

CERTAIN ESTIMATES OF NORMALIZED ANALYTIC FUNCTIONS

SWATI ANAND* — NAVEEN KUMAR JAIN**,c — SUSHIL KUMAR***

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ABSTRACT. Let ϕ be a normalized convex function defined on open unit disk $\mathbb{D}$. For a unified class of normalized analytic functions which satisfy the second order differential subordination $f'(z) + \alpha z f''(z) \prec \phi(z)$ for all $z \in \mathbb{D}$, we investigate the distortion theorem and growth theorem. Further, the bounds on initial logarithmic coefficients, inverse coefficient and the second Hankel determinant involving the inverse coefficients are examined.

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1. Introduction

The class of all normalized analytic functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \quad (1.1)$$

in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is denoted by $\mathcal{A}$. Denote by $\mathcal{S}$, the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{D}$. Let $\mathcal{P}$ be the class of analytic functions p defined on $\mathbb{D}$, normalized by the condition $p(0) = 1$ and satisfying $\operatorname{Re}(p(z)) > 0$. Let f and g be analytic in $\mathbb{D}$. Then f is subordinate to g , denoted by $f \prec g$, if there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{D}$ such that $f(z) = g(w(z))$. In particular, if g is univalent in $\mathbb{D}$ then f is subordinate to g when $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. In this paper we shall assume that ϕ is an analytic function with positive real part in $\mathbb{D}$ and normalized by the conditions $\phi(0) = 1$ and $\phi'(0) > 0$. It is noted that $\phi(\mathbb{D})$ is convex. The function ϕ is symmetric with respect to the real axis and it maps $\mathbb{D}$ onto a region starlike with respect to $\phi(0) = 1$. The Taylor series representation of the function ϕ is given by

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (1.2)$$

where $B_1 > 0$. For such ϕ , Ma and Minda [22] studied the unified subclasses $S^*(\phi)$ and $C(\phi)$ of starlike and convex functions respectively, analytically defined as

$$S^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\} \quad \text{and} \quad C(\phi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \right\}.$$

The authors investigated the growth, distortion and coefficient estimates for these classes. In particular, the class $S^*(\phi)$ reduces to some well known subclasses of starlike functions. For example, when $-1 \leq B < A \leq 1$, $S^*[A, B]$ is the class of Janowski starlike functions introduced by Janowski

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* Corresponding author.

[11]. For $0 \leq \alpha < 1$, $S^*[1 - 2\alpha, -1] = S_\alpha^*$ is the class of starlike functions of order α , introduced by Robertson [31]. The class $\mathcal{SL} = \mathcal{S}^*(\sqrt{1+z})$, introduced by Sokół and Stankiewicz [36], consists of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region $\Omega_L := \{w \in \mathbb{C} : |w^2 - 1| < 1\}$, that is, the right-half of the lemniscate of Bernoulli. Later, Mendiratta et al. [23] introduced the class $\mathcal{S}_e^* := \mathcal{S}^*(e^z)$ consists of functions $f \in \mathcal{A}$ satisfying the condition $|\log(zf'(z)/f'(z))| < 1$. In 2011, Ali et al. [3](see also [10]) studied the class of all those functions $f \in \mathcal{A}$ which satisfy the following third order differential equation

$$f(z) + \alpha f'(z) + \gamma z^2 f''(z) = g(z),$$

where the function g is subordinate to a convex function h . In [3], the best dominant on all solutions of the differential equation in terms of double integral were obtained. Some certain variations of the class $\mathcal{R}(\alpha, h) = \{f \in \mathcal{A} : f'(z) + \alpha z f''(z) \prec h(z), z \in \mathbb{D}\}$, where h is a convex function have been investigated by several authors [6, 24, 35, 37, 39]. On the basis of the above discussed works, we consider a unified class of all functions $f \in \mathcal{A}$ such that

$$f'(z) + \alpha z f''(z) \prec \phi(z) \quad (1.3)$$

for $z \in \mathbb{D}$ and where $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha \geq 0$. The class of such functions is denoted by $\mathcal{R}(\alpha, \phi)$. Since $f \in \mathcal{R}(\alpha, \phi)$, $f'(z) + \alpha z f''(z) \neq \phi(e^{i\theta})$, $\theta \in [0, 2\pi)$, it is observed that

$$f'(z) + \alpha z f''(z) = [(1 - \alpha)f(z) + \alpha(zf'(z))]'.$$

Also, we have $zf'(z) = f(z) * \frac{z}{(1-z)^2}$ and $f(z) = f(z) * \frac{z}{1-z}$. Thus,

$$\begin{aligned} f'(z) + \alpha z f''(z) &= \left((1 - \alpha)f(z) * \frac{z}{1-z} + \alpha f(z) * \frac{z}{(1-z)^2} \right)' \\ &= \left(f(z) * \left((1 - \alpha) \frac{z}{1-z} + \alpha \frac{z}{(1-z)^2} \right) \right)'. \end{aligned}$$

Therefore, we conclude that

$$\left(f(z) * \frac{z - z^2(1 - \alpha)}{(1 - z)^2} \right)' - \phi(e^{i\theta}) \neq 0$$

or equivalently

$$\frac{1}{z} \left(f(z) * \frac{z + z^2(2\alpha - 1)}{(1 - z)^3} \right) \neq \phi(e^{i\theta})$$

which is the necessary and sufficient conditions for a function $f \in \mathcal{A}$ to be in the class $\mathcal{R}(\alpha, \phi)$.

In this paper, we compute the distortion, growth inequalities for a function f in the class $\mathcal{R}(\alpha, \phi)$. The sharp bounds on initial logarithmic coefficients for such functions are also obtained. Next, we obtain the bounds on initial inverse coefficients of the function $f \in \mathcal{R}(\alpha, \phi)$ as well as bounds on Fekete Szegő functional and second Hankel determinant.

2. Distortion and growth theorem

The first theorem proves the distortion theorem of the functions f belonging to $\mathcal{R}(\alpha, \phi)$.

THEOREM 2.1. *Let $\alpha \in \mathbb{C}$, $\operatorname{Re} \alpha \geq 0$ and the function ϕ be defined by (1.2). If the function $f \in \mathcal{R}(\alpha, \phi)$, then*

$$1 + \sum_{n=1}^{\infty} \frac{|B_n|(-r)^n}{n \operatorname{Re} \alpha + 1} \leq |f'(z)| \leq 1 + \sum_{n=1}^{\infty} \frac{|B_n|r^n}{n \operatorname{Re} \alpha + 1}$$

for $|z| < r < 1$. The result is sharp.

We make use of the following lemma in order to prove some of our results.

LEMMA 2.1 ([7: Lemma 2, p. 192]). *Let h be a convex function with $\gamma \neq 0$ and $\operatorname{Re} \gamma \geq 0$. If $p(z)$ is regular in $\mathbb{D}$ and $p(0) = h(0)$, then*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z) \quad (2.1)$$

implies that $p(z) \prec q(z) \prec h(z)$, where

$$q(z) = \gamma z^{-\gamma} \int_0^z h(t) t^{\gamma-1} dt.$$

The function q is convex and the best dominant.

