofs apply to the n-dimensional case.

Axioms 1–3 are obvious consequences of the general properties of a distance function.

Ad 4. As we mentioned before, we can reduce to the case when $a = 0$ (using the isometry T_{-a}). Then it suffices to take $x = -q/\|q\| \sinh(d_h(b, c))$, for $q \neq 0$ (cf. item (C) in Theorem 5.1). If $q = 0 = a$, an x suitable for arbitrary $q \neq 0$ would be appropriate.

Ad 5. The Fixe-Segment Axiom follows from the metric 3-homogeneity. Under its assumptions, we can find a global isometry Φ such that $\Phi(a) = a'$, $\Phi(b) = b'$ and $\Phi(c) = c'$. If then $\Phi(d) = d'$, the assertion of the axiom is satisfied. But since $a \neq b$, the isometry Φ is uniquely determined on the line $L(a, b)$, hence indeed $\Phi(d) = d'$.

Ad 6. By the definition of betweenness, $2d_h(a, b) = d_h(a, a) = 0$ and then $a = b$.

Ad 7. Since any line on a hyperbolic plane $(H^2(\mathbb{R}), d_h)$ can be mapped onto a one-dimensional vector subspace by an isometry of the plane, its complement has two connected components.

Lemma 6.2. *Let L be a line in $(H^2(\mathbb{R}), d_h)$ and let C_1 and C_2 be the two connected components of $\mathbb{R}^2 \setminus L$. Fix points $x \in L$, $y \in C_1$ and $z \in \mathbb{R}^2$. Then*

(1) *z belongs to $C_1 \cup L$ provided that $B(xyz)$,*

(2) *z does not belong to C_1 provided that $B(yxz)$.*

Proof. By metric homogeneity, we may reduce to the case when $x = (0, 0)$ and $L = \{(t, 0) \colon t \in \mathbb{R}\}$. Since hyperbolic lines passing through 0 are simply vector subspaces, both cases are obvious. □

To prove that the Pasch Axiom is satisfied in $(H^2(\mathbb{R}), d_h)$, we observe that b is not in the same component of $\mathbb{R}^2 \setminus L(p, c)$ as a and q. (One can easily verify the degenerated cases when [some of] those points lie on $L(p, c)$.) Therefore $I(b, q)$ has to intersect $L(p, c)$ in a (unique) point, say x.

The same reasoning applied to c, $L(q, b)$, a and p leads to the conclusion that $I(c, p)$ intersects $L(b, q)$:

$$I(c, p) \cap L(b, q) = L(c, p) \cap L(b, q) = L(c, p) \cap I(c, p) = I(c, p) \cap I(b, q) = \{x\}.$$

Observe that the setting of the Pasch Axiom is contained within two intersecting lines, and therefore can be assumed to be contained in $\mathbb{R}^2 \times \{0\}^{n-2} \subset H^n(\mathbb{R})$, which is naturally isometric with $(H^2(\mathbb{R}), d_h)$. Hence the axiom is satisfied in higher dimensions as well.

Ad 8. Axiom 8 is witnessed e.g. by the triple $(0, 0)$, $(0, 1)$ and $(1, 0)$.

Ad 9. By an isometric transformation, we can reduce the issue to the case when $b = -a$. Observe that $2\langle x, a\rangle = \cosh(d_h(x, -a)) - \cosh(d_h(x, a))$, for any $x \in \mathbb{R}^n$. Hence all the points equidistant from a and $-a$ are vectors perpendicular to a. In $H^2(\mathbb{R})$ the set of all such points is a straight line and the axiom 9 follows.

Ad 10. That the Euclid's Axiom is violated in the hyperbolic plane, can be easily deduced from Theorem 5.4 or 5.5.

We pick two lines, K and L, intersecting in a point a. Then – thanks to Theorem 5.5 – we find a third line, M, disjoint from K and L. Then we pick an arbitrary point t on M and points b, c, d (with $b \in K$ and $c \in L$ close to a, and $d \in I(a, t) \cap I(b, c)$) to complete the setting of the axiom.

