DE GRUYTER

Mathematica Slovaca

DOI: 10.1515/ms-2022-0007
Math. Slovaca **72** (2022), No. 1, 103–110

OSCILLATION OF SECOND ORDER DELAY DIFFERENTIAL EQUATIONS WITH NONLINEAR NONPOSITIVE NEUTRAL TERM

Blanka Baculíková*,c — B. Sudha** — K. Thangavelu*** — E. Thandapani****

(*Communicated by Michal Fečkan*)

ABSTRACT. This paper deals with oscillation of a second order delay differential equations with a nonlinear nonpositive neutral term. Some new oscillation criteria and three examples are presented which improve and generalize the results reported in the literature.

1. Introduction

In this paper, we study the oscillatory behavior of a second order nonlinear neutral delay differential equations of the form

$$(a(t)(x(t)-p(t)x^{\alpha}(\tau(t)))')'+q(t)x^{\beta}(\sigma(t))=0, \qquad t\geq t_0>0 \tag{1.1}$$

subject to the following conditions:

(C_1) $0<\alpha\leq 1$, and β are ratio of odd positive integers;

(C_2) $a\in C^1([t_0,\infty),(0,\infty))$, and $p,q\in C([t_0,\infty),(0,\infty))$ and $p(t)$ is such that $0<p(t)\leq p<1$ for all $t\geq t_0$;

(C_3) $\tau\in C^1([t_0,\infty),\mathbb{R})$, $\sigma\in C^1([t_0,\infty),\mathbb{R})$, $\tau(t)\leq t$, $\sigma(t)\leq t$, $\tau'(t)>0$, $\sigma'(t)>0$, and $\lim\limits_{t\to\infty}\tau(t)=\lim\limits_{t\to\infty}\sigma(t)=\infty$.

By a solution of equation (1.1), we mean a function $x\in C([T_x,\infty),\mathbb{R})$, $T_x\geq t_0$, which has the property $a(t)(x(t)-p(t)x^{\alpha}(\tau(t)))'\in C^1([T_x,\infty),\mathbb{R})$ and satisfies equation (1.1) on $[T_x,\infty)$. We consider only those solutions x of equation (1.1) which satisfy $\sup\{|x(t)|:t\geq T\}>0$ for all $T\geq T_x$, and assume that equation (1.1) possesses such solutions. As usual, a solution of equation (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise it is called nonoscillatory.

In the last few years, there has been great interest in investigating the oscillatory behavior of solutions of different types of neutral differential equations since such equations have numerous applications in science and engineering, see [8,9]. We choose to investigate the oscillatory behavior of solutions of equation (1.1) since similar properties for delay differential equations with linear neutral term are studied in many papers, see [1,3–6,10–13,16,17] and the references cited therein.

2020 Mathematics Subject Classification: Primary 34C10, 34K11.
Keywords: Oscillation, neutral differential equation, nonpositive sublinear neutral term.
c Corresponding author.

Very recently, in [2, 14], the authors considered equation (1.1) with $p(t) < 0$ for all $t \geq t_0$, and established criteria for the oscillation of all solutions under the following condition

$$\int_{t_0}^{\infty} \frac{\mathrm{d}t}{a(t)} = \infty. \tag{1.2}$$

Motivated by the above observation, in this paper we derive some new oscillation results for the equation (1.1), which improve and complement those reported in [2, 11, 13–15].

2. Oscillation results

In this section, we present sufficient conditions for the oscillation of all solutions of equation (1.1). Define

$$z(t) = x(t) - p(t)x^{\alpha}(\tau(t)),$$

$$R(t) = \int_{t_1}^{t} \frac{\mathrm{d}s}{a(s)} \qquad \text{for } t \geq t_1 \geq t_0.$$

Note that from the assumptions and the form of equation (1.1), it is enough to state and prove the results for the case of positive solutions since the proof of the other case is similar.

Lemma 2.1. *Assume condition* (1.2) *holds. If x is a positive solution of equation* (1.1) *then the corresponding function z satisfies one of the following two cases:*

(I) $z(t) > 0$, $z'(t) > 0$, $(a(t)z'(t))' < 0$;

(II) $z(t) < 0$, $z'(t) > 0$, $(a(t)z'(t))' < 0$,

for all $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large.