Proof of Theorem 2.1. Let the function f be in the class $\mathcal{R}(\alpha, \phi)$. Then $f'(z) + \alpha z f''(z) \prec \phi(z)$. For $p(z) = f'(z)$ and $\gamma = 1/\alpha$, Lemma 2.1 yields

$$f'(z) \prec \frac{1}{\alpha} z^{-1/\alpha} \int_0^z \phi(t) t^{1/\alpha-1} dt.$$

On taking $t = z\zeta^\alpha$ in above relation, we get

$$f'(z) \prec \int_0^1 \phi(z\zeta^\alpha) d\zeta. \quad (2.2)$$

Since the function ϕ is symmetric with respect to real axis, $\phi(z)$ has real coefficients. Also $\phi'(0) > 0$ gives $\phi'(x)$ is increasing on $(0, 1)$. From [12], we have

$$\min_{|z|=r} \operatorname{Re} \phi(z) = \phi(-r) \quad \text{and} \quad \max_{|z|=r} \operatorname{Re} \phi(z) = \phi(r). \quad (2.3)$$

Using (2.3) and (2.2) for $|z| = r$, we have

$$\begin{aligned} |f'(z)| &\geq \operatorname{Re} f'(z) \geq \min_{|z|=r} \operatorname{Re} f'(z) \\ &\geq \min_{|z|=r} \operatorname{Re} \int_0^1 \phi(z\zeta^\alpha) d\zeta = \int_0^1 \min_{|z|=r} \operatorname{Re} \phi(z\zeta^\alpha) d\zeta = \int_0^1 \phi(-r\zeta^{\operatorname{Re} \alpha}) d\zeta. \end{aligned} \quad (2.4)$$

Similarly, we have

$$|f'(z)| \leq \int_0^1 \phi(r\zeta^{\operatorname{Re} \alpha}) d\zeta. \quad (2.5)$$

Since $\phi(z\zeta^{\operatorname{Re} \alpha}) = 1 + B_1 z\zeta^{\operatorname{Re} \alpha} + B_2 z^2 \zeta^{2\operatorname{Re} \alpha} + \dots$, a simple calculation yields

$$\begin{aligned} \int_0^1 \phi(z\zeta^{\operatorname{Re} \alpha}) d\zeta &= \int_0^1 (1 + B_1 z\zeta^{\operatorname{Re} \alpha} + B_2 z^2 \zeta^{2\operatorname{Re} \alpha} + B_3 z^3 \zeta^{3\operatorname{Re} \alpha} + \dots) d\zeta \\ &= 1 + \frac{B_1 z}{\operatorname{Re} \alpha + 1} + \frac{B_2 z^2}{2\operatorname{Re} \alpha + 1} + \frac{B_3 z^3}{3\operatorname{Re} \alpha + 1} + \dots \\ &= 1 + \sum_{n=1}^{\infty} \frac{B_n z^n}{n\operatorname{Re} \alpha + 1}. \end{aligned} \quad (2.6)$$

From (2.4), (2.5) and (2.6) the result follows. The result is sharp for the function $f: \mathbb{D} \rightarrow \mathbb{C}$ defined by

$$f(z) = z + \sum_{n=1}^{\infty} \frac{B_n z^{n+1}}{(n+1)(1+n \operatorname{Re} \alpha)}. \quad (2.7)$$

□

THEOREM 2.2. *Let $\alpha \in \mathbb{C}$ such that $\operatorname{Re} \alpha \geq 0$ and the function ϕ be as in (1.2). Then for the function $f \in R(\alpha, \phi)$, we have*

$$1 + \sum_{n=1}^{\infty} \frac{|B_n|(-r)^n}{(n+1)(n \operatorname{Re} \alpha + 1)} \leq \frac{|f(z)|}{|z|} \leq 1 + \sum_{n=1}^{\infty} \frac{|B_n|r^n}{(n+1)(n \operatorname{Re} \alpha + 1)} \quad (|z| < r < 1).$$

Proof. Let

$$H(z) = \int_0^1 \phi(z\zeta^\alpha) d\zeta \quad (2.8)$$

and

$$\Phi_\alpha(z) = \int_0^1 \frac{1}{1-zt^\alpha} dt = \sum_{n=0}^{\infty} \frac{z^n}{1+n\alpha}.$$

From [33: Theorem 5, p. 113], it is noted that $\Phi_\alpha(z)$ is convex with $\operatorname{Re} \alpha \geq 0$. Also,

$$\Phi_\alpha(z) * \phi(z) = \left(\sum_{n=0}^{\infty} \frac{z^n}{1+n\alpha} \right) * \left(1 + \sum_{n=1}^{\infty} B_n z^n \right) = 1 + \sum_{n=1}^{\infty} \left(\frac{B_n}{1+n\alpha} \right) z^n.$$

It view of above and (2.6), we have

$$\Phi_\alpha(z) * \phi(z) = \int_0^1 \phi(z\zeta^\alpha) d\zeta = H(z).$$

Since convolution of two convex functions is convex, the function H is convex and $H(0) = 1$. Putting $\gamma = 1$ and $h(z) = H(z)$ in Lemma 2.1, we get

$$p(z) \prec \frac{1}{z} \int_0^z H(t) dt. \quad (2.9)$$

Using (2.8), substituting $t = z\sigma$ and $p(z) = f(z)/z$ in (2.9), we have

$$\frac{f(z)}{z} \prec \int_0^1 \int_0^1 \phi(z\sigma\zeta^\alpha) d\sigma d\zeta. \quad (2.10)$$

Let $h(z) = f(z)/z$. Then (2.3) together with (2.10) yields

$$|h(z)| \geq \min_{|z|=r} \operatorname{Re} h(z) \geq \int_0^1 \int_0^1 \min_{|z|=r} \phi(z\sigma\zeta^\alpha) d\sigma d\zeta = \int_0^1 \int_0^1 \phi(-r\sigma\zeta^{\operatorname{Re} \alpha}) d\sigma d\zeta \quad (2.11)$$

and

$$|h(z)| \leq \max_{|z|=r} \operatorname{Re} h(z) \leq \int_0^1 \int_0^1 \max_{|z|=r} \phi(z\sigma\zeta^\alpha) d\sigma d\zeta = \int_0^1 \int_0^1 \phi(r\sigma\zeta^{\operatorname{Re} \alpha}) d\sigma d\zeta. \quad (2.12)$$

A simple calculation shows that

$$\begin{aligned} \int_0^1 \int_0^1 \phi(z\sigma\zeta^{\operatorname{Re}\alpha}) d\sigma d\zeta &= \int_0^1 \left(1 + \sum_{n=1}^{\infty} \left(\frac{B_n}{1+n\operatorname{Re}\alpha}\right) (z\sigma)^n\right) d\sigma \\ &= 1 + \sum_{n=1}^{\infty} \frac{B_n z^n}{(n+1)(1+n\operatorname{Re}\alpha)}. \end{aligned} \quad (2.13)$$

Now, the result follows from (2.11), (2.12) and (2.13). The result is sharp for the function f given by (2.7). $\square$

Remark 1. Letting $\phi(z) = (1 - (1 - 2\beta)z)/(1 - z)$, where $\beta < 1$ and $\alpha = 1$, Theorem 2.2 reduces to a result due to Silverman [34: Corollary 2, p. 250]. Further, for $\phi(z) = (1 - (1 - 2\beta)z)/(1 - z)$, where $\beta < 1$ and $\alpha > 0$, Theorem 2.3 yields [6: Corollary 3, p. 178].