Points x and y from the assertion cannot be found, since one of them has to belong to the other component of $\mathbb{R}^2 \setminus M$ than $K \cup L$ (by Lemma 6.2).

Ad 11. The continuity axiom is trivial if one of the sets consists of less than two points. In the other case both sets have to be contained in a hyperbolic line which is isometric to $\mathbb{R}$. On the real line the assertion is satisfied.

7. Absolute vs. 3-point homogeneity

Looking at a complicated formula for the hyperbolic metric, it is a natural temptation to search for a 'simpler' realisation of non-Euclidean (hyperbolic) geometry. Even if it is possible, a new model will not be as 'perfect' as the hyperbolic space described in this paper. More precisely, if a certain metric space (X, d) non-homothetic to $H^n(\mathbb{R})$ defined in Section 2 was used as a model for the hyperbolic geometry, then d would not be geodesic or else (X, d) would not be 3-point homogeneous. (Recall that, for a positive integer n, a metric space (X, d) is metrically *n-point homogeneous* if any isometric map $u\colon (A, d) \to (X, d)$ defined on a subset A of X that has at most n elements is extendable to an isometry $v\colon (X, d) \to (X, d)$.) The above statement is a consequence of deep achievements (which we neither discuss here in full details nor give their proofs) of the 50's of the 20th century due to Wang [18], Tits [17] and Freudenthal [8]. They classified *all* connected locally compact metric spaces that are 2-point homogeneous. All that follows is based on Freudenthal's work [8]; we also strongly recommend his paper [9] where main results of the former article are well discussed (see, e.g., Subsection 2.21 therein).

To formulate the main result on the classification, **up to isometry**, of all connected locally compact 3-point homogeneous metric spaces, let us introduce necessary notions.

For any positive integer n let $\mathbb{S}^n$ stand for the Euclidean unit sphere in $\mathbb{R}^{n+1}$:

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1}\colon\ \|x\| = 1\}.$$

We equip $\mathbb{S}^n$ with metric d_s of the *great-circle distance*, given by

$$d_s(x, y) = \frac{1}{\pi} \arccos(\langle x, y\rangle) = \frac{1}{\pi} \arccos \frac{2 - d_e(x, y)^2}{2}.$$

It is well-known (and easy to prove) that d_s is a metric equivalent to d_e (restricted to $\mathbb{S}^n$) such that the metric space $(\mathbb{S}^n, d_s)$ has the following properties:

- it is geodesic and absolutely homogeneous;
- it has diameter 1;
- any two points of $\mathbb{S}^n$ whose d_s-distance is smaller than 1 (that is, which are not antipodal) can be joint by a **unique** straight line segment.

Further, for any subinterval I of $[0, \infty)$ containing 0 let $\Omega(I)$ be the set of all continuous functions $\omega\colon I \to [0, \infty)$ that vanish at 0 and satisfy the following two conditions for all $x, y \in I$:

(ω1) $x < y \implies \omega(x) < \omega(y)$;

(ω2) $\omega(x + y) \leq \omega(x) + \omega(y)$ provided $x + y \in I$.

Finally, let $\Omega = \Omega_\infty = \Omega([0, \infty))$ and $\Omega_1 = \Omega([0, 1])$. For any $\omega \in \Omega$ there exists $\lim_{t\to\infty} \omega(t) \in [0, \infty]$, which will be denoted by $\omega(\infty)$.

After all these preparations, we are ready to formulate the main result of this section.

Theorem 7.1. *Each connected locally compact* 3*-point homogeneous metric space having more than one point is isometric to exactly one metric space* (X,d) *among all listed below (everywhere below* n *denotes a positive integer).*

- $X = \mathbb{R}^n$, $d = \omega \circ d_e$ *where* $\omega \in \Omega$ *and* $\omega(1) = \min(1, \frac{1}{2}\omega(\infty))$.
- $X = \mathbb{S}^n$, $d = \omega \circ d_s$ *where* $\omega \in \Omega_1$.
- $X = H^n(\mathbb{R})$, $d = \omega \circ d_h$ *where* $n > 1$ *and* $\omega \in \Omega$.