Proof. Assume that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$ for some $t_1 \geq t_0$. From the definition of $z(t)$ and equation (1.1), we have

$$(a(t)z'(t))' = -q(t)x^{\beta}(\sigma(t)) < 0. \tag{2.1}$$

Hence az' is decreasing and of one sign for large t, that is, there exists a $t_2 \geq t_1$ such that $z'(t) > 0$ or $z'(t) < 0$ for all $t \geq t_2$.

If $z'(t) < 0$ for $t \geq t_2$, then $a(t)z'(t) \leq -d < 0$ for $t \geq t_2$, where $d = -a(t_2)z'(t_2) > 0$. Thus, we have

$$z(t) \leq z(t_2) - d \int_{t_2}^{t} \frac{\mathrm{d}s}{a(s)}.$$

In view of condition (1.2), we see that $\lim_{t\to\infty} z(t) = -\infty$. Now we consider the following two cases separately.

<u>Case 1</u>. If x is unbounded, then there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} x(t_k) = \infty$ where $x(t_k) = \max\{x(s) : t_0 \leq s \leq t_k\}$. Since $\lim_{t\to\infty} \tau(t) = \infty$, $\tau(t_k) > t_0$ for all sufficiently large k and $\tau(t) \leq t$, we have

$$x(\tau(t_k)) = \max\{x(s) : t_0 \leq s \leq \tau(t_k)\} \leq \max\{x(s) : t_0 \leq s \leq t_k\} = x(t_k).$$

Hence

$$z(t_k) = x(t_k) - p(t_k)x^{\alpha}(\tau(t_k)) \geq (1 - p(t_k)x^{\alpha-1}(t_k))x(t_k) \to \infty$$

as $k \to \infty$ since $0 < \alpha \le 1$ and $p(t)$ is bounded, which contradicts the fact that $\lim_{t\to\infty} z(t) = -\infty$.
Case 2. If x is bounded then z is also bounded, since $p(t)$ is bounded, which contradicts $\lim_{t\to\infty} z(t) = -\infty$. Hence z satisfies one of the cases (I) and (II). This completes the proof. □

Lemma 2.2. *Assume condition* (1.2) *holds. Let x be a positive solution of equation* (1.1) *such that case* (I) *of* Lemma 2.1 *holds. Then*

$$x(t) > z(t) > R(t)a(t)z'(t) \tag{2.2}$$

for all $t \ge t_1$ and $z(t)/R(t)$ is eventually decreasing.

Proof. From the definition of z and (C_2) we have $x(t) > z(t)$ for all $t \ge t_1 \ge t_0$. In view of case (I), we obtain

$$\begin{aligned} z(t) &= z(t_1) + \int_{t_1}^{t} \frac{a(s)z'(s)}{a(s)} \mathrm{d}s \\ &> R(t)a(t)z'(t), \qquad t \ge t_1. \end{aligned} \tag{2.3}$$

Furthermore,

$$\left(\frac{z(t)}{R(t)}\right)' = \frac{a(t)R(t)z'(t) - z(t)}{a(t)R^2(t)}, \qquad t \ge t_1.$$

By (2.3), one obtains $\left(\frac{z(t)}{R(t)}\right)' < 0$ for all $t \ge t_1$. Thus, $\frac{z(t)}{R(t)}$ is strictly decreasing for all $t \ge t_1$. This completes the proof. □

Theorem 2.1. *Let $\beta < \alpha$, $\sigma(t) < \tau(t)$, and condition* (1.2) *hold. If*

$$\int_{t_1}^{\infty} q(t)R^{\beta}(\sigma(t))\mathrm{d}t = \infty \tag{2.4}$$

and

$$\limsup_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^{t} \frac{1}{a(s)} \int_{s}^{t} \frac{q(u)}{p^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(u)))} \mathrm{d}u\mathrm{d}s > 0 \tag{2.5}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Assume that there is a nonoscillatory solution x of equation (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \ge t_1 \ge t_0$. It follows from Lemma 2.1 that the corresponding function z satisfies either (I) or (II).