THEOREM 2.3. *Let $\alpha \in \mathbb{C}$ such that $\operatorname{Re}\alpha \geq 0$ and the function ϕ be as in (1.2) such that $f \in R(\alpha, \phi)$. Then*

$$|a_n| \leq \frac{B_1}{|n + n(n-1)\alpha|} \quad \text{for all } n \geq 2. \quad (2.14)$$

Proof. For $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$, set $f'(z) + \alpha z f''(z) = p(z)$, $z \in \mathbb{D}$. Since $f \in R(\alpha, \phi)$, $p(z) \prec \phi(z)$. A simple calculation gives

$$f'(z) + \alpha z f''(z) = 1 + \sum_{n=2}^{\infty} [n + n(n-1)\alpha] a_n z^{n-1} = 1 + \sum_{n=1}^{\infty} p_n z^n. \quad (2.15)$$

On comparing the coefficients of z^{n-1} , we get

$$(n + n(n-1)\alpha) a_n = p_{n-1} \quad \text{for all } n \geq 2.$$

By making use of [32: Theorem X, p. 70], we get $|p_n| \leq B_1$, for all $n \geq 1$. Hence, we get the desired inequality. The inequality (2.14) is sharp for the function f_n given by $f'_n(z) + \alpha z f''_n(z) = \phi(z^{n-1})$. $\square$

Remark 2. On taking $\phi(z) = (1 - (1 - 2\beta)z)/(1 - z)$, where $\beta < 1$ and $\alpha = 1$, Theorem 2.3 yields a result due to Silverman [34: Corollary 1, p. 250]. Further, letting $\phi(z) = (1 - (1 - 2\beta)z)/(1 - z)$, where $\beta < 1$ and $\alpha > 0$ Theorem 2.3 reduces to [6: Corollary 2, p. 178].

3. Bounds on initial logarithmic coefficient

For a function $f \in \mathcal{S}$, the logarithmic coefficients γ_n are defined by the following series expansion:

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n, \quad \log 1 := 0. \quad (3.1)$$

On comparing the coefficients of z on both the sides, we get the initial logarithmic coefficients

$$\gamma_1 = \frac{1}{2} a_2, \quad \gamma_2 = \frac{1}{2} \left(a_3 - \frac{1}{2} a_2^2 \right) \quad \gamma_3 = \frac{1}{2} \left(a_4 - a_2 a_3 + \frac{1}{3} a_2^3 \right) \quad (3.2)$$

In 1979, the authors [5] showed that the logarithmic coefficients γ_n of every function $f \in \mathcal{S}$ satisfy the inequality $\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \pi^2/6$, where the equality holds if and only if the function f is rotation

of the Koebe function $k(z) = z(1 - e^{i\theta})^{-2}$ for each θ . The n^{th} logarithmic coefficient of $k(z)$ is $\gamma_n = e^{in\theta}/n$ for each θ and $n \geq 1$. In [40], the logarithmic coefficients γ_n of each close-to-convex function $f \in S$ is bounded by $(A \log n)/n$ where A is an absolute constant. In 2018, the authors [4, 28] obtained the bounds on logarithmic coefficients of certain subclasses of the class of close-to-convex functions. Recently, Adegani et al. [1] investigated the bounds for the initial logarithmic coefficients of the generalized classes $S^*(\phi)$ and $C(\phi)$. To find the bounds on initial logarithmic coefficient for class $\mathcal{R}(\alpha, \phi)$, we shall use the following two lemmas.

LEMMA 3.1 ([26: p. 172]). *Assume that w is a Schwarz function so that $w(z) = \sum_{n=1}^{\infty} c_n z^n$. Then*

$$|c_1| \leq 1 \quad \text{and} \quad |c_n| \leq 1 - |c_1|^2, \quad n = 2, 3, \dots$$

LEMMA 3.2 ([29: Theorem 1]). *Let $w(z) = \sum_{n=1}^{\infty} c_n z^n$ be the Schwarz function. Then for any real numbers q_1 and q_2 , the following sharp inequality holds:*

$$|c_3 + q_1 c_1 c_2 + q_2 c_1^3| \leq H(q_1; q_2),$$

where

$$H(q_1; q_2) = \begin{cases} 1 & \text{if } (q_1, q_2) \in D_1 \cup D_2 \cup \{(2, 1)\}, \\ |q_2| & \text{if } (q_1, q_2) \in \cup_{k=1}^7 D_k, \\ \frac{2}{3}(|q_1| + 1) \left(\frac{1 + |q_1|}{3(|q_1| + 1 + q_2)} \right)^{\frac{1}{2}} & \text{if } (q_1, q_2) \in D_8 \cup D_9, \\ \frac{q_2}{3} \left(\frac{q_1^2 - 4}{q_1^2 - 4q_2} \right) \left(\frac{q_1^2 - 4}{3(q_2 - 1)} \right)^{\frac{1}{2}} & \text{if } (q_1, q_2) \in D_{10} \cup D_{11} \setminus \{(2, 1)\}, \\ \frac{2}{3}(|q_1| - 1) \left(\frac{|q_1| - 1}{3(|q_1| - 1 - q_2)} \right)^{\frac{1}{2}} & \text{if } (q_1, q_2) \in D_{12}, \end{cases}$$

and for $k = 1, 2, \dots, 12$, the sets D_k are defined as follows:

$$D_1 = \left\{ (q_1, q_2) : |q_1| \leq \frac{1}{2}, |q_2| \leq 1 \right\},$$

$$D_2 = \left\{ (q_1, q_2) : \frac{1}{2} \leq |q_1| \leq 2, \frac{4}{27}(|q_1| + 1)^3 - (|q_1| + 1) \leq q_2 \leq 1 \right\},$$

$$D_3 = \left\{ (q_1, q_2) : |q_1| \leq \frac{1}{2}, q_2 \leq -1 \right\},$$

$$D_4 = \left\{ (q_1, q_2) : |q_1| \geq \frac{1}{2}, q_2 \leq -\frac{2}{3}(|q_1| + 1) \right\},$$

$$D_5 = \{(q_1, q_2) : |q_1| \leq 2, q_2 \geq 1\},$$

$$D_6 = \left\{ (q_1, q_2) : 2 \leq |q_1| \leq 4, q_2 \geq \frac{1}{12}(q_1^2 + 8) \right\},$$

$$D_7 = \left\{ (q_1, q_2) : |q_1| \geq 4, q_2 \geq \frac{2}{3}(|q_1| - 1) \right\},$$

$$D_8 = \left\{ (q_1, q_2) : \frac{1}{2} \leq |q_1| \leq 2, -\frac{2}{3}(|q_1| + 1) \leq q_2 \leq \frac{4}{27}(|q_1| + 1)^3 - (|q_1| + 1) \right\},$$

$$\begin{aligned}
 D_9 &= \left\{ (q_1, q_2) : |q_1| \geq 2, -\frac{2}{3}(|q_1| + 1) \leq q_2 \leq \frac{2|q_1|(|q_1| + 1)}{q_1^2 + 2|q_1| + 4} \right\}, \\
 D_{10} &= \left\{ (q_1, q_2) : 2 \leq |q_1| \leq 4, \frac{2|q_1|(|q_1| + 1)}{q_1^2 + 2|q_1| + 4} \leq q_2 \leq \frac{1}{12}(q_1^2 + 8) \right\}, \\
 D_{11} &= \left\{ (q_1, q_2) : |q_1| \geq 4, \frac{2|q_1|(|q_1| + 1)}{q_1^2 + 2|q_1| + 4} \leq q_2 \leq \frac{2|q_1|(|q_1| - 1)}{q_1^2 - 2|q_1| + 4} \right\}, \\
 D_{12} &= \left\{ (q_1, q_2) : |q_1| \geq 4, \frac{2|q_1|(|q_1| - 1)}{q_1^2 - 2|q_1| + 4} \leq q_2 \leq \frac{2}{3}(|q_1| - 1) \right\}.
 \end{aligned}$$