In particular:

- *all connected locally compact* 3*-point homogeneous metric spaces are absolutely homogeneous, and each of them is homeomorphic either to a Euclidean space or to a Euclidean sphere (unless it has at most one point);*
- *each locally compact geodesic* 3*-point homogeneous metric space having more than one point is isometric to exactly one of the spaces:* $(\mathbb{R}^n, d_e)$, $(\mathbb{S}^n, rd_s)$ *(where* $r > 0$*),* $(H^n(\mathbb{R}), rd_h)$ *(where* $n > 1$ *and* $r > 0$*).*

Remark 7.2. Busemann's [5] and Birkhoff's [3, 4] results have a similar spirit. However, both of them assume, besides metric convexity, a sort of uniqueness of straight line segments joining sufficiently close two points, and their statements are less general.

It seems that the assertion of Theorem 7.1 (in this form) has never appeared in the literature. Nevertheless, we consider this result as Freudenthal's theorem – not ours. Such a thinking is justified by the fact that Theorem 7.1 easily follows from Freudenthal's theorem, stated below, proved in [8].

Denoting by $P^n(\mathbb{R})$ the n-dimensional real projective space (realised as the quotient space of $\mathbb{S}^n$ obtained by gluing antipodal points) equipped with a geodesic metric $d_p(\bar{u}, \bar{v}) = \frac{2}{\pi} \arccos(|\langle u, v\rangle|)$ (where $u, v \in \mathbb{S}^n$ and $\bar{u} = \{u, -u\}$ and $\bar{v} = \{v, -v\}$ denote their equivalence classes belonging to $P^n(\mathbb{R})$), we can formulate Freudenthal's *Hauptsatz IV* from [8] as follows.

Theorem 7.3. *Let* (Z, λ) *be a connected locally compact metric space that satisfies the following condition:*

> *There are two positive reals* γ *and* γ' *such that:*
> (a) $\gamma < \lambda(v, w)$ *for some* $v, w \in Z$;
> (b) $\gamma' < \lambda(a, b)$ *for some* $a, b \in Z$ *for which there is* $c \in Z$ *with* $\lambda(a, c) = \lambda(b, c) = \gamma$;
> (c) *any isometric map* $u\colon (A, \lambda) \to (Z, \lambda)$ *defined on an arbitrary subset* A *of* Z *of the form:*
> - $A = \{x, y\}$ *where* $\lambda(x, y) = \gamma$, *or*
> - $A = \{x, y, z\}$ *where* $\lambda(x, y) = \lambda(y, z) = \gamma$ *and* $\lambda(x, z) = \gamma'$
>
> *is extendable to an isometry* $v\colon (Z, \lambda) \to (Z, \lambda)$.

Then there is a unique space (M, ϱ) *among* $(\mathbb{R}^n, d_e)$ $(n > 0)$, $(\mathbb{S}^n, d_s)$ $(n > 0)$, $(H^n(\mathbb{R}), d_h)$ $(n > 1)$, $(P^n(\mathbb{R}), d_p)$ $(n > 1)$ *and a homeomorphism* $h\colon Z \to M$ *such that:*

$$\{h \circ u \circ h^{-1}\colon\ u \in \mathrm{Iso}(Z, \lambda)\} = \mathrm{Iso}(M, \varrho). \tag{7.1}$$

We prove Theorem 7.1 in few steps, each formulated as a separate lemma. All of them are already known but – for the reader's convenience – we give their short proofs.

Lemma 7.4. *For $n > 1$ the space $(P^n(\mathbb{R}), d_p)$ is 2-point, but not 3-point, homogeneous.*

Proof. We use here the notation introduced in the paragraph preceding the formulation of Theorem 7.3.