Let $z(t)$ satisfy case (I). From the definition of z, we have

$$x(t) \ge z(t)$$

and

$$x^{\beta}(\sigma(t)) \ge z^{\beta}(\sigma(t)).$$

Substituting above inequality in equation (1.1), we obtain

$$(a(t)z'(t))' + q(t)z^{\beta}(\sigma(t)) \le 0. \tag{2.6}$$

Now using (2.2) in (2.6) and letting $w(t) = a(t)z'(t)$, we see that $w(t)$ is a positive solution of the inequality

$$w'(t) + q(t)R^{\beta}(\sigma(t))w^{\beta}(\sigma(t)) \le 0, \qquad t \ge t_1. \tag{2.7}$$

On the other hand, by [6: Theorem 3.9.3] condition (2.4) guarantees that (2.7) has no eventually positive solution, a contradiction.

Let $z(t)$ satisfy case (II) of Lemma 2.1. From the definition of z, we have

$$x(\tau(t)) > \Big(-\frac{z(t)}{p(t)}\Big)^{1/\alpha}. \tag{2.8}$$

Using (2.8) in equation (1.1), we obtain

$$(a(t)z'(t))' - \frac{1}{p^{\beta/\alpha}(\tau^{-1}(\sigma(t)))}q(t)z^{\beta/\alpha}(\tau^{-1}(\sigma(t))) \le 0. \tag{2.9}$$

Since $z(t)$ is negative and increasing, we obtain $\lim\limits_{t\to\infty} z(t) = d \le 0$. We show that $d = 0$. If not, then $d < 0$ and $z(t) \le d$ and $z(\tau^{-1}(\sigma(t))) \le d$ for t large enough. Therefore

$$z^{\beta/\alpha}(\tau^{-1}(\sigma(t))) \le d^{\beta/\alpha}. \tag{2.10}$$

Integrating inequality (2.9) from t to ∞ and using (2.10) we have

$$-a(t)z'(t) \le \int_t^\infty \frac{q(u)}{p^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(u)))} z^{\beta/\alpha}(\tau^{-1}(\sigma(u)))\mathrm{d}u \le d^{\beta/\alpha}\int_t^\infty \frac{q(u)}{p^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(u)))}\mathrm{d}u.$$

Integrating once more the last inequality from t_1 to ∞, we obtain

$$z(t_1) \le d^{\beta/\alpha}\int_{t_1}^\infty \frac{1}{a(u)}\int_u^\infty \frac{q(s)}{p^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(s)))}\mathrm{d}s\mathrm{d}u,$$

which is a contradiction with (2.5), because from (2.5), we claim

$$\limsup_{t\to\infty}\int_{t_1}^\infty \frac{1}{a(u)}\int_u^\infty \frac{q(s)}{p^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(s)))}\mathrm{d}s\mathrm{d}u = \infty.$$

So, we have $\lim\limits_{t\to\infty} z(t) = 0$ and $z(t)$ is negative and increasing. Integrating (2.9) from s to t for $t > s$, we get

$$-a(s)z'(s) \le \int_s^t \frac{q(u)}{p^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(u)))} z^{\beta/\alpha}(\tau^{-1}(\sigma(u)))\mathrm{d}u.$$

Integrating above inequality from $\tau^{-1}(\sigma(t))$ to t for s and using that $z(t)$ is increasing, we have

$$z(\tau^{-1}(\sigma(t))) - z(t) \le z^{\beta/\alpha}(\tau^{-1}(\sigma(t)))\int_{\tau^{-1}(\sigma(t))}^t \frac{1}{a(s)}\int_s^t \frac{q(u)}{p^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(u)))}\mathrm{d}u\mathrm{d}s$$

or

$$\frac{z(\tau^{-1}(\sigma(t)))}{z^{\beta/\alpha}(\tau^{-1}(\sigma(t)))} \ge \int_{\tau^{-1}(\sigma(t))}^t \frac{1}{a(s)}\int_s^t \frac{q(u)}{p^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(u)))}\mathrm{d}u\mathrm{d}s. \tag{2.11}$$