THEOREM 3.1. *Let $\alpha \in \mathbb{C}$ such that $\operatorname{Re} \alpha \geq 0$ and the function ϕ be as in (1.2). Suppose $f \in \mathcal{R}(\alpha, \phi)$, then the initial logarithmic coefficients of f satisfy the following inequalities:*

$$(i) \quad |\gamma_1| \leq \frac{B_1}{4|1 + \alpha|}.$$

(ii)

$$|\gamma_2| \leq \begin{cases} \frac{B_1}{6|1 + 2\alpha|} & \text{if } |8(1 + \alpha)^2 B_2 - 3(1 + 2\alpha) B_1^2| \leq 8B_1|(1 + \alpha)^2|, \\ \frac{|8(1 + \alpha)^2 B_2 - 3(1 + 2\alpha) B_1^2|}{48|(1 + \alpha)^2||1 + 2\alpha|} & \text{if } |8(1 + \alpha)^2 B_2 - 3(1 + 2\alpha) B_1^2| > 8B_1|(1 + \alpha)^2|. \end{cases}$$

(iii) *If B_1, B_2, B_3 and α are real numbers, then*

$$|\gamma_3| \leq \frac{B_1}{8(1 + 3\alpha)} H(q_1; q_2),$$

where $H(q_1; q_2)$ is given in Lemma 3.2 such that

$$q_1 = \frac{2B_2}{B_1} - \frac{2B_1(1 + 3\alpha)}{3(1 + \alpha)(1 + 2\alpha)}$$

and

$$q_2 = \frac{B_3}{B_1} - \frac{2B_2(1 + 3\alpha)}{3(1 + \alpha)(1 + 2\alpha)} + \frac{B_1^2(1 + 3\alpha)}{6(1 + \alpha)^3}.$$

The bounds for γ_1 and γ_2 are sharp.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R}(\alpha, \phi)$ where ϕ is given by (1.2). Then $f'(z) + \alpha z f''(z) = \phi(w(z))$ for $z \in \mathbb{D}$, where $w(z) = \sum_{n=1}^{\infty} c_n z^n$ is the Schwarz function. A simple calculation yields

$$f'(z) + \alpha z f''(z) = 1 + B_1 c_1 z + (B_1 c_2 + B_2 c_1^2) z^2 + (B_1 c_3 + 2c_1 c_2 B_2 + B_3 c_1^3) z^3 + \cdots.$$

On comparing the coefficients of z , we obtain

$$\begin{aligned}
 2(1 + \alpha) a_2 &= B_1 c_1, \\
 3(1 + 2\alpha) a_3 &= B_1 c_2 + B_2 c_1^2, \\
 4(1 + 3\alpha) a_4 &= B_1 c_3 + 2B_2 c_1 c_2 + B_3 c_1^3.
 \end{aligned}$$

On substituting the above values of a_i ($i = 1, 2, 3$) in (3.2), we get

$$\begin{aligned}
 \gamma_1 &= \frac{B_1 c_1}{4(1+\alpha)}, \\
 \gamma_2 &= \frac{8(1+\alpha)^2 B_1 c_2 + (8(1+\alpha)^2 B_2 - 3(1+2\alpha) B_1^2) c_1^2}{48(1+\alpha)^2(1+2\alpha)}, \\
 \gamma_3 &= \frac{B_1}{8(1+3\alpha)} c_3 + \left(\frac{B_2}{4(1+3\alpha)} - \frac{B_1^2}{12(1+\alpha)(1+2\alpha)} \right) c_1 c_2 \\
 &\quad + \left(\frac{B_3}{8(1+3\alpha)} - \frac{B_1 B_2}{12(1+\alpha)(1+2\alpha)} + \frac{B_1^3}{48(1+\alpha)^3} \right) c_1^3.
 \end{aligned} \tag{3.3}$$

By using Lemma 3.1, we get the desired best possible estimate on γ_1 . The bound is sharp for $|c_1| = 1$.

$$\begin{aligned}
 |\gamma_2| &= \left| \frac{B_1}{6(1+2\alpha)} c_2 + \frac{8(1+\alpha)^2 B_2 - 3(1+2\alpha) B_1^2}{48(1+\alpha)^2(1+2\alpha)} c_1^2 \right| \\
 &\leq \frac{B_1}{6|1+2\alpha|} |c_2| + \frac{|8(1+\alpha)^2 B_2 - 3(1+2\alpha) B_1^2|}{48|(1+\alpha)^2||1+2\alpha|} |c_1^2| \\
 &\leq \frac{B_1}{6|1+2\alpha|} (1 - |c_1|^2) + \frac{|8(1+\alpha)^2 B_2 - 3(1+2\alpha) B_1^2|}{48|(1+\alpha)^2||1+2\alpha|} |c_1^2| \\
 &= \frac{B_1}{6|1+2\alpha|} + \left(\frac{|8(1+\alpha)^2 B_2 - 3(1+2\alpha) B_1^2|}{48|(1+\alpha)^2||1+2\alpha|} - \frac{B_1}{6|1+2\alpha|} \right) |c_1^2| \\
 &\leq \begin{cases} \frac{B_1}{6|1+2\alpha|} & \text{if } \frac{|8(1+\alpha)^2 B_2 - 3(1+2\alpha) B_1^2|}{48|(1+\alpha)^2||1+2\alpha|} \leq \frac{B_1}{6|1+2\alpha|} \\ \frac{|8(1+\alpha)^2 B_2 - 3(1+2\alpha) B_1^2|}{48|(1+\alpha)^2||1+2\alpha|} & \text{if } \frac{|8(1+\alpha)^2 B_2 - 3(1+2\alpha) B_1^2|}{48|(1+\alpha)^2||1+2\alpha|} > \frac{B_1}{6|1+2\alpha|}. \end{cases}
 \end{aligned}$$

These bounds are sharp for $|c_1| = 0$ and $|c_1| = 1$, respectively.

The third inequality follows by Lemma 3.1. Using Lemma 3.2 for γ_3 , we obtain

$$\begin{aligned}
 |\gamma_3| &= \left| \frac{B_1}{8(1+3\alpha)} c_3 + \left(\frac{B_2}{4(1+3\alpha)} - \frac{B_1^2}{12(1+\alpha)(1+2\alpha)} \right) c_1 c_2 \right. \\
 &\quad \left. + \left(\frac{B_3}{8(1+3\alpha)} - \frac{B_1 B_2}{12(1+\alpha)(1+2\alpha)} + \frac{B_1^3}{48(1+\alpha)^3} \right) c_1^3 \right| \\
 &= \frac{B_1}{8(1+3\alpha)} |c_3 + c_1 c_2 q_1 + c_1^3 q_2| \\
 &\leq \frac{B_1}{8(1+3\alpha)} H(q_1; q_2),
 \end{aligned}$$

where

$$\begin{aligned}
 q_1 &= 2 \left(\frac{B_2}{B_1} - \frac{B_1(1+3\alpha)}{3(1+\alpha)(1+2\alpha)} \right) \\
 \text{and } q_2 &= \frac{B_3}{B_1} - \frac{2B_2(1+3\alpha)}{3(1+\alpha)(1+2\alpha)} + \frac{B_1^2(1+3\alpha)}{6(1+\alpha)^3}.
 \end{aligned}$$

This completes the proof. $\square$

On taking $\phi(z) = e^z$, $\phi(z) = \sqrt{1+z}$ and $\phi(z) = (1+z)/(1-z)$, respectively in Theorem 3.1 the following corollaries follows immediately.