Let (u, v) and (x, y) be two pairs of points of $\mathbb{S}^n$ such that $d_p(\bar{u}, \bar{v}) = d_p(\bar{x}, \bar{y})$. This means that there is $\varepsilon \in \{1, -1\}$ such that $d_s(u, v) = d_s(x, \varepsilon y)$. Consequently, there exists $A \in O_{n+1}$ (which is automatically an isometry with respect to d_s) such that $A(u) = x$ and $A(v) = \varepsilon y$. Every member B of O_{n+1} naturally induces an isometry $\bar{B}\colon P^n(\mathbb{R}) \to P^n(\mathbb{R})$ that satisfies $\bar{B}(\bar{z}) = \overline{B(z)}$ for any $z \in \mathbb{S}^n$. Moreover, the assignment

$$O_{n+1} \ni B \mapsto \bar{B} \in \operatorname{Iso}(P^n(\mathbb{R}), d_p) \tag{7.2}$$

is surjective (this is classical, but non-trivial). So, $\bar{A} \in \operatorname{Iso}(P^n(\mathbb{R}), d_p)$ satisfies $\bar{A}(\bar{u}) = \bar{x}$ and $\bar{A}(\bar{v}) = \bar{y}$, which shows that $P^n(\mathbb{R})$ is 2-point homogeneous.

To convince oneself that $P^n(\mathbb{R})$ is not 3-point homogeneous for $n > 1$, it is enough to consider two triples (x, y, z_1) and (x, y, z_2) of points of $\mathbb{S}^n$ with $x = (1, 0, 0, \vec{0}\,)$, $y = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, \vec{0}\,)$, $z_1 = (\frac{1}{4}, \frac{1}{4}, \frac{\sqrt{14}}{4}, \vec{0}\,)$ and $z_2 = (\frac{1}{4}, -\frac{3}{4}, \frac{\sqrt{6}}{4}, \vec{0}\,)$ where $\vec{0}$ denotes the zero vector in $\mathbb{R}^{n-2}$ (which has to be omitted when $n = 2$). Observe that $\langle x, z_1 \rangle = \langle x, z_2 \rangle$ and $\langle y, z_1 \rangle = -\langle y, z_2 \rangle$. Consequently, $d_p(\bar{w}, \bar{z}_1) = d_p(\bar{w}, \bar{z}_2)$ for any $w \in \{x, y\}$. If $P^n(\mathbb{R})$ was 3-point homogeneous, there would exist an isometry of $P^n(\mathbb{R})$ that fixes x and y and sends z_1 onto z_2, which is impossible. To see this, assume – on the contrary – there exists such an isometry. It then follows from the surjectivity of (7.2) that there exists $B \in O_{n+1}$ such that $B(x) = \pm x$, $B(y) = \pm y$ and $B(z_1) = \pm z_2$. Then, replacing if needed B by $-B$, we may and do assume $B(x) = x$ and thus, since $\langle B(x), B(u) \rangle = \langle x, u \rangle$ for any $u \in \mathbb{R}^{n+1}$, also $B(y) = y$ and $B(z_1) = z_2$. But then $\langle y, z_2 \rangle = \langle B(y), B(z_1) \rangle = \langle y, z_1 \rangle$ which is false. □

Lemma 7.5. *Let (M, ϱ) be a 2-point homogeneous geodesic metric space having more than one point and I denote $\varrho(M \times M)$, and let $f\colon I \to [0, \infty)$ be a one-to-one function such that $\varrho_f = f \circ \varrho$ is a metric on M equivalent to ϱ. Then:*

(v0) *I is an interval and $f \in \Omega(I)$;*

(v1) *(M, ϱ_f) is 2-point homogeneous;*

(v2) $\operatorname{Iso}(M, \varrho_f) = \operatorname{Iso}(M, \varrho)$*;*

(v3) *(M, ϱ_f) is absolutely homogeneous iff so is (M, ϱ);*

(v4) *(M, ϱ_f) is geodesic iff $f(t) = ct$ for some positive constant c (and all $t \in I$).*