Since $\frac{z(\tau^{-1}(\sigma(t)))}{z^{\beta/\alpha}(\tau^{-1}(\sigma(t)))} = |z(\tau^{-1}(\sigma(t)))|^{1-\beta/\alpha}$ and $1 - \beta/\alpha > 0$, we have

$$\limsup_{t\to\infty}\int_{\tau^{-1}(\sigma(t))}^t \frac{1}{a(s)}\int_s^t \frac{q(u)}{p^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(u)))}\mathrm{d}u\mathrm{d}s \le 0,$$

which contradicts (2.5). This completes the proof of the theorem. □

Theorem 2.2. *Let $\beta = 1$, and condition* (1.2) *hold. If*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} q(s)R(\sigma(s))\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{2.12}$$

then every solution of equation (1.1) *is either oscillatory or tends to zero as $t \to \infty$.*

Proof. Assume that there exists a nonoscillatory solution x of equation (1.1), say, $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \ge t_1 \ge t_0$ so that for z one of the cases (I) and (II) holds.

Let $z(t)$ satisfy case (I) of Lemma 2.1. From the proof of case (I) of Theorem 2.1, we have for $\beta = 1$ that $w(t) = a(t)z'(t)$ is a positive solution of inequality

$$w'(t) + q(t)R(\sigma(t))w(\sigma(t)) \le 0. \tag{2.13}$$

On the other hand, by [6: Theorem 2.1.1] condition (2.12) guarantees that inequality (2.13) has no eventually positive solution, a contradiction.

Let $z(t)$ satisfy case (II) of Lemma 2.1. By $z < 0$ and $z' > 0$, we see that

$$\lim_{t\to\infty} z(t) = d \le 0$$

where d is a finite constant. That is, z is bounded and as in the proof of case (1) of Lemma 2.1, x is also bounded. Therefore $\lim_{t\to\infty} x(t) = \ell$, $0 \le \ell < \infty$. We claim that $\ell = 0$. If $\ell > 0$, there is a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} x(t_k) = \ell$. Now

$$z(t_k) = x(t_k) - p(t_k)x^{\alpha}(\tau(t_k))$$

and so

$$x(\tau(t_k)) = \frac{1}{p^{\frac{1}{\alpha}}(t_k)}(x(t_k) - z(t_k))^{\frac{1}{\alpha}}.$$

Letting $k \to \infty$, we obtain

$$\ell \ge \lim_{k\to\infty} x(\tau(t_k)) \ge \left(\frac{\ell}{p}\right)^{\frac{1}{\alpha}}.$$

Since $p \in (0,1)$, we see that $\ell = 0$, that is, $\lim_{t\to\infty} x(t) = 0$. This completes the proof of the theorem. □

Theorem 2.3. *Let $\beta > 1$, and condition* (1.2) *hold. If there exists a positive nondecreasing function $\rho(t)$ such that*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[\rho(s)q(s) - \frac{a(\sigma(s))(\rho'(s))^2}{4\beta K^{\beta-1}\rho(s)\sigma'(s)}\right] \mathrm{d}s = \infty \tag{2.14}$$

for any constant $K > 0$, then every solution of equation (1.1) *is either oscillatory or tends to zero as $t \to \infty$.*

Proof. Proceeding as in the proof of Theorem 2.1, we see that one of the cases of Lemma 2.1 holds for all $t \ge t_1$.

Case (I) Proceeding as in the proof of Theorem 2.1 (Case (I)), we obtain

$$(a(t)z'(t))' + q(t)z^{\beta}(\sigma(t)) \le 0 \tag{2.15}$$

for all $t \ge t_1$. Define

$$w(t) = \rho(t)\frac{a(t)z'(t)}{z^{\beta}(\sigma(t))}, \qquad t \ge t_1.$$

Then $w(t) > 0$, and from (2.15), we have

$$w'(t) \le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\beta a(t)z'(t)}{z^{\beta+1}(\sigma(t))}z'(\sigma(t))\sigma'(t)\rho(t). \tag{2.16}$$