COROLLARY 3.1.1. *Let $\alpha \in \mathbb{C}$ such that $\operatorname{Re} \alpha \geq 0$ and $\phi(z) = e^z$. Suppose $f \in \mathcal{R}(\alpha, \phi)$, then the initial logarithmic coefficients of f satisfy the following inequalities:*

- (i) $|\gamma_1| \leq \frac{1}{4|1+\alpha|},$
- (ii) $|\gamma_2| \leq \frac{1}{6|1+2\alpha|},$
- (iii) $|\gamma_3| \leq \frac{1}{8(1+3\alpha)}H(q_1; q_2),$

where $H(q_1; q_2)$ is given in Lemma 3.2 such that

$$q_1 = \frac{1+3\alpha(1+2\alpha)}{3(1+\alpha)(1+2\alpha)}$$

and

$$q_2 = \frac{1}{6} - \frac{(1+3\alpha)}{3(1+\alpha)(1+2\alpha)} + \frac{(1+3\alpha)}{6(1+\alpha)^3}.$$

COROLLARY 3.1.2. *Suppose that $f \in \mathcal{R}(\alpha, \phi)$ where $\operatorname{Re} \alpha \geq 0$ and $\phi(z) = \sqrt{1+z}$, then the initial logarithmic coefficients of f satisfy the following inequalities:*

- (i) $|\gamma_1| \leq \frac{1}{8|1+\alpha|},$
- (ii) $|\gamma_2| \leq \frac{1}{12|1+2\alpha|},$
- (iii) $|\gamma_3| \leq \frac{1}{16(1+3\alpha)}H(q_1; q_2),$

where $H(q_1; q_2)$ is given in Lemma 3.2 such that

$$q_1 = -\frac{1}{2} - \frac{(1+3\alpha)}{3(1+\alpha)(1+2\alpha)}$$

and

$$q_2 = \frac{1}{8} + \frac{1+3\alpha}{12(1+\alpha)(1+2\alpha)} + \frac{1+3\alpha}{24(1+\alpha)^3}.$$

COROLLARY 3.1.3. *Let the function $f \in \mathcal{R}(\alpha, \phi)$ where $\operatorname{Re} \alpha \geq 0$ and $\phi(z) = (1+z)/(1-z)$, then the initial logarithmic coefficients of f satisfy the following inequalities:*

- (i) $|\gamma_1| \leq \frac{1}{2|1+\alpha|},$
- (ii) $|\gamma_2| \leq \frac{1}{3|1+2\alpha|},$
- (iii) $|\gamma_3| \leq \frac{1}{4(1+3\alpha)}H(q_1; q_2),$

where $H(q_1; q_2)$ is given in Lemma 3.2 such that

$$q_1 = \frac{2(1+3\alpha(1+2\alpha))}{3(1+\alpha)(1+2\alpha)}$$

and

$$q_2 = 1 - \frac{4(1+3\alpha)}{3(1+\alpha)(1+2\alpha)} + \frac{2(1+3\alpha)}{3(1+\alpha)^3}.$$

4. Inverse coefficient estimates

From the Koebe one quarter theorem, the image of $\mathbb{D}$ under a function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Thus for every univalent function f there exist inverse function f^{-1} such that $f^{-1}(f(z)) = z$ for $z \in \mathbb{D}$ and $f(f^{-1}(\omega)) = \omega$ for $|\omega| < r_0(f)$ where $r_0(f) \geq 1/4$. The function f^{-1} has the Taylor series expansion $f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + \dots$ in some neighborhood of origin. In 1923, Löwner [21] initiated the problem of estimating the coefficients of inverse function and investigated the coefficient estimates for inverse function $f \in \mathcal{S}$. Later on, this lead to the study of inverse coefficient problem for several subclasses of $\mathcal{S}$ by various authors [2, 18–20, 29]. In [13, 16], authors obtained the initial inverse coefficients for the well known classes $\mathcal{C}$ and $\mathcal{S}^*(\alpha)$ ($0 \leq \alpha \leq 1$). Recently, Ravichandran and Verma [30] determined the bounds on inverse coefficient for the Janowski starlike functions.

In this section, we shall investigate the bounds on inverse coefficient. The following lemma is needed to obtain the coefficient bounds for the inverse function.

LEMMA 4.1 ([18: Lemma 3, p. 254]). *If $p(z) = 1 + p_1z + p_2z^2 + \dots$ is a function in the class $\mathcal{P}$, then for any complex number ν ,*

$$|p_2 - \nu p_1^2| \leq 2 \max\{1, |2\nu - 1|\}.$$

THEOREM 4.1. *Let $\alpha \in \mathbb{C}$ such that $\operatorname{Re} \alpha \geq 0$ and the function ϕ defined by (1.2). If function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R}(\alpha, \phi)$ and $f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} A_n \omega^n$ for all ω in some neighbourhood of the origin, then*

$$(i) \quad |A_2| \leq \frac{B_1}{2|1 + \alpha|},$$

$$(ii) \quad |A_3| \leq \frac{B_1}{3|1 + 2\alpha|} \max\{1, |\mu|\}, \text{ where } \mu = \frac{3B_1(1 + 2\alpha)}{2(1 + \alpha)^2} - \frac{B_2}{B_1},$$

(iii) *If B_1, B_2, B_3 and α are real numbers, then*

$$|A_4| \leq \frac{B_1}{4(1 + 3\alpha)} H(q_1; q_2),$$

where $H(q_1; q_2)$ is given in Lemma 3.2 such that

$$q_1 = 2 \left(\frac{B_2}{B_1} - \frac{5B_1(1 + 3\alpha)}{3(1 + \alpha)(1 + 2\alpha)} \right)$$

and

$$q_2 = \frac{B_3}{B_1} - \frac{10B_2(1 + 3\alpha)}{3(1 + \alpha)(1 + 2\alpha)} + \frac{5B_1^2(1 + 3\alpha)}{2(1 + \alpha)^3}.$$

Proof. Let $f \in \mathcal{R}(\alpha, \phi)$. Then

$$f'(z) + \alpha z f''(z) = \phi(w(z)), \tag{4.1}$$

where $w(z)$ is the analytic function w with $w(0) = 0$ and $|w(z)| < 1$. Let

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1z + p_2z^2 + p_3z^3 + \dots.$$

Since $w: \mathbb{D} \rightarrow \mathbb{D}$ is analytic thus p is a function with positive real part and

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2}p_1z + \frac{1}{2}(p_2 - \frac{p_1^2}{2})z^2 + \frac{1}{8}(p_1^3 - 4p_1p_2 + 4p_3)z^3 + \dots. \tag{4.2}$$

Then

$$\begin{aligned}\phi(w(z)) &= 1 + \frac{B_1 p_1}{2} z + \left(\frac{1}{4} B_2 p_1^2 + \frac{1}{2} B_1 (p_2 - \frac{1}{2} p_1^2) \right) z^2 \\ &\quad + \frac{1}{8} ((B_1 - 2B_2 + B_3) p_1^3 + 4(-B_1 + B_2) p_1 p_2 + 4B_1 p_3) z^3 + \dots\end{aligned}\quad (4.3)$$