Proof. Since f is one-to-one, any map $u\colon A \to M$ (where $A \subset M$) is isometric with respect to ϱ_f if and only if it is isometric with respect to ϱ. This implies (v1), (v2) and (v3). It follows from 1-point homogeneity of (M, ϱ) that $I = \{\varrho(a, x)\colon\ x \in M\}$ where a is arbitrarily fixed element of M. But this, combined with continuity of ϱ_f with respect to ϱ, yields that f is continuous at 0. Further, since M is geodesic, I is an interval and for any $x, y \in I$ with $x + y \in I$ there are points $a, b, c \in M$ such that $\varrho(a, c) = x + y$, $\varrho(a, b) = x$ and $\varrho(b, c) = y$. Then $f(x + y) = \varrho_f(a, c) \leq \varrho_f(a, b) + \varrho_f(b, c) = f(x) + f(y)$. So, $(\omega 2)$ holds. This implies that $|\omega(x) - \omega(y)| \leq \omega(|x - y|)$ for any $x, y \in I$. Thus f, being continuous at 0, is continuous (at each point of I). Finally, a continuous one-to-one function vanishing at 0 satisfies $(\omega 1)$ and therefore $f \in \Omega(I)$.

It remains to show that if ϱ_f is geodesic, then f is linear. Since $f \in \Omega(I)$, it is a bijection between I and $J = f(I)$, and J is an interval. Moreover, $\varrho = f^{-1} \circ \varrho_f$. So, assuming that ϱ_f is geodesic, it follows from the first part of the proof that also $f^{-1} \in \Omega(J)$. But if $f \in \Omega(I)$ and $f^{-1} \in \Omega(J)$, then $f(x + y) = f(x) + f(y)$ for all $x, y \in I$ with $x + y \in I$. The last equation implies that f is linear (for f is continuous). □

Lemma 7.6. *Assume (Z,λ) and (M,ϱ) are two 2-point homogeneous metric spaces having more than one point and satisfying the following three conditions:*

- *(M,ϱ) is geodesic;*
- *there exists $R \in \{1,\infty\}$ such that $\varrho(M \times M) = \{r \in \mathbb{R}\colon\ 0 \leq r \leq R\}$;*
- *there exists a homeomorphism $h\colon Z \to M$ such that (7.1) holds.*

Then there exists a unique $\omega \in \Omega_R$ such that $\lambda = \omega \circ \varrho \circ (h \times h)$. Moreover, (Z,λ) is 3-point or absolutely homogeneous iff so is (M,ϱ).

Proof. Recall that $h \times h\colon Z \times Z \to M \times M$ is given by $(h \times h)(x,y) = (h(x),h(y))$. Further, for simplicity, denote $I = \varrho(M \times M)$. It follows from our assumptions that $I = [0,1]$ (if $R = 1$) or $I = [0,\infty)$ (if $R = \infty$).

For any four points a, b, x and y in an arbitrary 2-point homogeneous metric space (Y,p) the following equivalence holds:

$$p(a,b) = p(x,y) \iff \exists \Phi \in \operatorname{Iso}(Y,p)\colon\ \Phi(a) = x \text{ and } \Phi(b) = y.$$

The above condition, applied for both (Z,λ) and (M,ϱ), combined with (7.1) yields that

$$\lambda(a,b) = \lambda(x,y) \iff \varrho(h(a),h(b)) = \varrho(h(x),h(y)) \qquad (a,b,x,y \in Z). \tag{7.3}$$

We infer that there exists a one-to-one function $\omega\colon I \to [0,\infty)$ such that $\lambda = \omega \circ \varrho \circ (h \times h)$. Since $\omega \circ \varrho = \lambda \circ (h^{-1} \times h^{-1})$ is a metric equivalent to ϱ, we conclude from Lemma 7.5 that $\omega \in \Omega_R$. The uniqueness of ω is trivial.

Finally, the remainder of the lemma (about 3-point or absolute homogeneity) follows from (7.3). Indeed, this condition implies that a map $u\colon (A,\lambda) \to (Z,\lambda)$ (where $A \subset Z$) is isometric iff the map $h \circ u \circ h^{-1}\big|_{h(A)}\colon (h(A),\varrho) \to (M,\varrho)$ is isometric. $\square$

Proof of Theorem 7.1. First of all, since each of the spaces

$$(\mathbb{R}^n,d_e)\ (n > 0),\ (\mathbb{S}^n,d_s)\ (n > 0),\ (H^n(\mathbb{R}),d_h)\ (n > 1) \tag{7.4}$$

is absolutely homogeneous, it follows from Lemma 7.5 that any space (X,d) listed in the statement of the theorem is also absolutely homogeneous (and, of course, connected, locally compact and has more than one point). Lemma 7.5 enables us recognizing those among them that are geodesic.