Since $a(t)z'(t)$ is nonincreasing we have $a(\sigma(t))z'(\sigma(t)) \ge a(t)z'(t)$ and so

$$w'(t) \le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\beta a(\sigma(t))\left(z'(\sigma(t))\right)^2}{z^{\beta+1}(\sigma(t))}\sigma'(t)\rho(t).$$

Using that $z(t)$ is positive increasing and $\beta > 1$ there is a constant $K > 0$ such that $z^{\beta-1}(t) \ge K^{\beta-1} > 0$ for all $t \ge t_2 \ge t_1$ and the last inequality implies

$$w'(t) \le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\beta K^{\beta-1}\sigma'(t)}{\rho(t)a(\sigma(t))}w^2(t).$$

Now using the completing the square, we have

$$w'(t) \le -\rho(t)q(t) + \frac{a(\sigma(t))(\rho'(t))^2}{4\beta K^{\beta-1}\rho(t)\sigma'(t)}, \qquad t \ge t_2.$$

Integrating the last inequality from t_2 to t, we obtain

$$\int_{t_2}^{t}\left[\rho(s)q(s) - \frac{a(\sigma(s))(\rho'(s))^2}{4\beta K^{\beta-1}\rho(s)\sigma'(s)}\right]\mathrm{d}s < w(t_2) < \infty.$$

Take lim sup as $t \to \infty$ in the last inequality, we obtain a contradiction to (2.14).

Case (II) In this case $z(t) < 0$ and $z'(t) > 0$ for all $t \ge t_1$. Then proceeding as in case (II) of Theorem 2.2, we obtain $\lim_{t\to\infty} x(t) = 0$. This completes the proof of the theorem. □

3. Examples

In this section, we present three examples to illustrate the main results.

***Example* 1.** Consider a second order neutral differential equation

$$\left(x(t) - px^{1/3}(t/2)\right)'' + 8tx^{1/5}(t/3) = 0, \qquad t \ge 1, \tag{3.1}$$

where $p \in (0,1)$ is a constant. Here $a(t) = 1$, $p(t) = p$, $q(t) = 8t$, $\tau(t) = t/2$, $\sigma(t) = t/3$ for $t \ge t_1 = 1$, $\alpha = 1/3$, $\beta = 1/5$ and $R(t) = t - 1$. Simple calculation shows that conditions (2.4) and (2.5) are satisfied. Therefore by Theorem 2.1 every solution of equation (3.1) is oscillatory.

***Example* 2.** Consider a second order neutral differential equation

$$\left(t\left(x(t) - px^{1/3}(t/2)\right)'\right)' + tx(t/3) = 0, \qquad t \ge 1, \tag{3.2}$$

where $p \in (0,1)$ is a constant. Here $a(t) = t$, $p(t) = p$, $q(t) = t$, $\tau(t) = t/2$, $\sigma(t) = t/3$ for $t \ge t_1 = 1$, $\alpha = 1/3$, $\beta = 1$ and $R(t) = \ln t$. Now one can easily verify that all conditions of Theorem 2.2 are satisfied. Hence every solution of equation (3.2) is either oscillatory or tends to zero as $t \to \infty$.

***Example* 3.** Consider a second order neutral differential equation

$$\left(t\left(x(t)-px^{1/3}(t-\pi/2)\right)'\right)'+\frac{1}{t}x^3(t-\pi)=0, \qquad t\geq 1, \tag{3.3}$$

where $p\in(0,1)$ is a constant. Here $a(t)=t$, $p(t)=p$, $q(t)=\frac{1}{t}$, $\tau(t)=t-\pi/2$, $\sigma(t)=t-\pi$ or $t\geq t_1=1$, $\alpha=1/3$, $\beta=3$ and $R(t)=\ln t$. By taking $\rho(t)=1$, one can easily see that all conditions of Theorem 2.3 are satisfied, and hence every solution of equation (3.3) is either oscillatory or tends to zero as $t\to\infty$.

We conclude this paper with the following remark.