Using expressions (4.3) and (4.1), we obtain

$$\begin{aligned}2(1 + \alpha) a_2 &= \frac{B_1 p_1}{2}, \\ 3(1 + 2\alpha) a_3 &= \frac{1}{4} B_2 p_1^2 + \frac{1}{2} B_1 (p_2 - \frac{1}{2} p_1^2), \quad \text{and} \\ 4(1 + 3\alpha) a_4 &= \frac{1}{8} ((B_1 - 2B_2 + B_3) p_1^3 + 4(-B_1 + B_2) p_1 p_2 + 4B_1 p_3).\end{aligned}\quad (4.4)$$

Since $f^{-1}(\omega) = \omega + A_2 \omega^2 + A_3 \omega^3 + A_4 \omega^4 + \dots$ in some neighbourhood of origin, so we have $f(f^{-1}(\omega)) = \omega$. That is

$$\begin{aligned}\omega &= f^{-1}(\omega) + a_2 (f^{-1}(\omega))^2 + a_3 (f^{-1}(\omega))^3 + \dots \\ &= \omega + A_2 \omega^2 + A_3 \omega^3 + A_4 \omega^4 + \dots + a_2 (\omega + A_2 \omega^2 + A_3 \omega^3 + A_4 \omega^4 + \dots)^2 \\ &\quad + a_3 (\omega + A_2 \omega^2 + A_3 \omega^3 + A_4 \omega^4 + \dots)^3.\end{aligned}$$

A simple calculation gives the following realtions:

$$\begin{aligned}A_2 &= -a_2, \\ A_3 &= 2a_2^2 - a_3, \quad \text{and} \\ A_4 &= -5a_2^3 + 5a_2 a_3 - a_4.\end{aligned}\quad (4.5)$$

On substituting the values of a_i from (4.4) into (4.5) and a simple calculation yields

$$\begin{aligned}A_2 &= \frac{-B_1}{4(1 + \alpha)} p_1, \\ A_3 &= \frac{-B_1}{6(1 + 2\alpha)} p_2 + \left(\frac{B_1^2}{8(1 + \alpha)^2} - \frac{B_2}{12(1 + 2\alpha)} + \frac{B_1}{12(1 + 2\alpha)} \right) p_1^2.\end{aligned}$$

In (4.2), on taking $c_1 = \frac{p_1}{2}$, $c_2 = \frac{1}{2}(p_2 - \frac{p_1^2}{2})$, $c_3 = \frac{1}{8}(p_1^3 - 4p_1 p_2 + 4p_3)$ and so on we get,

$$\begin{aligned}2(1 + \alpha) a_2 &= B_1 c_1, \\ 3(1 + 2\alpha) a_3 &= B_1 c_2 + B_2 c_1^2, \quad \text{and} \\ 4(1 + 3\alpha) a_4 &= B_1 c_3 + 2B_2 c_1 c_2 + B_3 c_1^3.\end{aligned}\quad (4.6)$$

On substituting the values of a_i from (4.6) in (4.5), we obtain

$$\begin{aligned}A_4 &= \frac{-B_1}{4(1 + 3\alpha)} c_3 + \left(\frac{5B_1^2}{6(1 + \alpha)(1 + 2\alpha)} - \frac{B_2}{4(1 + 3\alpha)} \right) c_1 c_2 \\ &\quad + \left(\frac{-B_3}{4(1 + 3\alpha)} + \frac{5B_1 B_2}{6(1 + \alpha)(1 + 2\alpha)} + \frac{-5B_1^3}{8(1 + \alpha)^3} \right) c_1^3.\end{aligned}$$

Since $|p_1| \leq 2$, we have

$$|A_2| \leq \frac{B_1}{2|1 + \alpha|}.$$

Consider

$$|A_3| = \frac{B_1}{6|1 + 2\alpha|} \left| p_2 - \left(\frac{3B_1(1 + 2\alpha)}{4(1 + \alpha)^2} - \frac{B_2}{2B_1} + \frac{1}{2} \right) p_1^2 \right|.$$

Then by Lemma 4.1, we get the desired estimate. Using Lemma 3.2 for A_4 , we obtain

$$\begin{aligned} |A_4| &= \left| \frac{-B_1}{4(1+3\alpha)} c_3 + \left(\frac{-B_2}{2(1+3\alpha)} + \frac{5B_1^2}{6(1+\alpha)(1+2\alpha)} \right) c_1 c_2 \right. \\ &\quad \left. + \left(\frac{-B_3}{4(1+3\alpha)} + \frac{5B_1 B_2}{6(1+\alpha)(1+2\alpha)} - \frac{5B_1^3}{8(1+\alpha)^3} \right) c_1^3 \right| \\ &= \frac{B_1}{4(1+3\alpha)} |c_3 + c_1 c_2 q_1 + c_1^3 q_2| \\ &\leq \frac{B_1}{4(1+3\alpha)} H(q_1; q_2), \end{aligned}$$

where

$$q_1 = 2 \left(\frac{B_2}{B_1} - \frac{5B_1(1+3\alpha)}{3(1+\alpha)(1+2\alpha)} \right)$$

and

$$q_2 = \frac{B_3}{B_1} - \frac{10B_2(1+3\alpha)}{3(1+\alpha)(1+2\alpha)} + \frac{5B_1^2(1+3\alpha)}{2(1+\alpha)^3}. \quad \square$$

The following corollaries are an immediate consequence of the Theorem 4.1 for $\phi(z) = e^z$, $\phi(z) = \sqrt{1+z}$ and $\phi(z) = (1+z)/(1-z)$, respectively.

COROLLARY 4.1.1. *Let $\alpha \in \mathbb{C}$ such that $\operatorname{Re} \alpha \geq 0$ and $\phi(z) = e^z$. If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R}(\alpha, \phi)$ and $f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} A_n \omega^n$ for all ω in some neighbourhood of the origin, then*

- (i) $|A_2| \leq \frac{1}{2|1+\alpha|},$
- (ii) $|A_3| \leq \frac{1}{3|1+2\alpha|} \max \left\{ 1, \left| \frac{3(1+2\alpha)}{2(1+\alpha)^2} - \frac{1}{2} \right| \right\},$
- (iii) $|A_4| \leq \frac{1}{4(1+3\alpha)} H(q_1; q_2),$

where α is real and $H(q_1; q_2)$ is given in Lemma 3.2 such that

$$q_1 = \frac{-7+3\alpha(-7+2\alpha)}{3(1+\alpha)(1+2\alpha)}$$

and

$$q_2 = \frac{1}{6} - \frac{5(1+3\alpha)}{3(1+\alpha)(1+2\alpha)} + \frac{5(1+3\alpha)}{2(1+\alpha)^3}.$$

COROLLARY 4.1.2. *Suppose that the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R}(\alpha, \phi)$, where $\operatorname{Re} \alpha \geq 0$ and $\phi(z) = \sqrt{1+z}$ and $f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} A_n \omega^n$ for all ω in some neighbourhood of the origin, then*

- (i) $|A_2| \leq \frac{1}{4|1+\alpha|},$
- (ii) $|A_3| \leq \frac{1}{6|1+2\alpha|} \max \left\{ 1, \left| \frac{3(1+2\alpha)}{4(1+\alpha)^2} + \frac{1}{4} \right| \right\},$
- (iii) $|A_4| \leq \frac{1}{8(1+3\alpha)} H(q_1; q_2),$

where α is real and $H(q_1; q_2)$ is given in Lemma 3.2 such that

$$q_1 = -\frac{1}{2} - \frac{5(1+3\alpha)}{3(1+\alpha)(1+2\alpha)}$$

and

$$q_2 = \frac{1}{8} + \frac{5(1+3\alpha)}{12(1+\alpha)(1+2\alpha)} + \frac{5(1+3\alpha)}{8(1+\alpha)^3}.$$