Further, if (Z,λ) is a connected locally compact 3-point homogeneous metric space having more than one point, we infer from Theorem 7.3 that there are a metric space (M,ϱ) and a homeomorphism $h\colon Z \to M$ such that (7.1) holds and either (M,ϱ) is listed in (7.4) or it is $(P^n(\mathbb{R}),d_p)$ with $n > 1$. However, since (M,ϱ) is 2-point homogeneous (cf. Lemma 7.4), it follows from Lemma 7.6 that it is also 3-point homogeneous (because (Z,λ) is so). Thus, Lemma 7.4 implies that (M,ϱ) is listed in (7.4), and we conclude from Lemma 7.6 that there is $\widetilde{\omega} \in \Omega(I)$ (where $I = \varrho(M \times M)$) for which

$$\lambda = \widetilde{\omega} \circ \varrho \circ (h \times h). \tag{7.5}$$

Now if (M,ϱ) is not a Euclidean space, put $\omega = \widetilde{\omega}$ and observe that $(X,d) = (M,\omega \circ \varrho)$ is listed in the statement of the theorem and $h\colon (Z,\lambda) \to (X,d)$ is an isometry, thanks to (7.5).

In the remaining case, when $(M,\varrho) = (\mathbb{R}^n,d_e)$, we proceed as follows. There is $\alpha > 0$ such that $\widetilde{\omega}(\alpha) = \min(1,\frac{1}{2}\widetilde{\omega}(\infty))$. We define $\omega \in \Omega$ by $\omega(t) = \widetilde{\omega}(\alpha t)$. Observe that then $\omega(\infty) = \widetilde{\omega}(\infty)$ and hence $\omega(1) = \min(1,\frac{1}{2}\omega(\infty))$. So, $(X,d) = (\mathbb{R}^n,\omega \circ d_e)$ is listed in the statement of the theorem. What is more, the map

$$(Z,\lambda) \ni z \mapsto \frac{1}{\alpha}h(z) \in (X,d)$$

is an isometry, again by (7.5) (and the definition of ω).

The last thing we need to prove is that all the metric spaces (X,d) listed in the statement of the theorem are pairwise non-isometric. To this end, let (M_1,ϱ_1) and (M_2,ϱ_2) be two spaces among listed in (7.4), ω_1 and ω_2 be two functions such that $(X,d) = (M_j,\omega_j \circ \varrho_j)$ for $j \in \{1,2\}$ is listed in the statement of the theorem; and let $h\colon (M_1,\omega_1 \circ \varrho_1) \to (M_2,\omega_2 \circ \varrho_2)$ be an isometry. Then

$$\mathrm{Iso}(M_1,\omega_1 \circ \varrho_1) = \mathrm{Iso}(M_1,\varrho_1) = \{h^{-1} \circ u \circ h\colon\ u \in \mathrm{Iso}(M_2,\varrho_2)\}.$$

So, we infer from the uniqueness in Theorem 7.3 that $(M_1,\varrho_1) = (M_2,\varrho_2)$. To simplify further arguments, we denote $(M,\varrho) = (M_1,\varrho_1)$. Thus, h is an isometry from $(M,\omega_1 \circ \varrho)$ onto $(M,\omega_2 \circ \varrho)$. We conclude that:

$$\varrho(a,b) = \varrho(x,y) \iff \varrho(h(a),h(b)) = \varrho(h(x),h(y)) \qquad (a,b,x,y \in M)$$

(because both ω_1 and ω_2 are one-to-one). The above condition implies that there is a one-to-one function $f\colon I \to I$ where $I = \varrho(M \times M)$ such that $\varrho \circ (h \times h) = f \circ \varrho$. Since both ϱ and $\varrho \circ (h \times h)$ are geodesic, it follows from Lemma 7.5 that there is a constant $c > 0$ such that $f(t) = ct$. So, h is a dilation on (M,ϱ). Now we consider two cases. First assume that (M,ϱ) is not a Euclidean space. Then $c = 1$, which for hyperbolic spaces follows from Theorem 3.5 and for spheres is trivial (just compare diameters). Thus, $h \in \mathrm{Iso}(M,\varrho)$ and therefore $\omega_1 \circ \varrho = \omega_2 \circ \varrho$. Consequently, $\omega_1 = \omega_2$.