Remark 1. Note that Theorem 2.1 guarantees that every solution of equation (1.1) is oscillatory, and from other theorems we see that every solution is either oscillatory or tends to zero as $t\to\infty$. Therefore it is interesting to improve Theorems 2.2, and 2.3 so that all solutions of equation (1.1) are oscillatory only. Further note that the results reported in [7, 11, 13, 17] cannot be applied to equations (3.1), (3.2) and (3.3) since the neutral term is not linear.

Acknowledgement. The paper is supported by the grant project KEGA 037TUKE-4/2020.

REFERENCES

[1] AGARWAL, R. P.—BOHNER, M.—LI, T.: *Nonoscillation and Oscillation Theory of Functional Differential Equations*, Marcel Dekker, New York, 2004.

[2] AGARWAL, R. P.—BOHNER, M.—LI T.—ZHANG, C.: *Oscillation of second order differential equations with a sublinear neutral term*, Carpathian J. Math. **30** (2014), 1–6.

[3] AGARWAL, R. P.—GRACE, S. R.: *Oscillation theorems for certain neutral differential equations*, Comput. Math. Appl. **38** (1999), 1–11.

[4] BACULÍKOVÁ, B.—DŽURINA, J.: *Oscillation theorems for second order nonlinear neutral differential equations*, Comput. Math. Appl. **61** (2011), 94–99.

[5] DONG, J. G.: *Oscillation behavior of second order nonlinear neutral differential equations with deviating arguments*, Comput. Math. Appl. **59** (2010), 3710–3717.

[6] ERBE, L. H.—KONG, Q.—ZHANG, B. G.: *Oscillation Theory For Functional Differential Equations*, Marcel Dekker, New York, 1995.

[7] GRACE, S. R.: *Oscillatory behavior of second-order nonlinear differential equations with a nonpositive neutral terms*, Mediterr. J. Math. **14** (2017), Art. 229.

[8] GYORI, I.—LADAS, G.: *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991.

[9] HALE, J. K.: *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1987.

[10] LI T.—AGARWAL R.P.—BOHNER M.: *Some oscillation results for second order neutral differential equations*, J. Indian Math. Soc. **79** (2012), 97–106.

[11] LI, Q.—WANG, W.—CHEN, F—LI, T.: *Oscillation of second order nonlinear delay differential equations with nonpositive neutral coefficients*, Adv. Difference Equ. **2015** (2015), Art. Id. 35.

[12] LIN, X.: *Oscillation of second order nonlinear neutral differential equations*, J. Math. Anal. Appl. **309** (2005), 442–452.

[13] QIN, H.—SHANG, N.—LU, Y.: *A note on oscillation criteria of second order nonlinear neutral delay differential equations*, Comput. Math. Appl. **56** (2008), 2987–2992.

[14] TAMILVANAN, S.—THANDAPANI, E.—DŽURINA, J.: *Oscillation of second order nonlinear differential eqation with sublinear neutral term*, Diff. Equ. Appl. **9** (2017), 29–35.

[15] TAMILVANAN, S.—THANDAPANI, E.—GRACE, S. R.: *Oscillation theorems for second order nonlinear differential equation with a nonlinear neutral term*, Int. J. Dyn. Syst. Differ. Equ. **7** (2017), 316–327.

[16] THANDAPANI, E.—RAMA, R.: *Comparison and oscillation theorems for second order nonlinear neutral differential equations*, Serdica Math. J. **39** (2013), 1–16.

[17] WONG, J. S. W: *Necessary and sufficient conditions for oscillation of second order neutral differential equations*, J. Math. Anal. Appl. **252** (2000), 342–352.

Received 15. 12. 2020
Accepted 27. 2. 2021

* *Department of Mathematics*
Faculty of Electrical Engineering and Informatics
Technical University of Košice
SLOVAKIA
E-mail: blanka.baculikova@tuke.sk

** *Department of Mathematics*
Institute of Science and Technology
Kattankulathur
INDIA
E-mail: sudhabala10@gmail.com

*** *Department of Mathematics*
Pachiyappas College
Chennai
INDIA
E-mail: kthangavelu14@gmail.com

**** *Ramanujan Institute For Advanced Study in Mathematics*
University of Madras
Chennai
INDIA
E-mail: ethandapani@yahoo.co.in