5. Hankel determinant

The problem involving coefficient bounds have attracted the interest of many authors in particular to second Hankel determinants and Fekete Szegő functional. The coefficient functional $|a_3 - \mu a_2^2|$ where μ is a complex number is called the Fekete Szegő functional. The Fekete Szegő problem involves maximizing the functional $|a_3 - \mu a_2^2|$. Some authors have investigated the Fekete Szegő problem for the coefficients of inverse function [2, 25, 38]. The Hankel determinant $|H_2(1)| = |a_3 - a_2^2|$ is a particular case of the Fekete Szegő functional and $H_2(2) = |a_2 a_4 - a_3^2|$ is called the second Hankel determinants. In 2013, Lee et. al. [17] obtained the bounds for the second Hankel determinant for the unified class of Ma-Minda starlike and convex functions. The authors [38] estimated the bounds on second Hankel determinant for the class of strongly convex functions of order α using the inverse coefficients. One may refer to [8, 9, 14, 15, 27] for more details. In the present section, we shall determine the Fekete Szegő functional for the inverse coefficient.

THEOREM 5.1. *Suppose $\alpha \in \mathbb{C}$ such that $\operatorname{Re} \alpha \geq 0$ and the function ϕ defined by (1.2). Let $f \in \mathcal{R}(\alpha, \phi)$ and $f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} A_n \omega^n$ for all ω in some neighbourhood of the origin. Then for any complex number μ , we have*

$$|A_3 - \mu A_2^2| \leq \frac{B_1}{3|1+2\alpha|} \max \left\{ 1, \left| \frac{3B_1(1+2\alpha)}{4(1+\alpha)^2} (\mu - 2) + \frac{B_2}{B_1} \right| \right\}.$$

Proof. In view of equations (4.4) and (4.5), we get

$$\begin{aligned} |A_3 - \mu A_2^2| &= \left| \frac{-B_1}{6(1+2\alpha)} p_2 + \left(\left(\frac{B_1^2}{8(1+\alpha)^2} + \frac{B_1 - B_2}{12(1+2\alpha)} \right) - \mu \frac{B_1^2}{16(1+\alpha)^2} \right) p_1^2 \right| \\ &= \left| \frac{B_1}{6(1+2\alpha)} \left(p_2 + \left(\frac{-3B_1(1+2\alpha)}{4(1+\alpha)^2} + \frac{B_2}{2B_1} - \frac{1}{2} + \mu \frac{3B_1(1+2\alpha)}{8(1+\alpha)^2} \right) p_1^2 \right) \right| \\ &= \left| \frac{B_1}{6(1+2\alpha)} (p_2 - \nu p_1^2) \right|, \end{aligned}$$

where $\nu = \frac{3B_1(1+2\alpha)}{8(1+\alpha)^2} (2 - \mu) - \frac{B_2}{2B_1} + \frac{1}{2}$. By Lemma 4.1, we get the required result. $\square$

THEOREM 5.2. *Let $\alpha \in \mathbb{C}$ such that $\operatorname{Re} \alpha \geq 0$ and the function ϕ defined by (1.2). Suppose function f in $\mathcal{R}(\alpha, \phi)$ and $f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} A_n \omega^n$ for all ω in some neighbourhood of the origin.*

(i) *If B_1, B_2 and B_3 satisfy the conditions*

$$4d_2 \leq d_3, \quad d_1 \leq \frac{B_1|1+\alpha|}{9|(1+2\alpha)^2|},$$

then

$$|A_2A_4 - A_3^2| \leq \frac{B_1^2}{9|(1+2\alpha)^2|}.$$

(ii) If B_1, B_2 and B_3 satisfy the conditions

$$4d_2 \geq d_3, \quad d_1 - \frac{d_2}{2} - \frac{B_1}{16|1+3\alpha|} \geq 0$$

or

$$4d_2 \leq d_3, \quad d_1 \geq \frac{B_1|1+\alpha|}{9|(1+2\alpha)^2|},$$

then

$$|A_2A_4 - A_3^2| \leq \frac{B_1d_1}{|1+\alpha|}.$$

(iii) If B_1, B_2 and B_3 satisfy the conditions

$$4d_1 > d_3, \quad d_1 - \frac{d_2}{2} - \frac{B_1}{16|1+3\alpha|} \leq 0,$$

then

$$|A_2A_4 - A_3^2| \leq \frac{\frac{B_1}{16|1+\alpha|} \left(\frac{16B_1|1+\alpha|}{9|(1+2\alpha)^2|} d_1 - \frac{B_1}{|1+3\alpha|} d_2 - 4d_2^2 - \frac{B_1^2}{16|(1+3\alpha)^2|} \right)}{d_1 - d_2 - \frac{B_1}{8|1+3\alpha|} + \frac{B_1|1+\alpha|}{9|(1+2\alpha)^2|}},$$

where

$$d_1 = \frac{B_1^3}{16|(1+\alpha)^3|} + \frac{B_2^2|1+\alpha|}{9B_1|(1+2\alpha)^2|} + \frac{B_1|B_2|}{12|1+\alpha||1+2\alpha|} + \frac{|B_3|}{8|1+3\alpha|},$$

$$d_2 = \frac{B_1^2}{12|1+\alpha||1+2\alpha|} + \frac{|B_2|}{4|1+3\alpha|} + \frac{2|B_2||1+\alpha|}{9|(1+2\alpha)^2|},$$

and

$$d_3 = \frac{8B_1|1+\alpha|}{9|(1+2\alpha)^2|} - \frac{B_1}{2|1+3\alpha|}.$$

In the proof of this result, the following lemma is needed.

LEMMA 5.1 ([18: Lemma 2, p. 254]). *If $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$, then*

$$2p_2 = p_1^2 + x(4 - p_1^2),$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z$$

for some x and z such that $|x| \leq 1$ and $|z| \leq 1$.

Remark 3. For real numbers P, Q and R , a standard computation gives

$$\max_{0 \leq t \leq 4} (Pt^2 + Qt + R) = \begin{cases} R, & Q \leq 0, P \leq -Q/4, \\ 16P + 4Q + R, & Q \geq 0, P \geq -Q/8 \text{ or } Q \leq 0, P \geq -Q/4, \\ \frac{4PR - Q^2}{4P}, & Q > 0, P \leq -Q/8. \end{cases} \quad (5.1)$$

Proof of Theorem 5.2. It follows from equations (4.4) and (4.5) that

$$\begin{aligned}
 A_2 A_4 - A_3^2 &= \frac{-B_1 p_1}{4(1+\alpha)} \left(\frac{-B_1}{8(1+3\alpha)} p_3 + \left(\frac{5B_1^2}{24(1+\alpha)(1+2\alpha)} + \frac{B_1 - B_2}{8(1+3\alpha)} \right) p_1 p_2 \right. \\
 &\quad \left. + \left(\frac{-5B_1^3}{6(1+\alpha)^3} - \frac{5B_1(B_2 - B_1)}{48(1+\alpha)(1+2\alpha)} - \frac{B_1 - 2B_2 + B_3}{32(1+3\alpha)} \right) p_1^3 \right) \\
 &\quad - \left(\left(\frac{B_1^2}{8(1+\alpha)^2} + \frac{(B_1 - B_2)}{12(1+2\alpha)} \right) p_1^2 - \frac{B_1 p_2}{6(1+2\alpha)} \right)^2 \\
 &= p_1^4 \left(\frac{B_1^4}{256(1+\alpha)^4} + \frac{B_1^2(B_1 - B_2)}{192(1+\alpha)^2(1+2\alpha)} + \frac{B_1(B_1 - 2B_2 + B_3)}{128(1+\alpha)(1+3\alpha)} \right. \\
 &\quad \left. - \frac{(B_1 - B_2)^2}{144(1+2\alpha)^2} \right) + \frac{B_1^2 p_1 p_3}{32(1+\alpha)(1+3\alpha)} - \frac{B_1^2 p_2^2}{36(1+2\alpha)^2} \\
 &\quad + p_1^2 p_2 \left(\frac{-B_1^3}{96(1+\alpha)^2(1+2\alpha)} - \frac{B_1(B_2 - B_1)}{32(1+\alpha)(1+3\alpha)} + \frac{B_1(B_1 - B_2)}{36(1+2\alpha)^2} \right).
 \end{aligned}$$