Finally, assume $(M,\varrho) = (\mathbb{R}^n,d_e)$. We have already known that $\varrho(h(x),h(y)) = c\varrho(x,y)$ for all $x,y \in \mathbb{R}^n$. Simultaneously, $\omega_2(\varrho(h(x),h(y))) = \omega_1(\varrho(x,y))$ for any $x,y \in \mathbb{R}^n$. Both these equations imply that $\omega_2(ct) = \omega_1(t)$ for any $t \geq 0$. In particular, $\omega_2(\infty) = \omega_1(\infty)$ and thus $\omega_2(c) = \omega_1(1) = \min(1,\frac{1}{2}\omega_1(\infty)) = \omega_2(1)$. Since ω_2 is one-to-one, we get $c = 1$ and hence $\omega_2 = \omega_1$. $\square$

REFERENCES

[1] BENZ, W.: *A characterization of dimension-free hyperbolic geometry and the functional equation of 2-point invariants*, Aequat. Math. **80** (2010), 5–11.

[2] BENZ, W.: *Classical Geometries in Modern Contexts. Geometry of Real Inner Product Spaces*, 3rd ed., Birkhaüser/Springer, Basel, 2012.

[3] BIRKHOFF, G.: *Metric foundations of geometry*, Proc. Natl. Acad. Sci. USA **27** (1941), 402–406.

[4] BIRKHOFF, G.: *Metric foundations of geometry I*, Trans. Amer. Math. Soc. **55** (1944), 465–492.

[5] BUSEMANN, H.: *On Leibniz's definition of planes*, Amer. J. Math. **63** (1941), 101–111.

[6] CHAVEL, I.: *Riemannian Geometry. A Modern Introduction*, 2nd ed., Cambridge University Press, New York, 2006.

[7] COXETER, H. S. M.: *Non-Euclidean Geometry*, 6th ed., AMS, Washington DC, 1998.

[8] FREUDENTHAL, H.: *Neuere Fassungen des Riemann-Helmholtz-Lieschen Raumproblems*, Math. Z. **63** (1956), 374–405 (in German).

[9] FREUDENTHAL, H.: *Lie groups in the foundations of geometry*, Adv. Math. **1** (1964), 145–190.

[10] KORDOS, M.: *On the syntactic form of dimension axiom for affine geometry*, Bull. Acad. Pol. Sci. Sér. Sci. Math. **18** (1969), 833–837.

[11] LOOMIS, L. H.: *Abstract congruence and the uniqueness of Haar measure*, Ann. Math. **46** (1945), 348–355.

[12] LOOMIS, L. H.: *Haar measure in uniform structures*, Duke Math. J. **16** (1949), 193–208.

[13] MORGAN, F.: *Riemannian Geometry. A Beginner's Guide*, Jones and Bartlett Publishers, Boston, 1993.

[14] REYNOLDS, W. F.: *Hyperbolic geometry on a hyperboloid*, Amer. Math. Monthly **100** (1993), 442–455.

[15] SCHWABHÄUSER, W.—SZMIELEW, W.—TARSKI, A.: *Metamathematische Methoden in der Geometrie*, Springer-Verlag, Berlin, 1983 (in German).

[16] TARSKI, A.—GIVANT, S.: *Tarski's system of geometry*, Bull. Symb. Logic **5** (1999), 175–214.

[17] TITS, J.: *Sur certaines classes d'espaces homogènes de groupes de Lie*, Mem. Acad. Royale Belgique, Classe des Sciences XXIX, Fasc. **3** (1955), 270 pp. (in French).

[18] WANG, H.-C.: *Two-point homogeneous spaces*, Ann. Math. **55** (1952), 177–191.

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