In view of Lemma 5.1, we obtain

$$\begin{aligned}
 A_2 A_4 - A_3^2 &= \frac{B_1}{16(1+\alpha)} \left[p_1^4 \left(\frac{B_1^3}{16(1+\alpha)^3} - \frac{B_2^2(1+\alpha)}{9B_1(1+2\alpha)^2} - \frac{B_1 B_2}{12(1+\alpha)(1+2\alpha)} + \frac{B_3}{8(1+3\alpha)} \right) \right. \\
 &\quad \left. + 2p_1^2(4 - p_1^2)x \left(-\frac{B_1^2}{24(1+\alpha)(1+2\alpha)} + \frac{B_2}{8(1+3\alpha)} - \frac{B_2(1+\alpha)}{9(1+2\alpha)^2} \right) \right. \\
 &\quad \left. + (4 - p_1^2)x^2 p_1^2 \left(\frac{-B_1}{8(1+3\alpha)} \right) - (4 - p_1^2)^2 x^2 \left(\frac{B_1(1+\alpha)}{9(1+2\alpha)^2} \right) \right. \\
 &\quad \left. + 2p_1(4 - p_1^2)(1 - |x|^2)z \frac{B_1}{8(1+3\alpha)} \right].
 \end{aligned}$$

Since $|p_1| \leq 2$ and it can be assumed that $p_1 > 0$ and thus we get that $p_1 \in [0, 2]$. Letting $p_1 = p$ and $|x| = \gamma$ in the above expression, we get

$$\begin{aligned}
 |A_2 A_4 - A_3^2| &\leq \frac{B_1}{16|1+\alpha|} \left[p^4 \left(\frac{B_1^3}{16|1+\alpha|^3} + \frac{B_2^2|1+\alpha|}{9B_1|(1+2\alpha)^2|} + \frac{B_1|B_2|}{12|1+\alpha||1+2\alpha|} + \frac{|B_3|}{8|1+3\alpha|} \right) \right. \\
 &\quad \left. + 2p^2(4 - p^2)\gamma \left(\frac{B_1^2}{24|1+\alpha||1+2\alpha|} + \frac{|B_2|}{8|1+3\alpha|} + \frac{|B_2||1+\alpha|}{9|(1+2\alpha)^2|} \right) \right. \\
 &\quad \left. + (4 - p^2)\gamma^2 p^2 \left(\frac{B_1}{8|1+3\alpha|} \right) + (4 - p^2)^2 \gamma^2 \left(\frac{B_1|1+\alpha|}{9|(1+2\alpha)^2|} \right) \right. \\
 &\quad \left. + 2p(4 - p^2)(1 - \gamma^2)|z| \frac{B_1}{8|1+3\alpha|} \right].
 \end{aligned}$$

$$=: G(p, \gamma).$$

Let p be fixed. Using the first derivative test in the region $\Omega = \{(p, \gamma) : 0 \leq p \leq 2, 0 \leq \gamma \leq 1\}$ we get that $G(p, \gamma)$ is an increasing function of γ where $\gamma \in [0, 1]$. Thus for fixed $p \in [0, 2]$, we obtain $\max_{0 \leq \gamma \leq 1} G(p, \gamma) = G(p, 1) =: F(p)$, where

$$\begin{aligned} F(p) = & \frac{B_1}{16|1+\alpha|} \left[p^4 \left(\frac{B_1^3}{16|(1+\alpha)^3|} + \frac{B_2^2|1+\alpha|}{9B_1|(1+2\alpha)^2|} + \frac{B_1|B_2|}{12|1+\alpha||1+2\alpha|} + \frac{|B_3|}{8|1+3\alpha|} \right. \right. \\ & \left. \left. - \frac{B_1^2}{12|1+\alpha||1+2\alpha|} - \frac{|B_2|}{4|1+3\alpha|} - \frac{2|B_2||1+\alpha|}{9|(1+2\alpha)^2|} - \frac{B_1}{8|1+3\alpha|} + \frac{B_1|1+\alpha|}{9|(1+2\alpha)^2|} \right) \right. \\ & \left. + p^2 \left(\frac{B_1^2}{3|1+\alpha||1+2\alpha|} + \frac{|B_2|}{|1+3\alpha|} + \frac{8|B_2||1+\alpha|}{9|(1+2\alpha)^2|} + \frac{B_1}{2|1+3\alpha|} - \frac{8B_1|1+\alpha|}{9|(1+2\alpha)^2|} \right) \right. \\ & \left. + \frac{16B_1|1+\alpha|}{9|(1+2\alpha)^2|} \right]. \end{aligned}$$

Let

$$\begin{aligned} P = & \frac{B_1}{16|1+\alpha|} \left[\left(\frac{B_1^3}{16|(1+\alpha)^3|} + \frac{B_2^2|1+\alpha|}{9B_1|(1+2\alpha)^2|} + \frac{B_1|B_2|}{12|1+\alpha||1+2\alpha|} + \frac{|B_3|}{8|1+3\alpha|} \right) \right. \\ & \left. - \left(\frac{B_1^2}{12|1+\alpha||1+2\alpha|} + \frac{|B_2|}{4|1+3\alpha|} + \frac{2|B_2||1+\alpha|}{9|(1+2\alpha)^2|} \right) + \left(-\frac{B_1}{8|1+3\alpha|} + \frac{B_1|1+\alpha|}{9|(1+2\alpha)^2|} \right) \right], \\ Q = & \frac{B_1}{16|1+\alpha|} \left[4 \left(\frac{B_1^2}{12|1+\alpha||1+2\alpha|} + \frac{B_2}{4|1+3\alpha|} + \frac{2|B_2||1+\alpha|}{9|(1+2\alpha)^2|} \right) \right. \\ & \left. - \left(\frac{8B_1|1+\alpha|}{9|(1+2\alpha)^2|} - \frac{B_1}{2|1+3\alpha|} \right) \right], \\ R = & \frac{B_1^2}{9|(1+2\alpha)^2|}, \quad \text{and} \quad t = p^2. \end{aligned}$$

Then $F(t) = Pt^2 + Qt + R$. Using the inequality (5.1), we get the required result. $\square$

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**Department of Mathematics*

University of Delhi

Delhi-110 007

INDIA

E-mail: swati_anand01@yahoo.com

***Department of Mathematics*

Aryabhatta College

Delhi-110021

INDIA

E-mail: naveenjain05@gmail.com

****Bharati Vidyapeeth's college of Engineering*

Delhi-110063

INDIA

E-mail: sushilkumar16n@gmail.com